DT-14.01.10

MODULE-I

SIGNAL :-

Anything that carries some information can be called as a signal

OR

It is defined as a physical quantity that varies with time, pressure, temperature or any independent variable.

Eg. Speech signal, video signal

If a signal depends only on one independent variable — One dimensional signal.

Eg. Speech signal, ac power signal
(amplitude varies with respect to time)

If a signal depends on two independent variab

Eg. Image or picture is a 2-D signal with vertical and horizontal coordinates.

Multidimensional Signal.

Many signals that we come across which are created naturally like air pressure that depends on latitude, longitude, elevation and time — four dimensional signal.

Classification of Signals:-

Signal Processing:-

It is an operation which changes the characterist of a signal (amplitude, shape, phase & frequency content)

Based on nature and characteristics in time domain Signals are classified as

Continuous Time Signal

Discrete Time Signal

Continuous Time Signal:

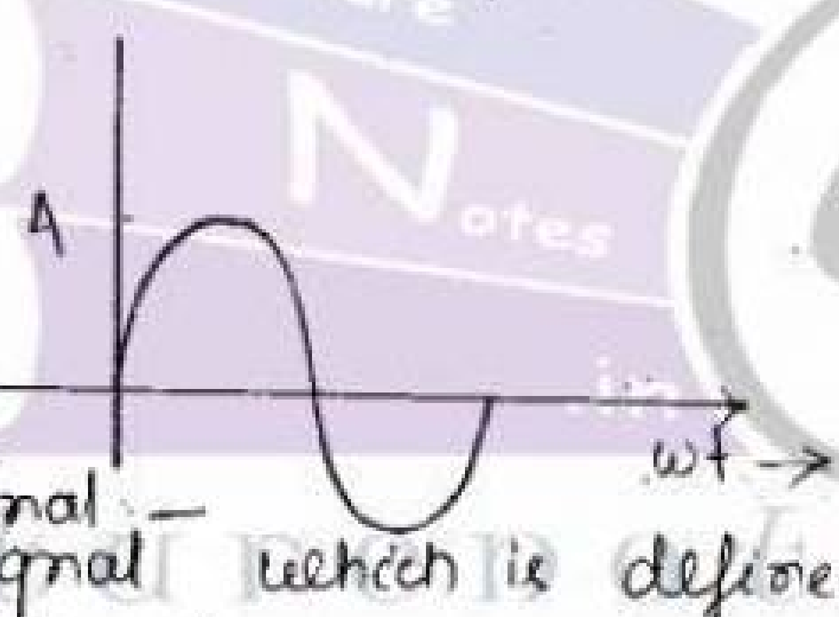
It is defined as a signal which is defined over a continuous range of time or which is defined over a preinstant of time.

Eg. All mathematical continuous signals like sine, cosine waves

represented by means of $x(t)$

x = Shape of the signal

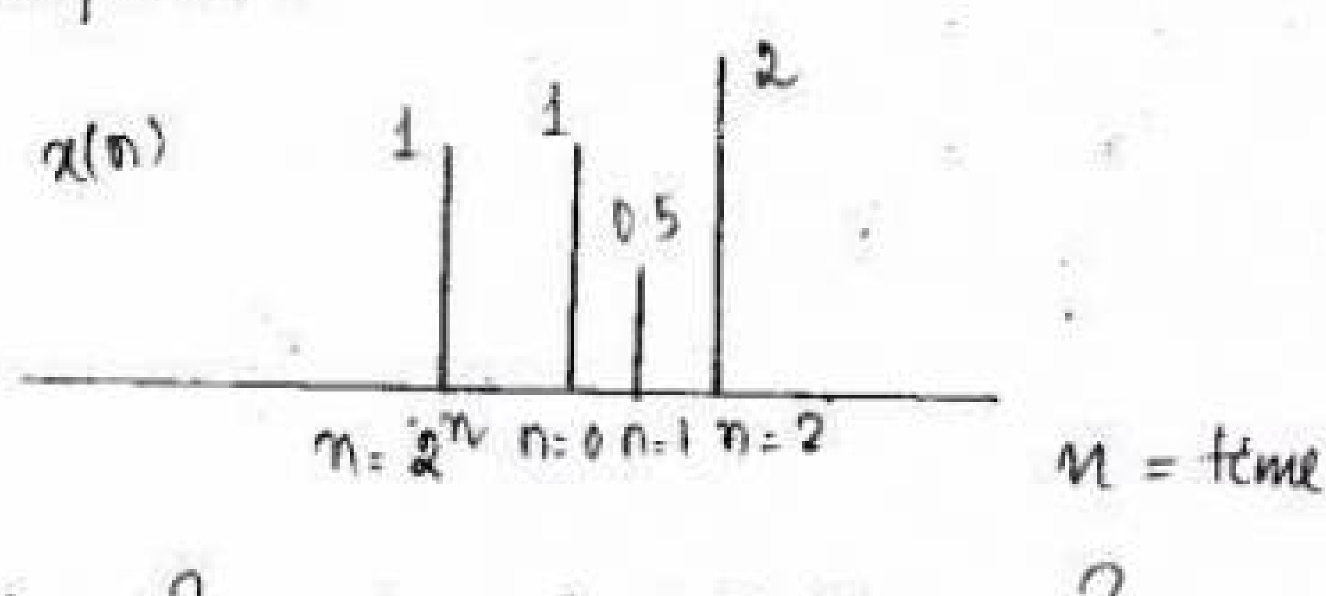
t = time (as a independent variable)



Discrete Time Signal -

It is a signal which is defined in discrete instant of time.

The amplitude between the 2 time intervals is just not defined.



$$x(n) = \{ \ldots 1, 0, \underset{\uparrow}{1}, 0.5, 2 \ldots \}$$

This arrow indicates amplitude 1 at $n=0$.

We may classify both continuous and discrete time signal.

Deterministic & Non-Deterministic Signal.
Periodic & Aperiodic signal.
Power & Energy Signal.
Even and Odd ~~symmetric~~ signal.
(symmetric) (anti-symmetric)

Deterministic Signal :-

→ It can be specified in time or we can say this signal has certain values for every instant of time.

The shape of this signal is regular and can be calculated mathematically (by any mathematical functions)

→ The nature and amplitude can be determined at any instant

$x(t) = B$ (Ramp signal)

$x(t) = \sin\omega t$ or $\cos\omega t$
(Sinusoidal)

$x(n) = 2$ for $n \leq 2$
$= 0$ for other values } amplitude is specified

Non-Deterministic Signal :-

→ It is one whose occurrence is random.

→ It is also called as a random signal.

Eg. Noise Signal.

A Periodic and Aperiodic Signal:

Periodic signal is one which repeats after a fixed time & Aperiodic Signal doesn't repeat after a fixed time

Consider a signal $x(t) = \sin(\Omega_0 t + \theta) \rightarrow$ discontinu

$x(n) = \sin(\omega_0 n + \theta) \rightarrow$ continuou

ω_0 = freq. in rad/sec

θ = in radian (phase)

$\rightarrow x(t) = \sin(\Omega_0 t + \theta)$ to be periodic if it satisfies

$x(t+T) = x(t)$

$x(t+T) = \sin\{\Omega_0(t+T) + \theta\}$

$= \sin\{\Omega_0 t + \Omega_0 T + \theta\}$

$=$ ~~Si~~ if ~~$\Omega_0 T + \theta$~~ $= \Omega_0 T = 2\pi$

$= \sin\{\Omega_0 t + \theta\}$. $\quad T = \frac{2\pi}{\Omega_0}$.

Hence, it is periodic $[x(t) = x(t+T)]$

$x(n) = \sin(\omega_0 n + \theta)$

$x(n+N) = \sin(\omega_0(n+N) + \theta)$

$= \sin(\omega_0 n + \omega_0 N + \theta)$

if $\omega_0 N$ = integer multiple of 2π

$= 2\pi K \quad N = \frac{2\pi}{\omega_0} K$

- We take a min value of K for which N will be an integer (Because N will not always be an integer)

Q) Find period of $x(n) = \cos \frac{2\pi}{3} n$.

$\omega_0 = 2\pi/3$.

$\therefore T = 2\pi/\omega_0 = \frac{2\pi}{2\pi/3} = 3$.

TEXTBOOK
Ramesh Babu
Salivahan

$x(n) = \cos \frac{3\pi}{5} n$

$\omega_0 = 3\pi/5$

$T = 2\pi/\omega_0 = \frac{10}{3} \neq$ integer.

So, we have to take k.

$T = \frac{2\pi k}{\omega_0} = \frac{10}{3}k$.

→ $x(t) = x_1(t) + x_2(t)$.

If T_1 is the time period of time signal $x_1(t)$.
T_2 is the time period of time signal $x_2(t)$

→ $\frac{T_1}{T_2}$ = Ratio of 2 integers.

Then the total time period of this composite signal is the LCM of 2 time periods of 2 signals $x_1(t)$ & $x_2(t)$

→ $\frac{T_1}{T_2}$ = irrational $\neq$ ~~not~~ a ratio of 2 integers

Then the total signal is aperiodic.

→ If T_1/T_2 has no common factor other than 1, then the total time period is $\Rightarrow BT_1 = AT_2$

$T_1/T_2 = 3/5 \Rightarrow 5T_1 = 3T_2$

If $T_1/T_2 = 6/8$, then the total time period of this signal is 24 (not 12)

$\Rightarrow$ LCM of 6, 8

In case of discrete time signal,

if $\kappa\ \frac{n_1}{n_2}$ = rational = ratio of 2 integers

Then the total time period is the LCM of two.

$$x(n) = \cos\frac{n}{8}$$

$$\omega = 1/8$$

$$T = 2\pi/\omega = 2\pi\times 8 = 16\pi \neq \text{rational no.}$$

$T \neq$ LCM of two signals.

If ω_0 contains π terms, then the discrete time signal is periodic

If ω_0 doesn't contains π terms - aperiodic.

Q) If check this signal is periodic or aperiodic

(a) $x(n) = e^{j6\pi n}$

(b) $x(n) = e^{j\,3/5(n+1/2)}$

(c) $x(n) = \cos(2\pi/5\, n)$

(d) $x(n) = \cos \pi/3\, n + \cos 3\pi/4\, n$

(e) $x(t) = \cos t + \sin\sqrt{2}t$

Ans- $x(t) = \cos t + \sin\sqrt{2}t$.

$$\omega = 1$$

$$T = 2\pi/1 = 2\pi$$

$$\omega = \sqrt{2}$$

$$T = 2\pi/\sqrt{2} = \sqrt{2}\pi$$

The signal is aperiodic

(a) $x(n) = e^{j6\pi n}$

$= [\cos 6\pi n + j\sin 6\pi n]$

$\omega = 6\pi$

$T = 2\pi/6\pi = 1/3$ = irrational.

we have to take $\cdot K$

$T = \frac{1}{3}K$ $\quad K = 3$.

(b) $x(n) = e^{j\,3/5(n+1/2)}$

$\omega = 3/5(n+1/2)$

$n\pi = 2\pi/\omega = \frac{2\pi}{3/5(n+1/2)}$. not a multiple of π so, ~~not a~~ aperiodic signal

(c) $x(n) = \cos(2\pi/5\, n)$

$\omega = \cancel{3/5}\; 2\pi/5 \cancel{n}$

$n\pi = 2\pi/\omega = \frac{2\pi}{2\pi/5 \cancel{n}} = \cancel{2/5}\; 5/\cancel{n}\; 5$

= periodic

(d) $x(n) = \cos \pi/3\, n + \cos 3\pi/4\, n$.

$\omega = \pi/3 \quad n_1 = 2\pi/\omega = 2\pi/\pi/3$

$= 2\pi \times 3/\pi = 6$.

$\omega = 3\pi/4 \quad n_2 = 2\pi \times \frac{4}{3\pi} = 8/3 \Rightarrow$ Here, $K = 3$

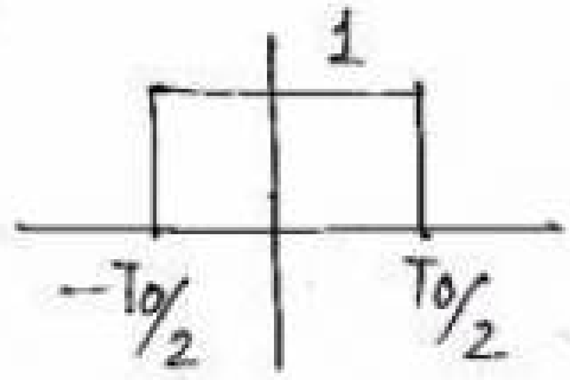
$\frac{n_1}{n_2} \cancel{\frac{T_1}{T_2}} = \frac{6 \times 3}{8} = 6/8$

$T = n =$ LCM of 6 & 8 = 24.

Q. $x(t) = \text{rect}(t/T_0)$

$= 1$ for $-T_0/2 \le t \le T_0/2$

$= 0$ for other value.



Dt-17.01.11

Energy Signal & Power Signal :-

For an energy signal, $0 < E < \infty$ & $P = 0$

For a power signal, $0 < P < \infty$ & $E = 0$.

For a real ~~distinct~~ discrete type signal,

$$\boxed{\begin{aligned} E &= \sum_{-\infty}^{\infty} |x(n)|^2 \\ P &= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N}^{N} |x(n)|^2 \end{aligned}}$$

$|x(n)|$ ↓ for complex type signal

For continuous time signal,

$$\boxed{\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 \\ P &= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \end{aligned}}$$

For an signal of infinite durat the signal is neither a power signal nor a energy signal.

Important Steps :

→ If a signal is periodic and of infinite duratio then the signal is power signal. Hence, we can calculate the power directly.

→ If the signal is periodic for a finite durat then the signal can be a energy signal.

→ If the signal is aperiodic, then the signal can be a energy signal.

(*) $\xrightarrow{}$ unit step signal

Q) $x(n) = u(n)$ → periodic signal & of infinite duration

So, the signal is a power signal.

Q) $x(n) = \left(\frac{1}{3}\right)^n u(n)$ $\qquad P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |x(n)|^2$

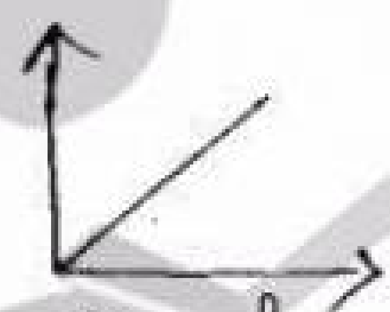
The signal is aperiodic & of infinite duration $\qquad P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{0}^{N} 1$

$E = \sum_{-\infty}^{\infty} x^2(n)$ $\qquad = \lim_{N\to\infty} \frac{1}{2N+1}(N+1)$

$= \sum_{0}^{\infty} 1$ $\qquad = \frac{1}{2}$

$= 1+1+1+\dots+\infty$

$= \infty$

Q) $x(t) = t\,u(t)$ (st. line) → 

Neither a energy nor a power signal

Q) $x(n) = \left(\frac{1}{3}\right)^n u(n)$ → aperiodic & of infinite duration

So, it is a energy signal.

$$E = \sum_{-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{0}^{\infty} \left(\frac{1}{3}\right)^{2n} = \sum_{0}^{\infty} \left(\frac{1}{9}\right)^n$$

$$= 1 + \frac{1}{9} + \frac{1}{9^2} + \dots \infty$$

$$= \frac{1}{1-\frac{1}{9}} = \frac{9}{8}$$

$x(t) = a + u(t)$

$$E = \sum_{-\infty}^{\infty} x_n^2$$

$$= \sum_{0}^{\infty} 7.1 = 0 + \ldots + \infty = 0$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N}^{N} x_n^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{0}^{N} T^2$$

$$= \frac{1}{2T} \times T^2 =$$

→ When a signal is processed, the signal undergo some manipulations.
Some of the manipulations are

(1) Shifting the ~~time~~ signal in time domain.

(2) Folding the signal in the time domain & Scaling the signal in time domain.

(3) Time scaling

(4) When a signal is processed, the signal may be shifted in the time domain i.e the signal can be advanced ~~& de~~ or delayed in the time domain and the shifted signal is represented by

If $x(n)$ = discrete time sequence

n = independent variable

It is represented by $x(n-k)$ where K = is an integer

→ If K value is +ve, i.e the signal is delayed in the time domain or the signal is shifted to the right by 'K' instant of time in time domain

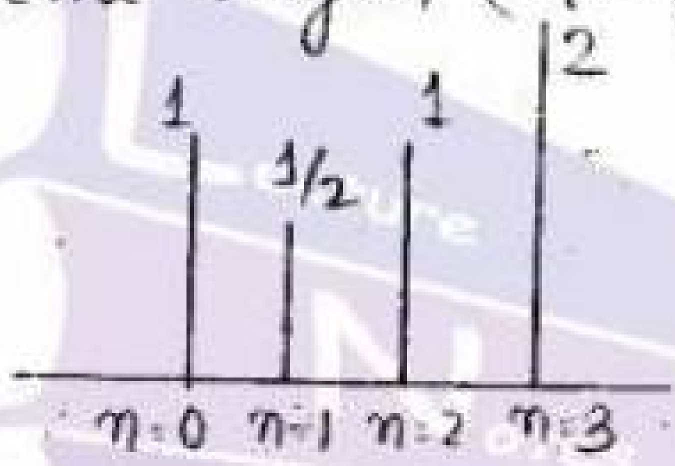
→ If K value is -ve, then the signal is shifted to the left in the time domain i.e the signal is advanced.

→ Folding

The folding operation is done by simply replacing $x(n)$ by $x(-n)$

Folding is the ~~rep~~ reflection of the signal about time orgin $(n=0)$

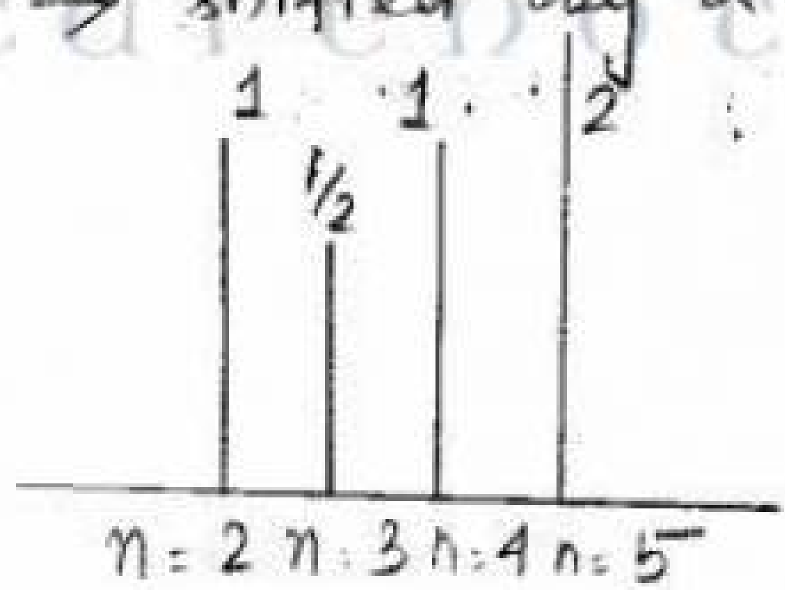
Q)



$x(n) = \{1, \frac{1}{2}, 1, 2\}$

$x(n-2) \rightarrow$ shifted by 2 units (but amplitude remains same)

$K = +2$



Right shifting.

$y(n) = x(n-2)$

$y(0) = x(-2) = 0$

$y(1) = x(-1) = 0$

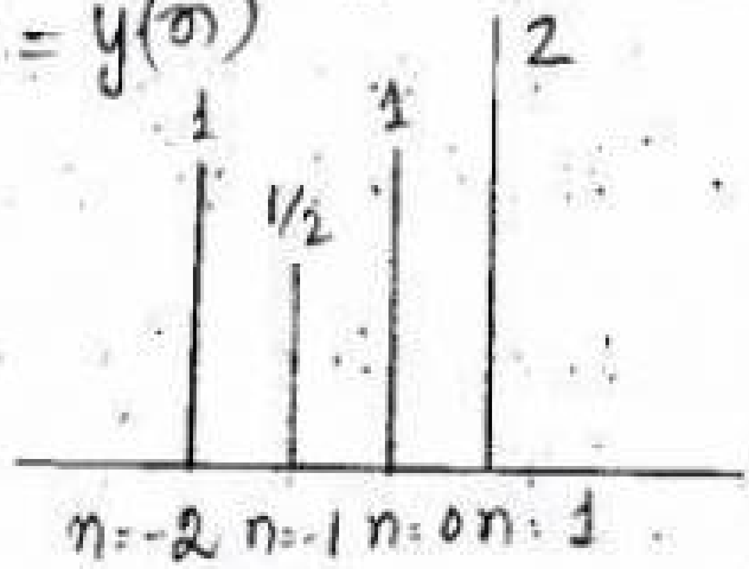
$y(2) = x(0) = 1$

$y(3) = x(1) = \frac{1}{2}$

$y(4) = x(2) = 1$

$y(5) = x(3) = 2$

$x(n+2) = y(n)$



Left shifting
$K = -2$

$y(0) = x(2) = 1$

$y(1) = x(3) = 2$

$y(2) = x(4) = 0$

$y(-1) = x(1) = 1/2$

$y(-2) = x(0) = 1$

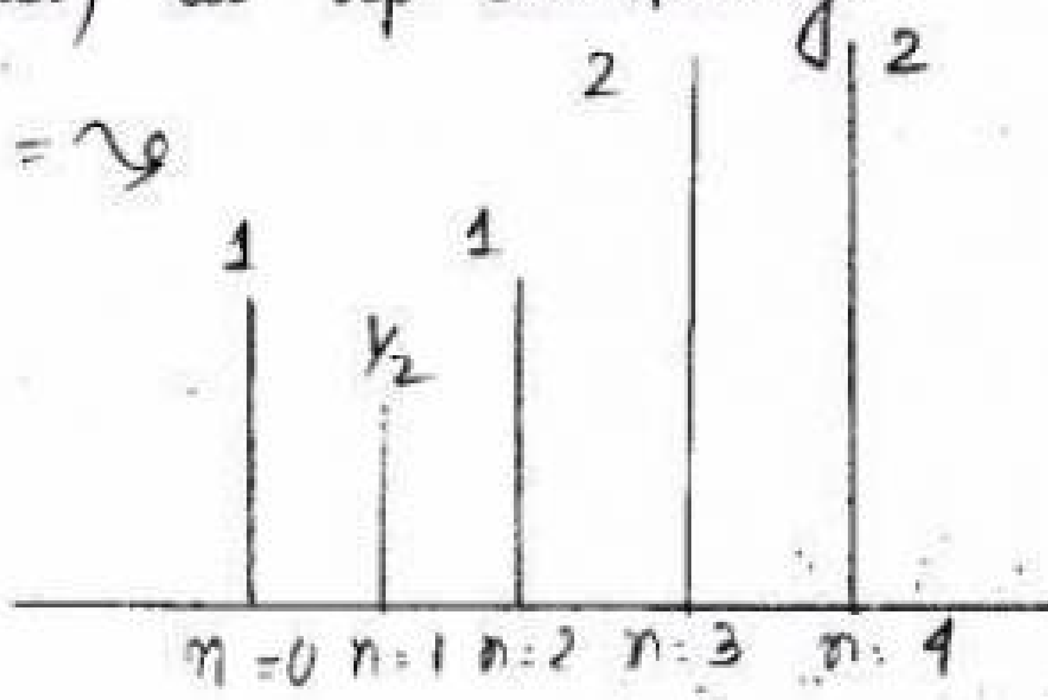
Time Scaling:-

The operation is done by simply replacing n/K into n.

If K = integer $\Rightarrow$ Time scaling is known as down sampling

If $K \neq$ integer ($1/3$, $1/2$) $\Rightarrow$ Time scaling is known as up sampling

$x(n) =$



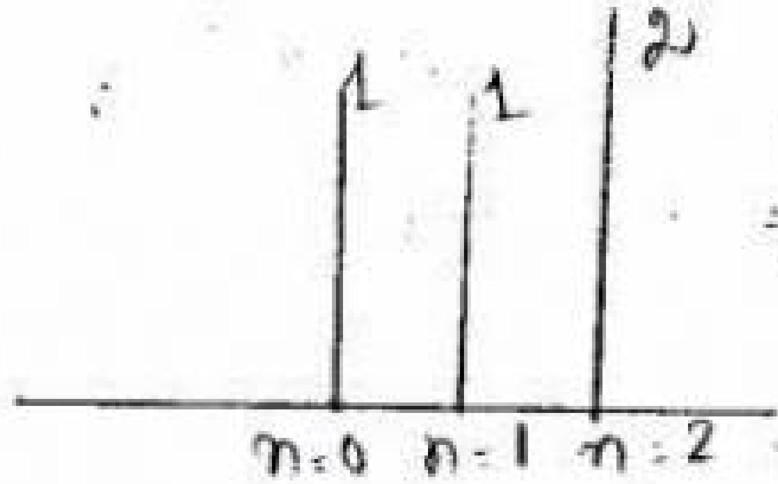
$x(2n) =$ Let $y(n) = x(2n)$

$y(0) = x(0) = 1$

$y(1) = x(2) = 1$

$y(2) = x(4) = 2$

$y(-1) = x(-2) = 0$



$\Rightarrow x(2n)$

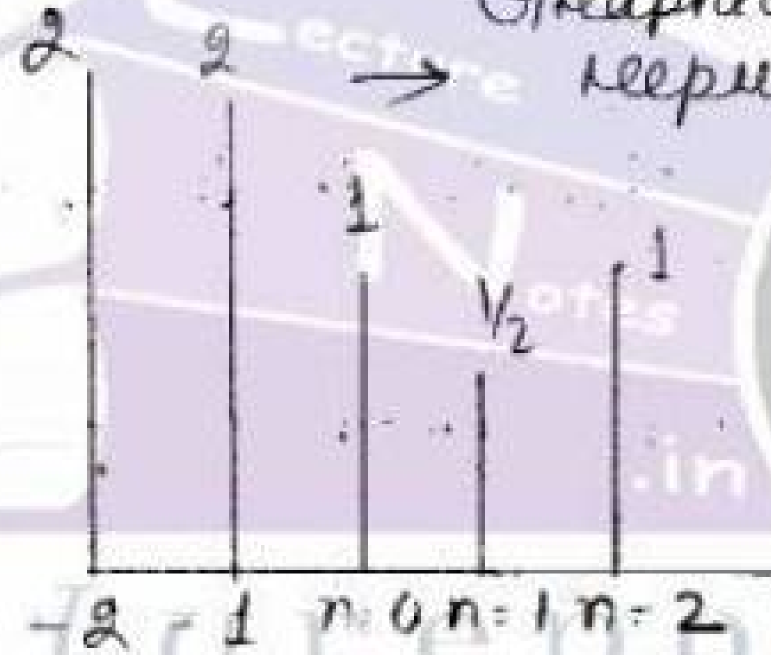
$\underline{x(-n+2)}$ = ~~x~~(steps)

~~shift~~ flip

→ fold the sequence $x(n)$ & then ~~fold/the sequence~~ ~~by~~/ shift the ~~sequ~~ folded sequence to the right side by '2' instant of time.

$\underline{x(-n-2)}$

Graphical → representation



$\Rightarrow x(-n)$

$y(n) = x(-n+2)$

$y(0) = x(2)$

$y(1) = x(1)$

$y(2) = x(0)$

$y(3) = x(-1)$

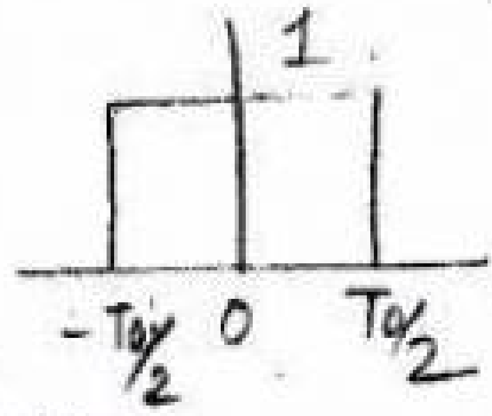
$y(4) = x(-2)$

$y(-1) = x(3)$

$y(-2) = x(4)$

Q) $x(t) = \mathrm{rect}(t/T_0) = 1 \cdot \mathrm{rect}(t/T_0)$.

Check whether the signal is energy or power signal.



Amplitude = 1

It is a energy signal.

bounded function (finite)

$$E = \int_{-\infty}$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2$$

$$= \int_{-T_0/2}^{T_0/2}$$

rms value of the signal is the root of the power.

E_{rms} = mean square value = power.

→ A discrete ~~time~~ type signal can be represented by 4 ways

(1) Functional representation

(2) Graphical "

(3) Sequencial " $\longrightarrow \{1, 2, 1, 1\}$

(4) Tabulate "

If no arrow is given, the sequence is starting from $n=0$.

$$\{1, \underset{\uparrow}{2}, 1, 1$$

finite duration sequence.

$\{ \ldots\ldots 1, 2, 1, 1 \ldots \}$ → infinite duration sequence.

Tabular sequence :-

n	$x(n)$
$n = 0$	1
$n = 1$	2
$n = 2$	1
$n = 3$	1

* $\frac{d}{dt}[u(t)] = \delta(t)$.

$u(t) = \int \cancel{dt)\,dt)}\ \delta(t)\, dt$

$\frac{d}{dt}[x(t)] = u(t)$.

$\Rightarrow x(t) = \int u(t)$.

$= \int\int \delta(t)\, dt$

$\frac{d}{dt}\left[\frac{d\,\delta(t)}{dt}\right]$

$= \frac{d^2}{dt^2}[\delta(t)]$

$=$

DT- 23.01.11

SYSTEM :-

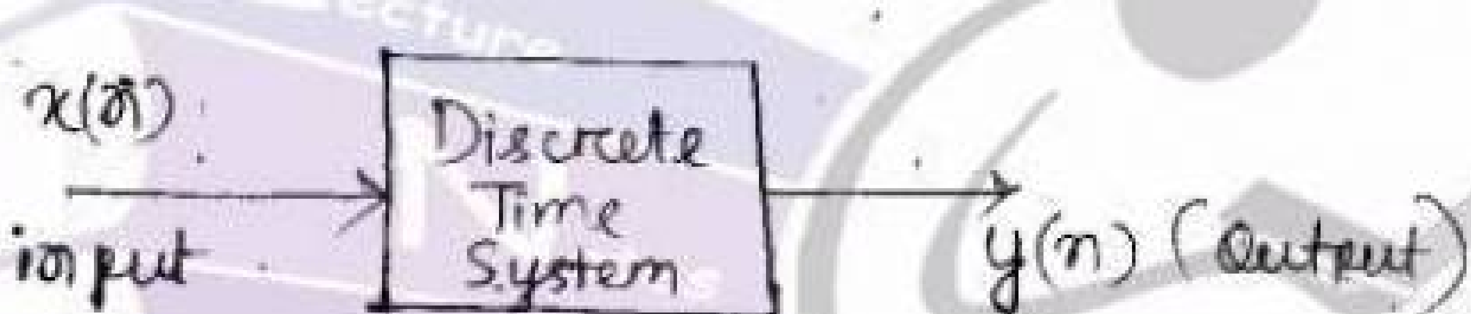
A system is defined as a physical device that performs an operation on a input signal.

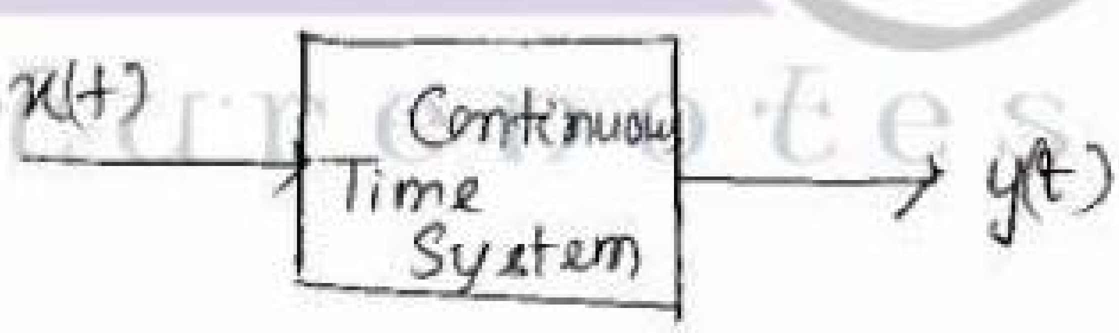
2 types :-

- Continuous
- Discrete

Discrete Type System :-

It is defined as a physical device / algorithm that operates on discrete time system according to some well defined rules and produces an discrete output signal





Continuous Type System

It is defined as a physical device / algorithm which operates on ~~continuous~~ continuous time I/P signal.

Types of Discrete time System:-

(1) Static and Dynamic

(2) Causal & non-causal

(3) Linear & non-Linear

(4) Time variant & Time invariant system

(5) FIR (finite impulse response) & IIR system (infinite

Static System:-

A system is said to be a static system if the O/P of the system at a time instant n depends on the input samples of that time instant. Eg. $y(n) = x^2(n)$

Dynamic System:-

A system is said to be a dynamic system if the output of the system at a time instant n depends on the both past & present values of input

$$y(n) = x(n) + x(n-1)$$

Causal & Non-Causal System:-

A system is said to be causal system if the O/P of the system at any instant n depends on the past and present value of the input and doesn't depend on the future value.

A system is said to be non-causal system if the O/P of the system depends on the ~~past~~ present and future value of the input.

$$\underset{\text{present}}{y(1)} = \underset{\text{present}}{x(1)} + \underset{\text{past}}{x(0)}$$

$$y(n) = x(n) + x(-n+1)$$

$n=-1 \quad y(-1) = x(-1) + x(2)$

$$y(1) = y(1) + x(0)$$

↳ future value

Linear and Non-Linear System :-

A system that satisfies superposition principle is known as a linear system. Superposition principle states that the if the response of the system to the weighted sum of the input signals be equal to the corresponding sum of the O/P of the system to each of the input signal.

Eg. $y(n) = x^2(n)$

for input, $x_1(n)$

$$y_1(n) = x_1^2(n)$$

for input $x_2(n)$

$$y_2(n) = x_2^2(n)$$

$$a_1 y_1(n) + a_2 y_2(n) = T[a_1 x_1(n) + a_2 x_2(n)]$$

$$y_3(n) = [a_1 x_1(n) + a_2 x_2(n)]^2$$

Q) $y(n) = x(n) + \frac{1}{x(n-1)}$

$$y_1(n) = x_1(n) + \frac{1}{x_1(n-1)}$$

$$y_2(n) = x_2(n) + \frac{1}{x_2(n-1)}$$

$a_1 y_1(n) + a_2 y_2(n) = T\left[a_1 x_1(n) + \frac{\cancel{a_1}}{a_1 x_1(n-1)} + a_2 x_2(n) + \frac{\cancel{a_2}}{a_2 x_2(n-1)}\right]$

(Weighted sum of the output)

$\cancel{y_3(n) = a_1 x_1(n) + a_2 x_2(n) +}$

w.r.t. Weighted sum of input $= [a_1 x_1(n) + a_2 x_2(n)]$

$$y_3(n) = a_1 x_1(n) + a_2 x_2(n) + \frac{1}{a_1 x_1(n-1) + a_2 x_2(n-1)}$$

The system is not a linear system.

Q) $y(n) = n\,x(n)$

$y_1(n) = n \cdot x_1(n) = T[x_1(n)]$

$y_2(n) = n\,x_2(n) = T[x_2(n)]$

$a_1 y_1(n) + a_2 y_2(n) = T[a_1 n x_1(n) + a_2 n x_2(n)]$

$= n[a_1 x_1(n) + a_2 x_2(n)]$

$y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)]$

$= n[a_1 x_1(n) + a_2 x_2(n)]$

So, $a_1 y_1(n) + a_2 y_2(n) = y_3(n)$

System is a linear system

Time Variant & Time non variant system

$$y(n) = x(-n)$$

If the input is delayed by 'k' samples of tim

$$y(n,k) = x(-n-k) \quad \text{--- (I)}$$

If the output is delayed by 'k' samples of time

$$y(n-k) = x[-(n-k)]$$

$$= x(-n+k). \quad \text{--- (II)}$$

The system is time variant system

Q) $y(n) = n\,x(n)$

Ans- $y(n,k) = n\,x(n-k) \quad \text{--- (I)}$

$$y(n-k) = (n-k)\,x(n-k) \quad \text{--- (II)}$$

$$y(n,k) \neq y(n-k)$$

Time The system is time variant.

FIR System

The system is said to be a FIR system if its impulse response is finite one.

IIR System

The system is said to be a IIR system if its impulse response is an infinite one.

Causal & Non-Causal Signal & Anticausal Signa

A signal $x(n)$ is causal, if $x(n) = 0$,
for $n < 0$ [$x(n) = \{\underset{\uparrow}{1}, 2, 3,$

A signal $x(n)$ is anticausal, if $x(n) = 0$ for $n > 0$.
[$x(n) = \{1, 2, 3, \underset{\uparrow}{2}$

A signal $x(n)$ is non-causal signal.

$$x(n) = \{1, \underset{\uparrow}{2}, 3, 2\}.$$

DT-31.01.11

Z-TRANSFORM

→ Z-transform is the mathematical tool for the analysis of a linear time-invariant (LIT) discrete time signal in the frequency domain.

→ In z-domain, the convulution of 2 time domain signal is equal to the multiplication of their corresponding Z-transform.

$$X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n} \quad \text{---} (1)$$

where z is a complex variable.

$$z = re^{j\omega}.$$

Relation between fourier & Z-transform :-

→ Fourier transform of a sequence $x(n)$ is given by

$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x(n) e^{-j\omega n} \quad \cdots (2)$$

Comparing eq-(1) & (2)

Putting $z = re^{j\omega}$ in eq-(1)

$$\Rightarrow X(re^{j\omega}) = \sum_{-\infty}^{\infty} x(n) (re^{j\omega})^{-n}$$

$$= \sum_{-\infty}^{\infty} x(n) r^{-n} e^{-j\omega n}$$

→ If $r = 1$. $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x(n) e^{-j\omega n}$

Then the Z-transform of sequence $x(n)$ is ~~is~~ reduced to fourier transform

Region of Convergence :-

The region of convergence of $X(z)$ is all the set of all values of z for which $X(z)$ attains a finite value.

$-\infty$ to ∞ - two sided z-transform.
0 to ∞ - one sided z-transform.

$$x(n) = \{\underset{\uparrow}{1}, 0, 3, -1, 2\}$$

find

Q) ROC of causal sequence.

$$x(n) = \{\underset{\uparrow}{1}, 0, 3, -1, 2\}$$

$$X(z) = \sum_{-\infty}^{\infty} x(n)\, z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + \ldots$$

$$= 1 + 0 \times z^{-1} + 3z^{-2} - z^{-3} + 2z^{-4}$$

All values of z except zero, is the region of convergence.

Q). $x(n) = \{-3, -2, -1, 0, \underset{\uparrow}{1}\}$ Left hand side sequence.

$$x(z) = \sum_{-\infty}^{0} x(n)\, z^{-n}$$

$$= x(0) + x(-1)z + x(-2)z^{2} + x(-3)z^{3} + x(-4)z^{4}$$

$$= 1 + 0 \times z - 1z^{2} - 2z^{3} - 3z^{4}$$

All values of z except $z = \infty$, is the region of convergence.

Z transform & ROC of two sided finite sequence

$$x(n) = \{2, -1, 3, 2; \underset{\uparrow}{1}, 0, 2, 3, -1\}$$

All values of Z except $Z = 0$ and ∞ is the region of convergence.

Z transform of infinite duration causal sequence

$$x(n) = a^n u(n)$$

$$X(Z) = \sum_{-\infty}^{\infty} x(n) z^{-n}$$

$$X(Z) = \sum_{-\infty}^{\infty} a^n u(n) z^{-n}$$

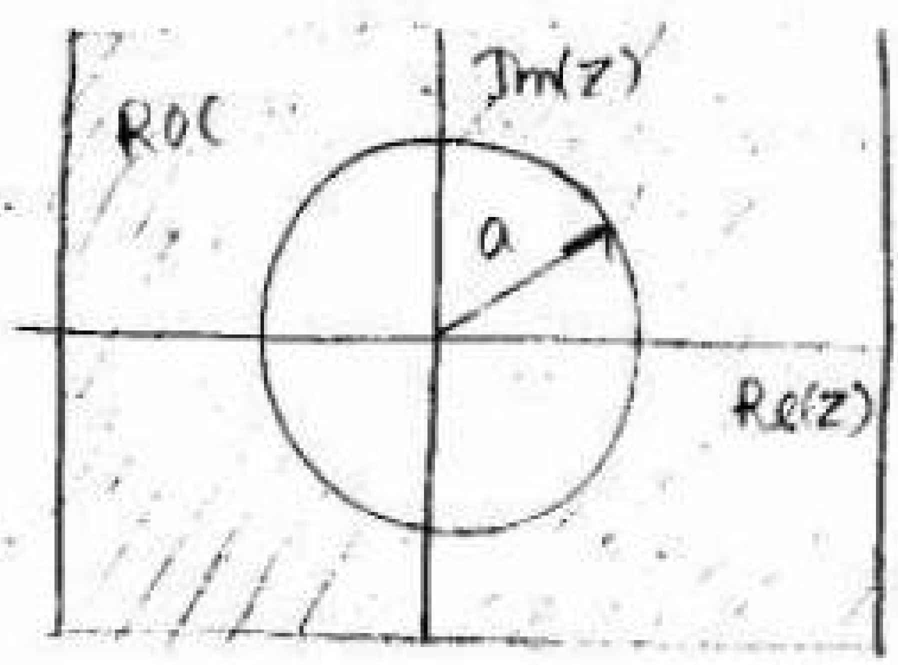
$$= \sum_{0}^{\infty} a^n z^{-n}$$

$$\sum_{0}^{\infty} (a z^{-1})^n$$ ⇝ this value is less than 1

$$= \frac{1}{1 - a z^{-1}} = \frac{Z}{Z - a}$$

ROC of this sequence is $|a z^{-1}| < 1$

$|Z| > |a|$



* $x(n) = -b^{n} u(-n-1) \rightarrow$ non-causal sequence
anti-causal.

$$X(z) = \sum_{-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{-\infty}^{\infty} -b^{n} u(-n-1)\, z^{-n}$$

$$= \cancel{\sum_{0}^{\infty} -b^{n}} - \sum_{-\infty}^{-1} b^{n} z^{-n}$$

$$= -\sum_{-\infty}^{-1} \underline{b^{n} z^{-n}}$$

$$X(z) = -\left[b^{-1}z^{-1} + b^{-2}z^{-2} + b^{-3}z^{-3} + \cdots + b^{-\infty}z^{-\infty}\right]$$

0 to +∞
0 to −∞

$$= -\left[\sum_{0}^{-\infty} b^{n} z^{-n} - b^{0} z^{-0}\right]$$

$\frac{1}{1-a}$

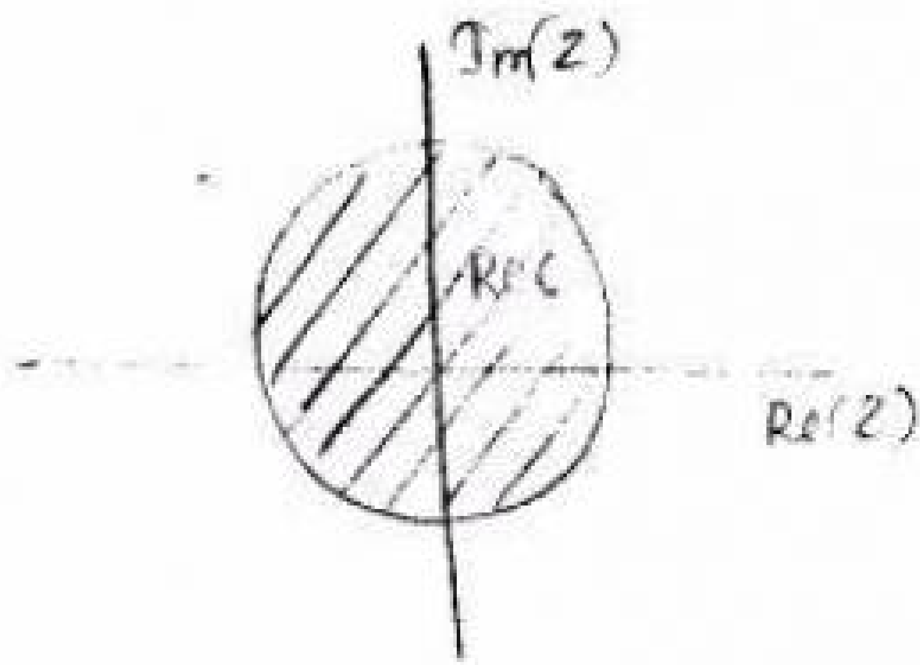
$$= -\left[\sum_{0}^{-\infty} (b^{-1}z)^{-n} - 1\right]$$

$$= -\left[\frac{1}{1-b^{-1}z} - 1\right]$$

$$= \frac{z}{z-b}.$$

ROC of this sequence is $|b^{-1}z| < 1$

$|z| < |b|$



8) $x(n) = a^n u(n) - b^n u(-n-1)$

$$X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{-\infty}^{\infty} \{a^n u(n) - b^n u(-n-1)\} z^{-n}$$

$$= \sum_{-\infty}^{\infty} a^n u(n) z^{-n} - \sum_{-\infty}^{\infty} b^n u(-n-1) z^{-n}$$

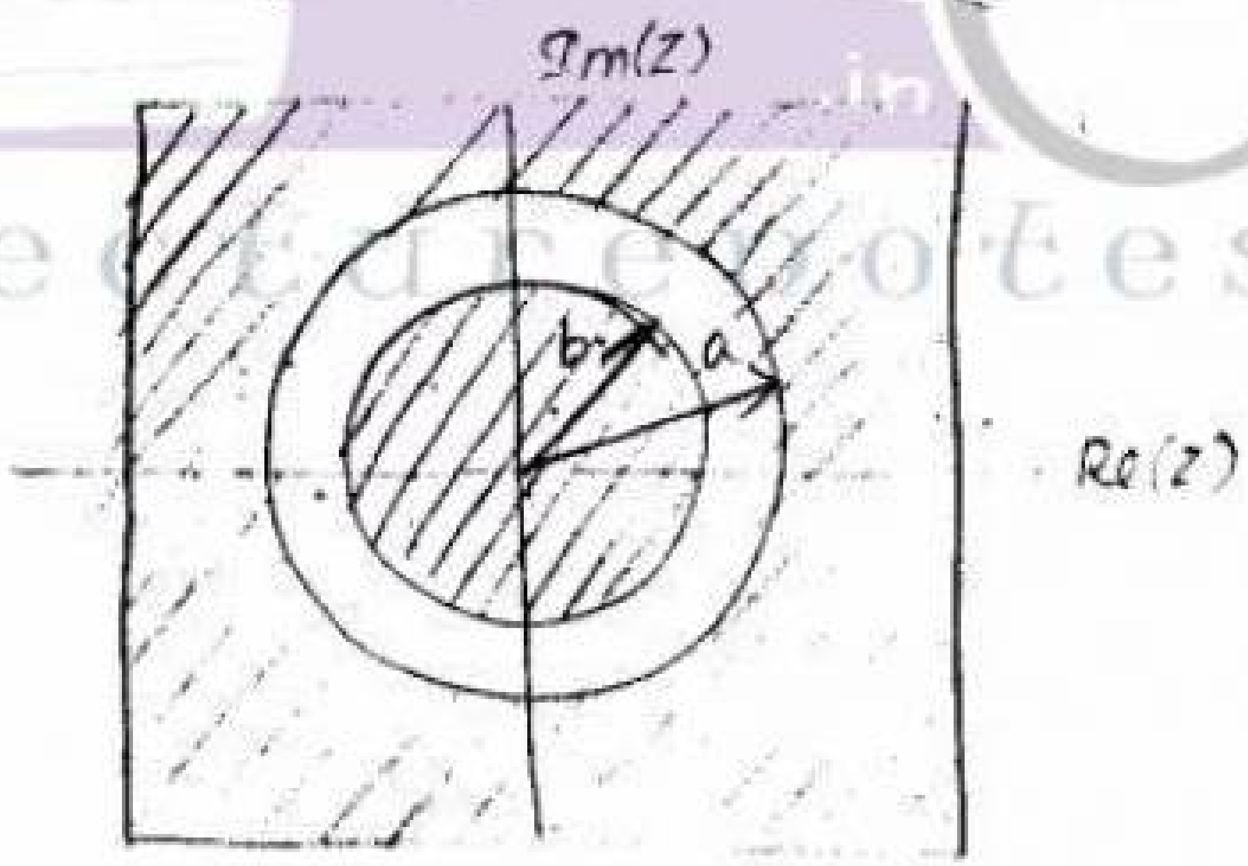
$$= \sum_{0}^{\infty} a^n z^{-n} - \sum_{-\infty}^{-1} b^n z^{-n}$$

$$= \frac{z}{z-a} + \frac{z}{z-b}$$

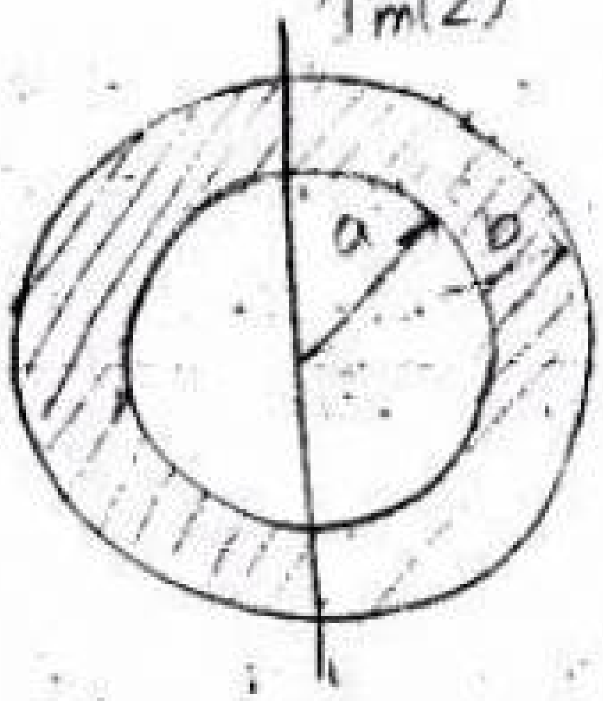
converges for $|z| > |a|$

converges for $|z| < |b|$

If $|b| < |a|$,



If $|a| < |b|$,

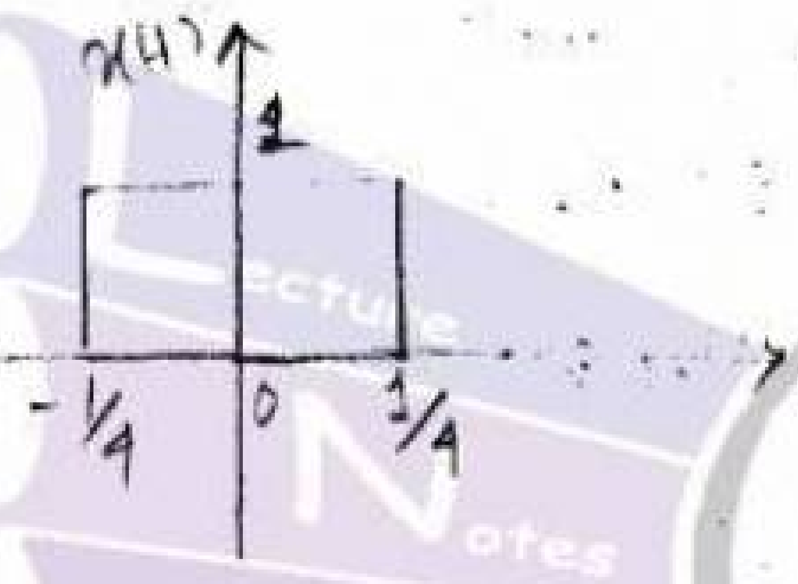


ROC

$|a| < |z| < |b|$

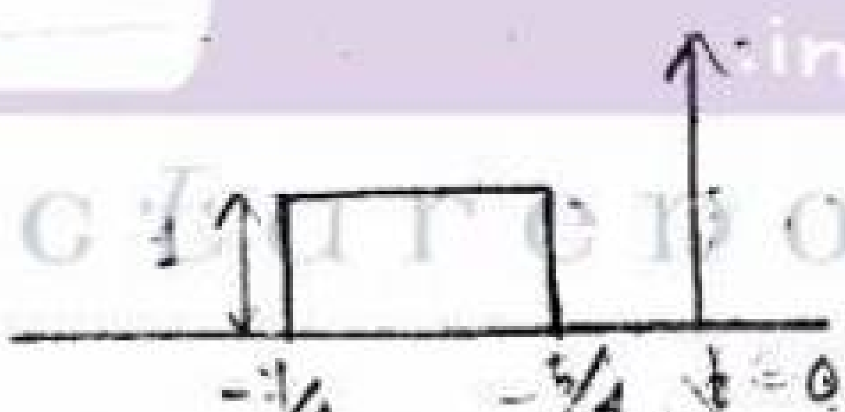
3) ~~$x(n)$~~ = $x(t) = \text{rect}\{2(t + 3/2)\}$

Plot the sequence



$\text{rect}\left\{\frac{t + 3/2}{1/2}\right\}$

$\text{rect}\left(t/_{1/2}\right)$



DT-

Stability and ROC

Let $h(n)$ is an impulse function of a causal or non-causal linear time invariant system. $H(z)$ be the system function, then the stability of the system can be found from ROC in the following theorem.

A system is said to be BIBO stable, if all the poles of the system function lies inside the unit circle or ROC contains in the in unit circle.

$$h(n) = 2^n u(n)$$

$$\sum_{-\infty}^{\infty} h(n) < \infty$$

$$= \sum_{-\infty}^{\infty} 2^n u(n)$$

$$= \sum_{0}^{\infty} 2^n$$

The series cannot convert. So, it is unstable

$$H(z) = \sum_{-\infty}^{\infty} 2^n u(n) z^{-n}$$

$$= \sum_{0}^{\infty} 2^n z^{-n}$$

$$= \sum_{0}^{\infty} (2z^{-1})^n$$

$$= \frac{1}{1-2z^{-1}} = \frac{z}{z-2}$$

$|2z^{-1}| < 1 \Rightarrow |z| > 2$

doesn't contain in the unit circle

Properties of ROC

→ ROC is a rim or disc in the z plane centered at origin.

→ ROC must be a connected region.

→ ROC doesn't contain any pole.

→ If $x(n)$ is a causal sequence, then ROC is all the values of z except $z = 0$

→ If $x(n)$ is a non-causal sequence, then ROC is all the values of z except $z = \infty$

→ ROC of any linear time invariant contains unit circle.

Properties of z transform:-

(1) Linearity :-

If $X(z) = Z[x(n)]$

$$Z[a_1 x_1(n) + a_2 x_2(n)] = a_1 X_1(z) + a_2 X_2(z)$$

$$\Rightarrow X_1(z) = Z[x_1(n)]$$

$$X_2(z) = Z[x_2(n)]$$

Let $|R_2| < |z| < |R_1|$ is the ROC of total sequence.

then $[a_1 x_1(n) + a_2 x_2(n)] = a_1 X_1(z) + a_2 X_2(z)$

Proof :-

$$Z[a_1 x_1(n) + a_2 x_2(n)]$$

$$= \sum_{-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] z^{-n}$$

$$= \sum_{-\infty}^{\infty} a_1 x_1(n) z^{-n} + \sum_{-\infty}^{\infty} a_2 x_2(n) z^{-n}$$

$$= a_1 X_1(z) + a_2 X_2(z)$$

(2) Time shift :

If $X(z) = Z[x(n)]$

$$Z[x(n-m)] = z^{-m}X(z)$$

$$= \sum_{-\infty}^{\infty} x(n-m) z^{-n}$$

$$= \sum_{l=-\infty}^{\infty} x(l) z^{-(l+m)} \qquad (n-m) = l, \quad n = l+m$$

$$= \sum_{l=-\infty}^{\infty} z^{-m} x(l) z^{-l} = z^{-m}X(z)$$

$X_+(z) = Z[x(n)]$, then one-sided

z transform ~~of x(n)~~

Two sided z transform of $x(n-m)$

double

$$Z[x(n-m)] = \sum_{-\infty}^{\infty} x(n-m) z^{-n}$$

$$= \sum_{0}^{\infty} x(l) z$$

$$= \sum_{0}^{\infty} z^{-m} x(l) z^{-l} + \sum_{-\infty}^{-1} x(l) z^{-m} z^{-l}$$

$$= z^{-m}\left[X_+(z) + \sum_{-\infty}^{-1} x(l) z^{-l}\right]$$

(3) Time Reversal

If $X(z) = Z[x(n)]$

$$Z[x(-n)] = X(z^{-1})$$

Proof :-

$$Z[x(-n)] = \sum_{-\infty}^{\infty} x(-n) z^{-n}$$

$$= \sum_{\infty}^{-\infty} x(l) z^{l} = \sum_{-\infty}^{\infty} x(l) (z^{-1})^{l}$$

$$= X(z^{-1}).$$

(4) Multiplication by a exponential sequence:

$z[u(n)]$

$X(z) = \frac{z}{z-1}$

$z[u(-n)] = \frac{z^{-1}}{z^{-1}-1}$

~~$z[a^n u(n)]$~~

If $X(z) = z[x(n)]$

$z[a^n x(n)] = X[a^{-1}z]$

$z[a^n u(n)] = \frac{a^{-1}z}{a^{-1}z-1} = \frac{z/a}{z/a - 1}$

$= \frac{z}{z-a}$

<u>Proof</u> :-

$z[a^n x(n)]$

$= \sum_{-\infty}^{\infty} a^n x(n) z^{-n}$

$= \sum_{-\infty}^{\infty} x(n) (a^{-1}z)^{-n}$

$= X(a^{-1}z)$

(5) Differentiation

If $X(z) = z[x(n)]$

then $z[n\, x(n)]$

$= -z \frac{d}{dz}[X(z)]$

$= -z \frac{d}{dz}\left[\sum_{n=-\infty}^{\infty} x(n) z^{-n}\right]$

$= -z \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz}\left[z^{-n}\right]$

$= -z \sum_{n=-\infty}^{\infty} x(n)(-n) z^{-n-1}$

~~(8) find out the z-transform of $z\left[\frac{1}{n+2}\right]$~~

$$\cancel{= \sum_{-\infty}^{\infty} x(n) z^{-n}}$$

$$\cancel{= \sum_{-\infty}^{\infty} 1}$$

(6) Convulution property :-

If $X(z) = z[x(n)]$ & $H(z) = z[h(n)]$

$$z[x(n) * h(n)] = X(z) \cdot H(z)$$

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Proof :-

L.H.S $= z[x(n) * h(n)]$

$$= z\left[\sum_{k=-\infty}^{\infty} x(k)\, h(n-k)\right]$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)\, z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, z^{-k} \sum_{n=-\infty}^{\infty} h(n-k)\, z^{-(n-k)}$$

$$= X(z) \cdot H(z)$$

(Q) Find Z transform of & also ROC.

(1) $u(n)$
(2) $\delta(n)$
(3) $\delta(n-k)$
(4) $u(-n-1)$
(5) $a^n u(n)$
(6) $-a^n u(-n-1)$

Ans - (1) $u(n)$.

$$\cancel{z[u(n)} \; z[x(n)] = \sum_{-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{0}^{\infty} u(n) z^{-n}$$

$$= \sum_{0}^{\infty} z^{-n}$$

$$= \sum_{0}^{\infty} (z^{-1})^n = 1$$

ROC of the sequence $= |z| > 1$.

(2) $\delta(n) = x(n)$

$$z[x(n)] = \sum_{-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{-\infty}^{\infty} \delta(n) z^{-n}$$

$$= \sum_{0}^{\infty} z^{-n} = z^{-0} + z^{-1} + z^{-2}$$

$$= 1$$

ROC of the sequence = all values of z except $z = 0$.

(3) $\delta(n-k)$

$$Z[x(n)] = \sum_{-\infty}^{\infty} \delta(n-k)\, z^{-n}$$

$$= \sum_{-\infty}^{\infty} \delta(n-k)\, z^{-k}\, z^{-(n-k)}$$

$$= z^{-k}$$

ROC = all values of z except $z=0$ if $k>1$
" " " " except $z=\infty$ if $k<1$

(4) $u(-n-1)$

$$Z[x(n)] = \sum_{-\infty}^{\infty} u(-n-1)\, z^{-n}$$

$$= \sum_{-\infty}^{\infty} u(-n-1)\, z^{-(n-1)}\, z^{-1}$$

$$= \sum_{0}^{\infty} z^{-(n-1)}\, u(-n-1)\, z^{-1}$$

$$= \frac{z}{z-1} \quad \text{ROC} = |z| < 1$$

(5) $a^n u(n)$

$$Z[x(n)] = \sum_{-\infty}^{\infty} a^n \cdot u(n)\, z^{-n}$$

$$= \sum_{0}^{\infty} a^n z^{-n}$$

$$= \sum_{0}^{\infty} (a z^{-1})^n$$

$$= \frac{z}{z-a}$$

$|z| > |a| := \text{ROC}$

(6) $-a^n u(-n-1)$

$$Z[x(n)] = \sum_{-\infty}^{\infty} -a^n u(-n-1) z^{-n}$$

$$= -\sum_{-\infty}^{\infty} a^n z^{-(n-1)} z^{-1} u(-n-1)$$

$$= \sum_{0}^{\infty} (az^{-1})^n$$

$$= \frac{z}{z-a}$$

ROC $= |z| < |a|$

Q) $x(n) = \left(\frac{1}{3}\right)^{n-1} u(n-1)$

$$Z[x(n)] = \left[\left(\frac{1}{3}\right)^{n-1} u(n-1)\right]$$

$$= z^{-1} \frac{z}{z-\frac{1}{3}} = \frac{1}{z-\frac{1}{3}}$$

Q) $x(n) = n \underbrace{a^n u(n)}_{y(n)}$

$\cancel{x(n)} = Z[n\, x(n)]$

$$y(n) = a^n u(n)$$

$$Z[y(n)] = Y(z) = \frac{z}{z-a}$$

$$Z[n\, x(n)] = -z \frac{d}{dz}[Y(z)]$$

$$= -z \frac{d}{dz}\left[\frac{z}{z-a}\right]$$

$$= -z \left[\frac{\cancel{z}-a-\cancel{z}}{z-a}\right]$$

$$= \frac{az}{z-a}$$

Q) $x(z) = \log(1-0.5z^{-1})$, $|z| > 0.5$

Find inverse z transform using differentiation property.

Ans -

$$X(z) = \log(1-0.5z^{-1})$$

$$Z[n\,x(n)] = -z\frac{d}{dz}[X(z)]$$

$$= -z\frac{d}{dz}[\log(1-0.5z^{-1})]$$

$$= \frac{0.5 \cdot -z}{1-0.5z^{-1}} \qquad -z^{-1+2}$$

$$= \frac{0.5z^{-2}(-z)}{1-0.5z^{-1}}$$

$$= \frac{0.5\frac{1}{z^2} \times -z}{1-0.5/z} = \frac{\frac{-0.5}{z}}{\frac{z-0.5}{z}}$$

$$= \frac{-0.5}{z-0.5}$$

$$= \frac{-z^{-1} \cdot 0.5}{1-0.5z^{-1}} = -0.5\,Z[(0.5)^{n-1} \cdot u(n-1)]$$

$$Z[(0.5)^{n-1}u(n-1)] = z^{-1}\frac{z}{z-0.5}$$

$$n\,x(n) = -0.5(0.5)^{n-1}u(n-1)$$

$$x(n) = -\frac{1}{n}(0.5)^{n}u(n-1).$$

5) $x(n) = \cos\omega_0 n \, u(n)$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\cos\omega_0 n = \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}$$

$$Z[x(n)] = \sum_{-\infty}^{\infty} \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} \cdot z^{-n} \cdot u(n)$$

$$= \frac{1}{2}\left[\sum_{0}^{\infty} (e^{j\omega_0} z^{-1})^n + \sum_{0}^{\infty} (e^{-j\omega_0} z^{-1})^n\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0} z^{-1}} + \frac{1}{1-e^{-j\omega_0} z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-j\omega_0} z^{-1} + 1 - e^{j\omega_0} z^{-1}}{(1-e^{j\omega_0} z^{-1})(1-e^{-j\omega_0} z^{-1})}\right]$$

$$= \frac{1}{2}\left[\frac{2 - 2z^{-1}\left(\frac{e^{j\omega_0} + e^{-j\omega_0}}{2}\right)}{1 - e^{-j\omega_0} z^{-1} - e^{j\omega_0} z^{-1} + z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{2 - 2z^{-1}\cos\omega_0}{1 + z^{-2} - 2z^{-1}\left(\frac{e^{-j\omega_0} + e^{j\omega_0}}{2}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{2 - 2z^{-1}(\cos\omega_0)}{1 + z^{-2} - 2z^{-1}(\cos\omega_0)}\right]$$

$$= \frac{1 - z^{-1}\cos(\omega_0)}{1 - (2\cos\omega_0) z^{-1} + z^{-2}}$$

* Unit of z^{-1} is secs. (delay element)

$$Z[r^n \cos\omega_0 n \, u(n)] = \frac{1 - z^{-1}(r\cos\omega_0)}{1 - (2r\cos\omega_0) z^{-1} + r^2 z^{-2}}$$

8) $x(n) = \sin\omega_0 n \, u(n)$

$$Z[x(n)] = \sum_{-\infty}^{\infty} \sin\omega_0 n \, u(n) \, z^{-n}$$

$$= \sum_{0}^{\infty} \left[\frac{e^{-j\omega_0 n} - e^{j\omega_0 n}}{2}\right] \cancel{u(n)} \, z^{-n}$$

$=$

Q) $x(n) = \left[3(3)^n - 4(2)^n\right] u(n)$.

Find z-transform of & ROC of $x(n)$.

Ans – $Z[x(n)] = \sum_{-\infty}^{\infty} \left[3(3)^n - 4(2)^n\right] u(n) \cdot z^{-n}$

$$= \sum_{0}^{\infty} \left[3(3)^n z^{-n}\right] . \sum_{0}^{\infty} \left[4(2)^n z^{-n}\right]$$

$$= 3\sum_{0}^{\infty} (3z^{-1})^n \, 4 \sum_{0}^{\infty} (2z^{-1})^n$$

$$= \frac{3}{1-3z^{-1}} \cdot \frac{2}{1-2z^{-1}}$$

$$= \frac{3z}{z-3} \cdot \frac{2z}{z-2}$$

$$= \frac{6z^2}{(z-3)\cdot(z-2)}$$

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INVERSE Z TRANSFORM :-

(1) Partial Expansion Method.
(2) Residue Method.
(3) Convulution method
(4) Long division method.

Partial Expansion method :-

Let $X(z) = \dfrac{z^{n} + b_0 z^{n-1} + b_1 z^{n-2} + \dots + b_m z^{n-m-1}}{z^{n} + a_1 z^{n-1} + a_2 z^{n-2} + \dots\ a_z}$

$= \dfrac{z^{n+1} + b_0 z^{n-1} + \dots + b_{m-1} z^{n-m-1}}{(z-P_1)(z-P_2)(z-P_3)\dots}$

$$\frac{X(z)}{z} = \frac{C_1}{z-P_1} + \frac{C_2}{z-P_2} + \frac{C_3}{z-P_3} + \dots$$

For distinct poles $C_K = (z-P_K)\dfrac{X(z)}{z}\Big|_{z=P_K}$

$k = 1, 2, 3 \dots$

$c_1, c_2, c_3 \rightarrow$ coefficients.

Q) $X(z) = \dfrac{z^3 + z^2}{(z-1)(z-3)}$

$$\frac{X(z)}{z} = \frac{z^2 + z}{(z-1)(z-3)} = \frac{z(z+1)}{(z-1)(z-3)}$$

$$\begin{array}{r} z^2 - 4z + 3 \,\big)\, z^2 + z \,\big(\, 1 \\ z^2 - 4z + 3 \\ \hline 5z - 3 \end{array}$$

$$= 1 + \frac{5z-3}{(z-1)(z-3)}$$

$$H(z) = \frac{5z-3}{(z-1)(z-3)}$$

$$\frac{H(z)}{z} = \frac{5z-3}{z(z-1)(z-3)}$$

$$= \frac{C_1}{z} + \frac{C_2}{z-1} + \frac{C_3}{z-3}$$

$$C_1 = z\frac{H(z)}{z}\bigg|_{z=0} = \cancel{z}\frac{5z-3}{\cancel{z}(z-1)(z-3)}\bigg|_{z=0}$$

$$= \frac{-3}{(-3)(-1)} = -1$$

$$C_2 = (z-1)\frac{H(z)}{z}\bigg|_{z=1} = \frac{5z-3}{z(z-3)}\bigg|_{z=1}$$

$$= \frac{2}{-2} = -1.$$

$$C_3 = (z-3)\frac{H(z)}{z}\bigg|_{z=3}$$

$$= \frac{(5z-3)}{z(z-1)}\bigg|_{z=3} = \frac{12}{3\times 2} = 2.$$

$$\cancel{\frac{X(z)}{z}} \Rightarrow \frac{H(z)}{z} = -\frac{1}{z} - \frac{1}{z-1} + \frac{2}{z-3}.$$

$$\frac{X(z)}{z} = 1 - \frac{1}{z} - \frac{1}{z-1} + \frac{2}{z-3}.$$

$$X(z) = z - 1 - \frac{z}{z-1} + \frac{2z}{z-3}$$

Taking inverse z transform

$$x(n) = -\delta(n+1) \mp \delta(n) - u(n)$$
$$+ 2(3)^n u(n).$$

Q) $\dfrac{X(z)}{z} = \dfrac{1}{(z-p_1)^2(z-p_2)}$

$$= \frac{C_1}{z-p_2} + \frac{C_2}{z-p_1} + \frac{C_3}{(z-p_1)^2}$$

$$C_1 = (z-p_2)\frac{X(z)}{z}\Big|_{z=p_2}$$

$$C_2 = \frac{1}{(2-1)!}\frac{d}{dz}\left[(z-p_1)^2\frac{X(z)}{z}\right]_{z=p_1}$$

$$C_3 = (z-p_1)^2\frac{X(z)}{z}\Big|_{z=p_1}$$

Q) $X(z) = \dfrac{1}{(2-z^{-1})(1-z^{-1})^2}$

$$= \frac{1}{(2-\frac{1}{z})(1-\frac{1}{z})^2} = \frac{z^3}{(2z-1)(z-1)^2}$$

$$= \frac{1}{2}\left[\frac{z^3}{(z-\frac{1}{2})(z-1)^2}\right]$$

$$\frac{X(z)}{z} = \frac{z^2}{(2z-1)(z-1)^2} \qquad \frac{1}{2}\left[\frac{z^2}{(z-\frac{1}{2})(z-1)^2}\right]$$

$$= \cancel{\frac{C_1}{(2z-1)} + \frac{C_2}{(z-1)} + \frac{C_3}{(z-1)^2}}$$

$$= \frac{1}{2}\left[\frac{C_1}{(z-\frac{1}{2})} + \frac{C_2}{(z-1)} + \frac{C_3}{(z-1)^2}\right]$$

$$C_1 = \frac{1}{2}\left(z-\frac{1}{2}\right)\frac{X(z)}{z}\Big|_{z=\frac{1}{2}}$$

$$= \frac{1}{2}\,\frac{(\frac{1}{2})^2}{(\frac{1}{2}-1)^2} = 1$$

$$c_2 = \frac{1}{2}\frac{d}{dz}\left[(z-1)^2\frac{X(z)}{z}\right]_{z=1}.$$

$$= \frac{1}{2}\frac{d}{dz}\left[\cancel{(z-1)^2}\cdot\frac{z^2}{(z-1/2)\cancel{(z-1)^2}}\right]$$

$$= \frac{1}{2}\frac{d}{dz}\left[\frac{z^2}{z-1/2}\right]$$

$$= \frac{1}{2}\left[\frac{(z-1/2)\times 2z - z^2}{(z-1/2)^2}\right]_{z=1}$$

$$= \frac{1}{2}\,\frac{1/2\times 2 - (1)^2}{(1-1/2)^2} = 0 .$$

$$C_2 = \frac{1}{2}(z-1)^2\frac{X(z)}{z}\Big|_{z=1}.$$

$$= \frac{1}{2}\cancel{(z-1)^2}\times\frac{z^2}{(z-1/2)\cancel{(z-1)^2}}\Big|_{z=1}$$

$$= \frac{1}{2}\times\frac{1}{1/2} = 1 .$$

$$X(z) = \frac{1}{2}\frac{z}{z-1/2} + \frac{1}{2}\frac{z}{(z-1)^2} \qquad |z|>1 .$$

$$x(n) = \frac{1}{2}\left[\left(\tfrac{1}{2}\right)^n u(n) + n\,u(n)\right].$$

$$* \; Z[a^n u(n)] = \frac{z}{z-a}.$$

$$Z[n a^{n-1} u(n)] = \frac{z}{(z-a)^2}$$

$$Z[n(n-1) a^{n-2} u(n)] = \frac{z}{(z-a)^3}.$$

(2) Residue Method

If $x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

$= \sum$ Residue of all the poles within closed circle of $X(z) z^{n-1}$

Q) $X(z) = \dfrac{2}{z(z-1/2)}$ So, all the poles

ROC: $|z| > 1/2$ $\quad x(n) = \dfrac{1}{2\pi j} \oint_c X(z) z^{n-1} dz$

$$x(n) = \frac{1}{2\pi j} \oint_c \frac{2}{z(z-1/2)} z^{n-1} dz$$

$$= \cancel{(z-0)} \cancel{X(z)} \frac{2 z^{n-1}}{\cancel{z}(z-1/2)}\Bigg|_{z=0} + \cancel{(z-1/2)} \frac{2}{z\cancel{(z-1/2)}} z^{n-1}\Bigg|_{z=1/2}$$

$$= 0 + \frac{2(1/2)^{n-1}}{1/2}$$

$$= 4(1/2)^{n-1} u(n)$$

→ First check ~~a~~ whether all the poles are within the ROC

→ If the poles are within the ROC, then $x(n)$ must be a causal sequence.

$$\sum \text{Residue of all poles} = +ve.$$

→ If the poles are outside the ROC, then $x(n)$ must be a anti-causal sequence.

$$\sum \text{Residue of all poles} = -ve.$$

→ If one pole lies inside ROC and other pol lies outside ROC, then ~~$x(n)$~~

$$x(n) = \underbrace{\sum \text{causal part}}_{\text{inside ROC}} + \underbrace{\sum \text{anti-causal part}}_{\text{outside ROC}}.$$

Q) $x(z) = \dfrac{2}{z(z-1/2)}$.

$$x(n) = \frac{1}{2\pi j}\oint_c X(z)\, z^{n-1}\, dz$$

$$= \frac{1}{2\pi j}\oint_c \frac{2}{z(z-1/2)}\, z^{n-1} \qquad f(z) = \frac{2}{z-1/2}$$

if we take $n=0$. $\qquad \dfrac{f(z)}{z^2}$

$$= \frac{1}{2\pi j}\oint_c \frac{2}{z(z-1/2)\, z}$$

$$= \left(z-1/2\right)\frac{2}{z^2\cancel{(z-1/2)}}\Bigg|_{z=1/2} + \frac{1}{\lfloor 2-1}\frac{d}{dz}\left[\frac{2}{z-1/2}\right]_{z=0}$$

$$= \frac{2}{(1/2)^2} + 2(-1)\frac{1}{(z-1/2)^2}\Bigg]_{z=0}$$

$$= 8 - 2\times\frac{1}{(\ )^2} = 8 - 8$$

DT - 11.02.11

Q) Find inverse Z-transform of

$$X(z) = \frac{z^2+z}{(z-1/2)^3(z-1/4)}, \quad |z| > 1/2$$

Using partial expansion of Cauchy's Residue method

Q) Find $x(n)$, $X(z) = \frac{z+3}{z^7(z-1/2)}$ without using residue & partial expansion method

Ans-

$$X(z) = \frac{z^2+z}{(z-1/2)^3(z-1/4)}$$

$$\frac{X(z)}{z} = \frac{z+1}{(z-1/2)^3(z-1/4)}$$

$$= \frac{A}{z-1/2} + \frac{Bs}{(z-1/2)^2} + \frac{Cs^2}{(z-1/2)^3} + \frac{D}{z-1/4}$$

$$= A(z-1/2)^2(z-1/4) + Bs(z-1/2)(z-1/4) + Cs^2(z-1/4) + D(z-1/2)^3 = z+1$$

$$= \frac{c_1}{z-1/2} + \frac{c_2}{(z-1/2)^2} + \frac{c_3}{(z-1/2)^3} + \frac{c_4}{z-1/4}$$

$$\cancel{c_1 = (z-1/2)\frac{X(z)}{z}\Big|_{z=1/2}}$$

$$C_4 = \left(z - \tfrac{1}{4}\right)\frac{X(z)}{z}\Big|_{z=1/4}$$

$$= \frac{z+1}{\left(z-\tfrac{1}{2}\right)^3} = \frac{\tfrac{1}{4}+1}{\left(\tfrac{1}{4}-\tfrac{1}{2}\right)^3} = \frac{5/4}{\left(-\tfrac{1}{4}\right)^3}$$

$$= \frac{5}{\cancel{4}} \times \frac{-\cancel{4}^{3}}{1} = -80$$

$$C_1 = \frac{1}{2!}\frac{d^2}{dz^2}\left[\left(z-\tfrac{1}{2}\right)^3 \frac{X(z)}{z}\right]_{z=1/2}$$

$$= 80$$

$$C_2 = \frac{d}{dz}\left[\left(z-\tfrac{1}{2}\right)^2 \frac{X(z)}{z}\right]_{z=1/2} = -20$$

$$C_3 = \left(z-\tfrac{1}{2}\right)^3 \frac{X(z)}{z}\Big|_{z=1/2} = 6$$

$$X(z) = -\frac{80z}{z-1/4} + \frac{80z}{z-1/2} - \frac{20}{\left(z-\tfrac{1}{2}\right)^2} + \frac{6z}{\left(z-\tfrac{1}{2}\right)^3}$$

Taking inverse z-transform

$$\Rightarrow x(n) = -80\left(\tfrac{1}{4}\right)^n u(n) + 80\left(\tfrac{1}{2}\right)^n u(n)$$

$$-20\left(\tfrac{1}{2}\right)^{n-1} u(n) + \cancel{6\left(\tfrac{1}{2}\right)^n u(n)}$$

$$6\,\frac{n(n-1)}{2!}\left(\tfrac{1}{2}\right)^{n-2} u(n)$$

2) $X(z) = \frac{z+3}{z^7(z-1/2)}$ $\qquad X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n}$.

$$= \frac{z[1+3z^{-1}]}{z^7(z-1/2)}$$

$$= (z^{-7}+3z^{-8}) \frac{z}{z-1/2}$$

$$X_1(z) = \frac{z}{z-1/2}$$

$$x_1(n) = (1/2)^n u(n)$$

$$X(z) = (z^{-7}+3z^{-8}) X_1(z)$$

Using inverse z-transform

$$\Rightarrow \cancel{X(z)}\ x(n) = [x_1(n-7) + 3x_1(n-8)] (1/2)^n u(n)$$

$$= (1/2)^{n-7} u(n-7) + 3(1/2)^{n-8} u(n-8)$$

Q) Find $x(n)$ using residue method.

$$X(z) = \frac{z}{(z-2)(z-3)} \quad |z|<2$$

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

$$= (z-2) X(z) z^{n-1} \Big|_{z=2} + (z-3) X(z) z^{n-1} \Big|_{z=3}$$

$$= \frac{z}{z-3} z^{n-1} \Big|_{z=2} + \frac{z}{z-2} z^{n-1} \Big|_{z=3}$$

$$= \cancel{-2 \cdot 2^{n-1}} \ \cancel{2 \cdot 2^{n-1}} +$$

$$= -2 \cdot 2^{n-1} + 3 \cdot 3^{n-1} =$$

Inverse Z-transform using convulution method

$$X(z) = \frac{(1 - \frac{1}{4} z^{-1})}{(1 - \frac{1}{9} z^{-2})}$$

$$= \frac{1 - \frac{1}{4z}}{1 - \frac{1}{9z^2}}$$

$$= \frac{(4z-1)}{4z} \times \frac{9z^2}{9z^2 - 1}$$

$$= \frac{\cancel{4}(z - \frac{1}{4})}{\cancel{4}z} \cdot \frac{\cancel{9}z^2}{\cancel{9}(z^2 - \frac{1}{9})}$$

$$= \frac{z(z - \frac{1}{4})}{(z^2 - \frac{1}{9})} = \frac{z(z - \frac{1}{4})}{(z + \frac{1}{3})(z - \frac{1}{3})}$$

$$= (z - \tfrac{1}{4}) \frac{z}{(z + \frac{1}{3})(z - \frac{1}{3})}$$

$$= H_1(z) \cdot H_2(z)$$

~~Let $H_1(z) = z - \frac{1}{4}$~~ $= \left(\frac{z}{z - \frac{1}{3}}\right) \frac{(z - \frac{1}{4})}{(z + \frac{1}{3})}$

~~$h_1(n) =$~~

$$= X_1(z) \cdot X_2(z)$$

Let $x_1(z) = \frac{z}{z - \frac{1}{3}}$

$$x_1(n) = \left(\tfrac{1}{3}\right)^n u(n)$$

$$x_2(z) = \frac{z - \frac{1}{4}}{z + \frac{1}{3}} = \frac{z}{z + \frac{1}{3}} - \frac{\frac{1}{4}}{z + \frac{1}{3}}$$

$$x_2(n) = \left(-\tfrac{1}{3}\right)^n u(n) - \frac{1}{4}\left(-\tfrac{1}{3}\right)^{n-1} u(n-1)$$

$$x(n) = x_1(n) * x_2(n)$$
$$= \left(\tfrac{1}{3}\right)^n u(n) * \left[(-3)^n u(n) - \frac{n}{4}(-3)^n u(n)\right]$$
$$= \left(\tfrac{1}{3}\right)^n u(n) * (-3)^n u(n) - \left(\tfrac{1}{3}\right)^n u(n) * \tfrac{n}{4}(-3)^n u(n)$$

$$x(n) = x_1(n) * x_2(n)$$
$$= \left(\tfrac{1}{3}\right)^n u(n) * \left[\left(-\tfrac{1}{3}\right)^n u(n) - \frac{1}{4}\left(-\tfrac{1}{3}\right)^{n-1} u(n-1)\right]$$
$$= \left[\left(\tfrac{1}{3}\right)^n u(n) * \left(-\tfrac{1}{3}\right)^n u(n) - \left(\tfrac{1}{3}\right)^n u(n) * \tfrac{1}{4}\left(-\tfrac{1}{3}\right)^{n-1} u(n-1)\right]$$

for $\left(\tfrac{1}{3}\right)^n u(n) * \left(-\tfrac{1}{3}\right)^n u(n)$

$$= \sum_{k=-\infty}^{\infty} \left(\tfrac{1}{3}\right)^k u(k) \left(-\tfrac{1}{3}\right)^{n-k} u(n-k)$$
$$= \sum_{k=0}^{n} \left(\tfrac{1}{3}\right)^k (-3)^{n-k}$$
$$= \sum_{k=0}^{n} \left(\tfrac{1}{3}\right)^k (-3)^n \cdot (-3)^{-k}$$
$$= \sum_{k=0}^{n} 3^{-k} (-1)^{-k} (3^{-k})$$
$$= \sum_{k=0}^{\infty} (-1)^{-k} 3^{-2k}$$
$$= \sum_{k=0}^{\infty} \left(\tfrac{1}{3}\right)^k \left(-\tfrac{1}{3}\right)^{n-k}$$
$$= \sum_{k=0}^{n} \left(\tfrac{1}{3}\right)^k \left(-\tfrac{1}{3}\right)^n \left(-\tfrac{1}{3}\right)^{-k}$$
$$= \left(-\tfrac{1}{3}\right)^n \sum_{k=0}^{\infty} 3^{-k} (-1)^{-k} \left(\tfrac{1}{3}\right)^{-k}$$
$$= \left(-\tfrac{1}{3}\right)^n \sum_{k=0}^{\infty} (-1)^{-k} \left(\tfrac{1}{3}\right)^k 3^k$$
$$= \left(-\tfrac{1}{3}\right)^n \sum_{k=0}^{\infty} (-1)^{-k} 3^{k-k} = \left(-\tfrac{1}{3}\right)^n \sum_{k=0}^{n} \frac{1}{(-1)^k}$$

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$$= \left(-\tfrac{1}{3}\right)^n \sum_{k=0}^{n} (-1)^k = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

~~1 0 − 1 + 1 − 1~~

8) $$X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+4z^{-2}}$$

$$X(z) = \frac{1+\frac{2}{z}}{1-\frac{2}{z}+\frac{4}{z^2}}$$

$$= \frac{\frac{z+2}{z}}{\frac{z^2-2z+4}{z^2}} = \frac{z(z+2)}{z^2-2z+4} = \frac{z(z+2)}{z^2-2z+4} \quad -(1)$$

$$Z[r^n \cos \omega_0 n \cdot u(n)] = \frac{1-z^{-1}(r\cos\omega_0)}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$$

$$\frac{1-\frac{1}{z}(r\cos\omega_0)}{\frac{z^2-2rz\cos\omega_0+r^2}{z^2}} = \frac{z(z-r\cos\omega_0)}{z^2-2rz\cos\omega_0+r^2}$$

Comparing this with eq-(1)

$r\cos\omega_0 = 1$ $\cos\omega_0 = \frac{1}{2}$

$r^2 = 4$, $r = 2$

$$z^2+2z = z[z+2]$$
$$= z\left[z+2\times\tfrac{1}{2}\right]+z$$

$$= \frac{z[z + 2\cos\omega_0]}{z^2 - 4z \times (\cos\omega_0) + (2)^2} + (z)$$

$$z = Az\sin\omega_0$$
$$= 2 \times z \times 1/2$$
$$\sin\omega_0 = 1/2$$
$$A = 2$$

$$= \frac{z[z + 2\cos\omega_0]}{z^2 - 2 \cdot 2(1/2) \cdot z + (2)^2} +$$

$$\frac{z}{z^2 - 2 \cdot (2)(1/2) z + (2)^2}$$

$$\omega_0 = \pi/3$$

$$= (2)^n \cos(\pi/3)n + (2)^n \sin(\pi/6)^n$$

<u>Unilateral z-transform</u> :- (One-sided z-transform)

$$z[y(n-k)] = \sum_{n=0}^{\infty} y(n-k) z^{-n}$$

$$= \sum_{l=-k}^{\infty} y(l) z^{-(l+k)} \qquad n-k = l, \quad n = l+k$$

$$= z^{-k} \sum_{l=-k}^{\infty} y(l) z^{-l}$$

$$= z^{-k} \left[\sum_{l=0}^{\infty} y(l) z^{-l} + \sum_{l=-k}^{-1} y(l) z^{-l} \right]$$

$$= z^{-k} \left[Y(z) + \sum_{n=k}^{1} y(-n) z^{n} \right] \qquad (n = -l$$

$$z[y(n-1)] = z^{-1}[Y(z) + y(-1) z]$$

DT-12.02.11

SYSTEM FUNCTION

→ A system function is described by linear time constant coefficient differential equation.

$$y(n) = -a_K \sum_{k=1}^{N} y(n-k) + b_K \sum_{k=0}^{M} x(n-k)$$

$$Y(z) = -a_K \sum_{k=1}^{N} z^{-k} Y(z) + b_K \sum_{k=0}^{M} z^{-k} X(z)$$

$$Y(z)\left[1 + a_K \sum_{k=1}^{N} z^{-k}\right] = b_K\, X(z) \sum_{k=0}^{M} z^{-k} X(z)$$

$$\boxed{\frac{Y(z)}{X(z)} = H(z) = \frac{b_K \sum_{k=0}^{M} z^{-k} X(z)}{1 + a_K \sum_{k=1}^{N} z^{-k}}}$$

Solution of difference equation

Q) Determine the unit step response of the system to differential equation i.e

~~$y(n) - 0.7(y - 2)$~~

$$y(n) - 0.7\, y(n-1) + 0.12\, y(n-2) = x(n-1) + x(n-2)$$

$$\text{if } y(-1) = y(-2) = 1.$$

Ans - $\mathcal{Z}[y(n-k)] = z^{-k}\left[Y(z) + \sum_{n=k}^{1} y(-n) z^{n}\right]$

For initial conditions, we take ~~z-trai~~ one-sided z-transform.

$$\Rightarrow \mathcal{Z}[y(n-1)] = z^{-1}\left[Y(z) + \sum_{n=1}^{k=1} y(-1) z^{1}\right]$$

$$= z^{-1} Y(z) + y(-1)$$

$$Z[y(n-2)] = z^{-2}\left[Y(z) + \sum_{n=2}^{1} y(-n) z^{n}\right]$$

$$= z^{-2}\left[Y(z) + y(-1)z + y(-2)z^{2}\right]$$

$$= z^{-2}Y(z) + z^{-1}y(-1) + y(-2).$$

$$\Rightarrow Y(z) - 0.7\left[z^{-1}Y(z) + y(-1)\right] + 0.12\left[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right]$$

$$= z^{-1}X(z) + x(-1) + z^{-2}X(z) + z^{-1}x(-1) + x(-2)$$

$$\Rightarrow Y(z)\left[1 - 0.7z^{-1} + 0.12z^{-2}\right] - 0.7 + 0.12z^{-1} + 0.1\ldots$$

$$= z^{-1}X(z) + \cancel{x(-1)} + z^{-2}X(z) - z^{-1}\cancel{x(-1)} + x(-2)$$

$$\Rightarrow Y(z)\left[1 - 0.7z^{-1} + 0.12z^{-2}\right] = X(z)\left[z^{-1} + z^{-2}\right] + 0.7 - 0.12z^{-1} \ldots$$

$$= X(z)\left[z^{-1} + z^{-2}\right] + 0.58 - 0.12z^{-1}$$

$$\Rightarrow Y(z)\left[1 - 0.7z^{-1} + 0.12z^{-2}\right] = \frac{z}{z-1} + 0.58 - 0.12z^{-1}$$

$$\Rightarrow Y(z)\left[(1-0.4z^{-1})(1-0.3z^{-1})\right] = 0.58 + \frac{z}{z-1} - \frac{0.12}{z}$$

$$\Rightarrow Y(z)\left[\left(1-\frac{0.4}{z}\right)\left(1-\frac{0.3}{z}\right)\right] = \frac{0.58z^{2} - 0.58z + z^{2} - 0.12z + 0.12}{z(z-1)}$$

$$= \frac{0.68z^{2} - 0.7z + 0.12}{z(z-1)}$$

$$\Rightarrow Y(z) = \frac{0.6z^{2} - 0.7z + 0.12}{z(z-1)(z-0.4)(z-0.3)}$$

Q) The unit step response of a LIT system is given by

$$s(n) = \left(\frac{1}{3}\right)^{n-2} u(n+2)$$ find out the system function $x(n) = n\,u(n)$.

Ans- $y(n) = x(n) * h(n)$.

Here, $y(n) = s(n)$.

$$s(n) = x(n) * h(n)$$

$$S(z) = X(z) \cdot H(z)$$

$$\cancel{X(z) = \left(\tfrac{1}{3}\right)^{n-2}}$$

$$s(n) = \left(\tfrac{1}{3}\right)^{n-2} u(n+2)$$

$$\cancel{S(z) =}\ s(n) = \left(\tfrac{1}{3}\right)^{n-2} u(n+2)$$

$$S(z) = \sum_{-\infty}^{\infty} \left(\tfrac{1}{3}\right)^{n-2} u(n+2)\, z^{-n}$$

$$= \sum_{-2}^{\infty} \left(\tfrac{1}{3}\right)^{n-2} z^{-n}$$

$$= \sum_{-2}^{\infty} \left(\tfrac{1}{3}\right)^{n} \cdot 3^{2}\, z^{-n}$$

$$= 3^2 \sum_{-2}^{0} \left(\tfrac{1}{3} z^{-1}\right)^{n} + \sum_{0}^{\infty} \left(\tfrac{1}{3}\right)^{n-2} z^{-n}$$

$$= 9\left(\frac{z}{z-\frac{1}{3}}\right) \qquad = \frac{1}{1-\frac{1}{3}z^{-1}}$$

$$= \frac{1}{1-\frac{1}{3z}} = \frac{3z}{3z-1}$$

$$= 9 \times \frac{3z}{3z-1}$$

$$= \frac{27z}{3z-1} +$$

$$= \sum_{-2}^{0}\left(\tfrac{1}{3}\right)^{n-2} z^{-n} + \sum_{0}^{\infty}\left(\tfrac{1}{3}\right)^{n-2} z^{-n}$$

$$= \left(\tfrac{1}{3}\right)^{-2-2} z^{2} + \left(\tfrac{1}{3}\right)^{-1-2} z^{1} + \left(\tfrac{1}{3}\right)^{0-2} z^{0}$$

$$+ \sum_{0}^{\infty}\left(\tfrac{1}{3}\right)^{n}\left(\tfrac{1}{3}\right)^{-2} \cdot z^{-n}$$

$$= 9 \sum_{-2}^{\infty} 3^{-n} z^{-n}$$

$$= 9\left[\left(\tfrac{1}{3z}\right)^{-2} + \left(\tfrac{1}{3z}\right)^{-1} + \left(\tfrac{1}{3z}\right)^{0} + \left(\tfrac{1}{3z}\right)^{1} + \left(\tfrac{1}{3z}\right)^{2} + \cdots \infty\right]$$

$$= 9\left(\tfrac{1}{3z}\right)^{-2}\left[1 + \left(\tfrac{1}{3z}\right)^{1} + \left(\tfrac{1}{3z}\right)^{2} + \cdots \infty\right]$$

$$Y(z) = \frac{9\left(\tfrac{1}{3z}\right)^{-2}}{\left(1 - \tfrac{1}{3z}\right)} = \frac{9}{\cancel{(3z)^{2}}} \frac{9 \times (3z)^{2}}{(3z-1)} = \frac{81 z^{3}}{\cancel{3}\left(z-\tfrac{1}{3}\right)}$$

$$= \frac{81 z^{3}}{\left(z-\tfrac{1}{3}\right)}$$

$$Y(z) = X(z) \cdot H(z)$$

$$\frac{81 z^{3}}{\left(z-\tfrac{1}{3}\right)} = \frac{z}{z-1} \cdot H(z)$$

$$H(z) = \frac{81(z-1) z^{2}}{\left(z-\tfrac{1}{3}\right)}$$

$$h(n) = z^{-1}\left[H(z)\right]$$

$$= z^{-1}\left[\frac{81(z-1) z^{2}}{\left(z-\tfrac{1}{3}\right)}\right]$$

DT-21.02.11

Four types of Fourier representation :-

Time domain Signal	Periodic Signal	Non-periodic Signal
1. Continuous	**Fourier Series** $x(t) = \sum_{k=-\infty}^{\infty} X(k) e^{jk\omega_0 t}$ $X(k) = \frac{1}{T}\int_{-T}^{T} x(t) e^{-j\omega_0 k t}$ T = time period $\omega_0 = 2\pi/T$	**Fourier transform** $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} dt$ $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
2. Discrete	**DTFS** $x(n) = \sum_{k=\langle N\rangle} X(k) e^{jk\omega_0 n}$ $X(k) = \frac{1}{N}\sum_{n=\langle n\rangle} x(n) e^{-jk\omega_0 n}$ Both have period N, $\omega_0 = 2\pi/N$.	**DTFT** $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})$ $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

Fourier	Representation
Time Domain Property	Frequency domain Property
Continuous $\longrightarrow$	Non-periodic
Discrete $\longrightarrow$	Periodic
Periodic $\longrightarrow$	Discrete
Non-periodic $\longrightarrow$	Continuous

Frequency Domain

on-periodic & ~~discrete~~ continuous.

on-periodic & discrete.

Periodic & discrete.

Periodic & continuous.

Frequency analysis of discrete ~~type~~ time signal :-

→ $x(n) = e^{Jw_0 n}$

The system is periodic, only when ω_0 = integral multiple of 2π :

$$x(n \pm N) = e^{J(w_0 \pm 2\pi)n}$$
$$\downarrow$$
$$N = 2\pi$$
$$= e^{Jw_0 n} \cdot e^{J2\pi n}$$
$$= e^{Jw_0 n} \cdot 1$$

w_0 = fundament frequency

N = " period

→ Any periodic signal can be represented by discrete time fourier series with fundamental frequency, $\omega_0 = \frac{2\pi}{N}$ and its harmonics $\frac{2\pi}{N} \times K$

→ The frequency representation of periodic signal consists of $e^{Jw_0 nK}$ where $-\infty < K < \infty$

So, it is clear that there are N independent harmonics out of infinite no. of harmonics i.e the first harmonic is identical to the (N+1) harmonics.

2nd harmonic is identical to the (N+2) harmonic.

→ 'N' independent harmonics range over 2π. are ~~represented~~ separated by $\omega_0 = 2\pi/N$ (fundamental period).

→ 'N' independent harmonics can be taken in the range of $0 \le K \le N-1$ or $-1 \le K \le N$..

Total summation = N

→ The ~~periodic~~ frequency fourier series representation of periodic signal can be written as a linear sum of

$$x(n) = \sum_{k=\langle N \rangle} C_K e^{j\omega_0 nk} \quad \text{—} (1)$$

We can write $C_K = x(k)$

→ As $x(n)$ is distinct on the range of N, then the above eqn can be written as

$$x(n) = \sum_{k=0}^{N-1} C_K e^{j\omega_0 nk}$$

→ To find C_K, multiply both sides $e^{-j\omega_0 mn}$ in eq-(1), & take the ~~sm~~ summation.

$$\Rightarrow \sum_{n=0}^{N-1} x(n) e^{-jm\omega_0 n} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} C_K e^{j\omega_0 nk} \cdot e^{-j\omega_0 mn}$$

$$= \sum_{k=0}^{N-1} C_K \sum_{n=0}^{N-1} e^{j 2\pi/N (k-m) n}$$

$$\sum_{n=0}^{N-1} e^{j2\pi/N (k-m)n} = N, \quad \text{if } (k-m) = \pm N, \pm 2N, \pm 3N$$
$$= 0 \quad \text{Otherwise}$$

$$= \sum_{k=0}^{N-1} C_k \cdot N$$

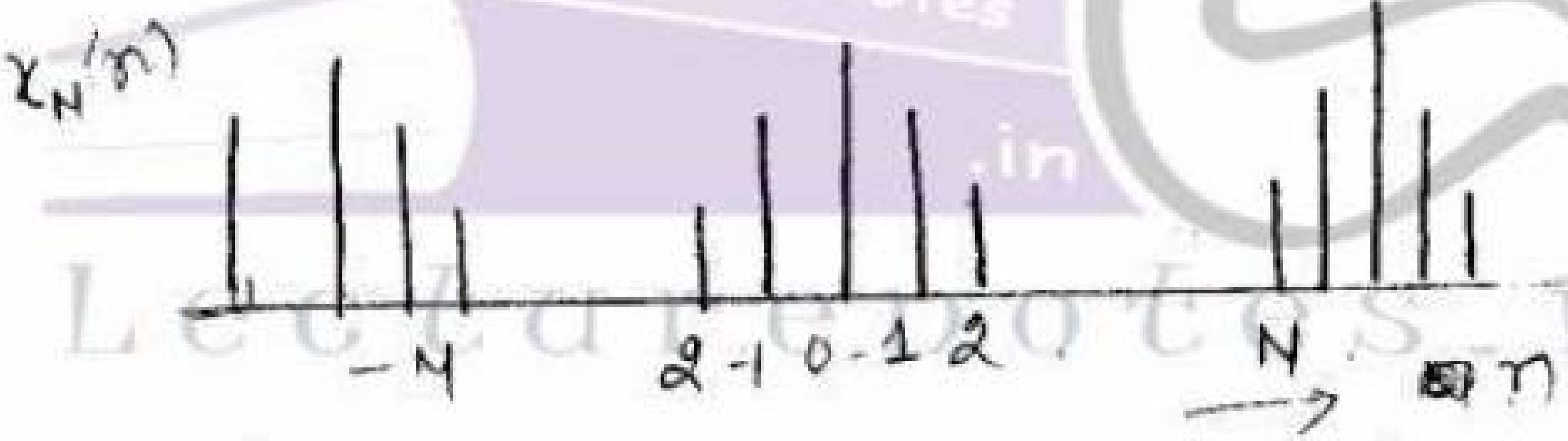
$$= \sum_{n=0}^{N-1} x(n) e^{-j\omega_0 mn} = N \sum_{m=0}^{N-1} C_m = N \cdot C_m$$

(we can write C_k or else C_m → It is same)

$$\boxed{C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j\omega_0 kn}}$$

~~Let $\lim_{l \to \infty} x_N(n)$~~

Let $\lim_{N \to \infty} x_N(n) = x(n)$ = aperiodic ~~representation~~ sequence.



So, $N \to \infty$, aperiodic signal

$C_k \to 0$

So, new fourier coeffecient is represented by

$N C_k = X(e^{j2\pi k/N})$

~~Let $\omega_0 k = \frac{2\pi}{N}$~~ $2\pi/k$

$$C_k = \frac{1}{N} \sum_{n=0}^{N} x(n) \cdot e^{-j\omega_0 kn} \qquad \omega = \text{continuous}$$

$n = -(N - N_2)$ to $N/2$

$$\underline{Nc_K} =$$

$$= N \cdot \frac{1}{N} \int_{-\infty}^{\infty} x(n)\, e^{-j\omega_0 k n}$$

$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x(n)\, e^{-j\omega_0 k n}$$

$$\cancel{X(e^{j\omega}) = \sum_{-\infty}^{\infty} x(n)\, e^{-j}}$$

$$\cancel{X(e^{j\omega}) = \sum_{-\infty}^{\infty} x(n)\, e^{-j}}$$

$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x(n)\, e^{-j\omega_0 n}$$

$$\Downarrow$$

Continuous ~~of~~ signal.

$$\underline{Nc_K}$$

$$= N \cdot 1/N \int_{-\infty}^{\infty} x(n)\, e^{-j\omega_0 k n}$$

$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x(n)\, e^{-j\omega_0 k n}$$

$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$\Downarrow$ ~~continuous~~ Continuous domain

IDFT :-

$$x(n) = \sum_{n=0}^{N-1} C_K \cdot e^{-j\omega_0 kn}$$

$$= \frac{1}{2\pi} \sum_{n=0}^{N-1} N \cdot C_K \cdot e^{-j\omega_0 kn} \cdot 2\pi/N$$

$$= \cancel{\tfrac{1}{2\pi}\int_0^{2\pi}} \frac{1}{2\pi} \sum_{n=0}^{N-1} X(e^{j\omega_0 k}) e^{-j\omega_0 kn}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

dc analysis – we considered complex term $(e^{j\omega})$

~~Q) $\cos \pi/2$~~ =

DT-25.02.11

DISCRETE FOURIER SERIES :-

$$C_K = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

↓

discrete / line spectra of $x(n)$.

fourier series coefficient

$$C_{K+N} = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi(K+N)n}{N}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} \cdot e^{-j2\pi n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$= C_K$$

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}.$$

c_k appears at .

$$\omega_k = 2\pi/N \, k.$$

$$\omega_k = 2\pi/3 \times 0,\ 2\pi/3 \times 1,\ 2\pi/3 \times 2 \ldots$$

If $N = 3$.

c_k appears at $\omega = 0,\ 2\pi/3,\ 4\pi/3$.

$$\boxed{\omega = 2\pi}.$$

Q) Find the DFS and DFT of $x(n) = \cos \pi/3 \, n$

Ans – $x(n) = \cos \pi/3 \, n$.

$\omega = \pi/3$

$N = 2\pi/\omega$

$= \frac{2\pi \times 3}{\pi} = 6.$

$N = 6.$ $\cancel{\omega = \pi/3}$

$$x(n) = \frac{1}{2}\left[e^{j\pi/3 \, n} + e^{-j\pi/3 \, n}\right]$$

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

$$x(n) = \sum_{k=\langle N\rangle} c_k e^{-j2\pi kn/N}.$$

$$= \sum_{k=\langle 6\rangle} c_k e^{-j2\pi kn/6}.$$

$$= \sum_{k=-2}^{3} c_k e^{-j\pi kn/3}$$

$c_k = 1/2$ for $k = -1, 1$.

$= 0$ for $-2, 0, 2, 3$

$\cancel{x(0)}$ $x(0) = 1$

$x(1) = 1/2$

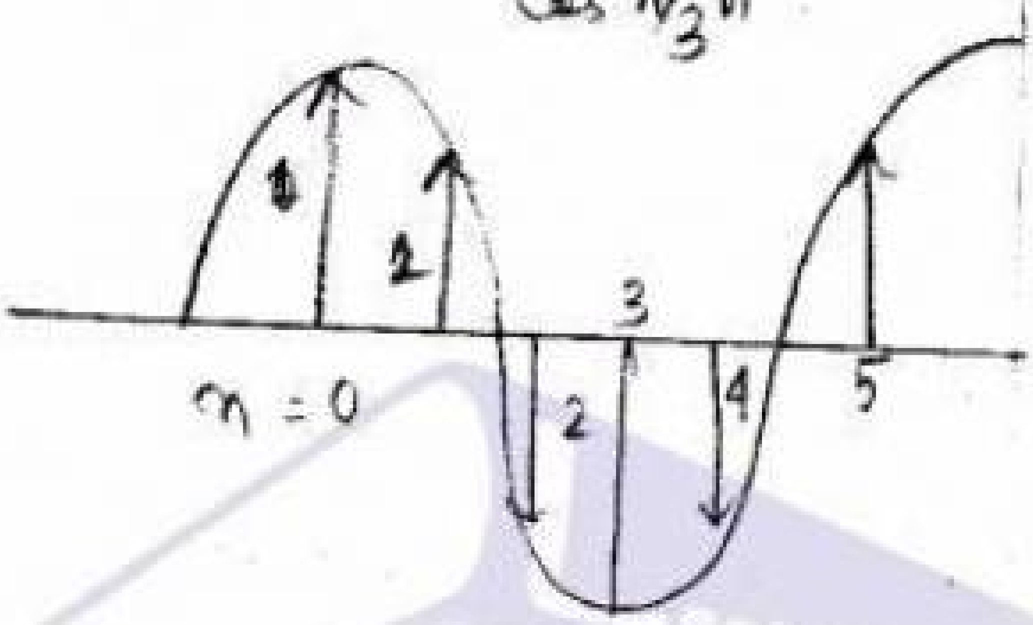
$x(2) = -1/2$

$x(3) = -1$ $\qquad x(n) = \{ 1$

$x(4) = -1/2$

$x(5) = 1/2$

$\cos \pi/3\, n$



$X(k) = DFS[x(n)]$ for one time N.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{5} x(n) e^{-j\pi/3\, kn}$$

$X(0) =$

$\cancel{X(0) = x}$ $X(1) = x(0) + x(1)e^{-j\pi/3} + x(2)e^{-j2\pi/3} + x(3)e^{-j\pi} + x(4)e^{-j4\pi/3} + x(5)e^{-j5\pi/3}$

$$= 1 + \frac{1}{2} e^{-j\pi/3} - \frac{1}{2} e^{-j2\pi/3} - e^{-j\pi} - \frac{1}{2} e^{-j4\pi/3} + \frac{1}{2} e^{-j5\pi/3}$$

$$= 1 + \frac{1}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}j\right) - \frac{1}{2}\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

$x(e^{j\omega}) = \cancel{\text{dTFT of}}\ x(n)$

$X(5) = 3$

Only for $X(1)$ & $x(5) = 3$, for all other it is zero.

$x(e^{j\omega}) = \underline{\text{dTFT}}$ of $x(n)$

↳ discrete time fourier transform

$X(K) = X(e^{j\omega})$ at $\omega = 2\pi K/N$

$$= \sum_{-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{0}^{N-1} x(n)\, e^{-j\, 2\pi/N\, kn}$$

Q) Find DFS of $x(n) = 3\cos 4\pi n + 5\sin \frac{\pi}{2} n$,

Ans - C_k = periodic

$N = 2\pi/\omega = 2\pi/\frac{\pi}{2}$

$$x(n) = 3\left[\frac{e^{-j4\pi n} + e^{j4\pi n}}{2}\right] + 5\left[\frac{e^{+j\pi/2\, n} - e^{-j\pi/2\, n}}{2j}\right]$$

$\cancel{N = 2\pi}$ $\quad \omega = \frac{2\pi}{N} \times K$

$\frac{2\pi}{0.4\pi} = \frac{2 \times 10}{4} = 5$

$N_1 = 2\pi/\omega \times k = \frac{2\pi}{4\pi} \times \frac{1}{2} \times 2$

$= 5$

$N_2 = 4 \qquad \frac{2\pi}{\pi/2} = 4.$

Common value of $N = 20$.

$$x(n) = \sum_{k=\langle 20\rangle} C_k e^{-j2\pi kn/20}$$

$$= \sum_{k=\langle 20\rangle} C_k e^{-j\pi kn/10}$$

$\cancel{C_k = 3/2}$ $\quad C_4 = C_{-4} = 3/2$

$C_5 = -5/2j$

$C_{-5} = +5/2j$

Discrete spectra

phase

$\angle C_k$

DT-26.02.11

DFT CALCULATION BY MATRIX METHOD :-

The DFT of N point sequence $x(n)$ is expressed by

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{nK}$$

where $W_N = e^{-j2\pi/N}$.

W_N = Twiddle factor.

Twiddle factor :-

Twiddle factor is a vector on a unit circle.

→ The magnitude of $|W_N| = |e^{-j2\pi/N}| = 1$

→ Phase angle $= -2\pi/N$

→ If $N = 4$

If the sequence is N-point, then W_N is $N \times N$ matrix which is known as DFT matrix.

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4 & W_4^1 & W_4^2 \\ W_4^0 & W_4^2 & W_4^3 & W_4^4 \\ W_4^0 & W_4^3 & W_4^6 & W_4^8 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$X(K) = [W_N] \begin{bmatrix} x(0) \\ \vdots \\ x(N) \end{bmatrix}$$

$W_N = e$

$W_4^0 = e^{-j 2\pi/4 \times 0}$

$W_4^1 = e^{-j 2\pi/4 \times 1}$

$$X(0) = \sum_{n=0}^{3} x(n) W_4^0$$

$$= x(0) W_4^0 + x(1) W_4^0 + x(2) W_4^0 + x(3) W_4^0$$

$$X(1) = x(0) W_4^0 + x(1) W_4^1 + x(2) W_4^2 + x(3) W_4^3$$

Normalised case W_N

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ 1 & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

$$X(k) = \sum_{n=0}^{3} x(n) W_4^{nk}$$

$$X(1) = \sum_{n=0}^{3} x(n) W_4^{n}$$

$$X(2) = \sum_{n=0}^{3} x(n) W_4^{2n}$$

$$X(3) = \sum_{n=0}^{3} x(n) W_4^{3n}$$

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -J & -1 & J \\ 1 & -1 & 1 & -1 \\ 1 & J & -1 & -J \end{bmatrix}$$

<u>To find IDFT :-</u>

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(n) \end{bmatrix} = \frac{1}{N} \left[W_N^{-1} \right] \begin{bmatrix} X(0) \\ X(1) \\ X(k) \end{bmatrix}$$

$$W_4^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & J & -1 & -J \\ 1 & -1 & 1 & -1 \\ 1 & -J & -1 & -J \end{bmatrix}$$

Complex conjugate of W_4

Q) The sequence $[1\ 1\ 0\ 0]$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & J & -1 & -J \\ 1 & -1 & 1 & -1 \\ 1 & -J & -1 & -J \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$X(0) = 2$ $\quad X(2) = 0$ $\quad X(3) = 1+J$

$X(1) = 1-J$

Properties of Twiddle factor :-

→ **Periodicity** :-

$$W_N^{r} = W_N^{r \pm N}$$

$$W_N^{1} = W_N^{1 \pm N}$$

→ **Symmetric Property**

$$W_N^{r} = -W_N^{r \pm N/2}$$

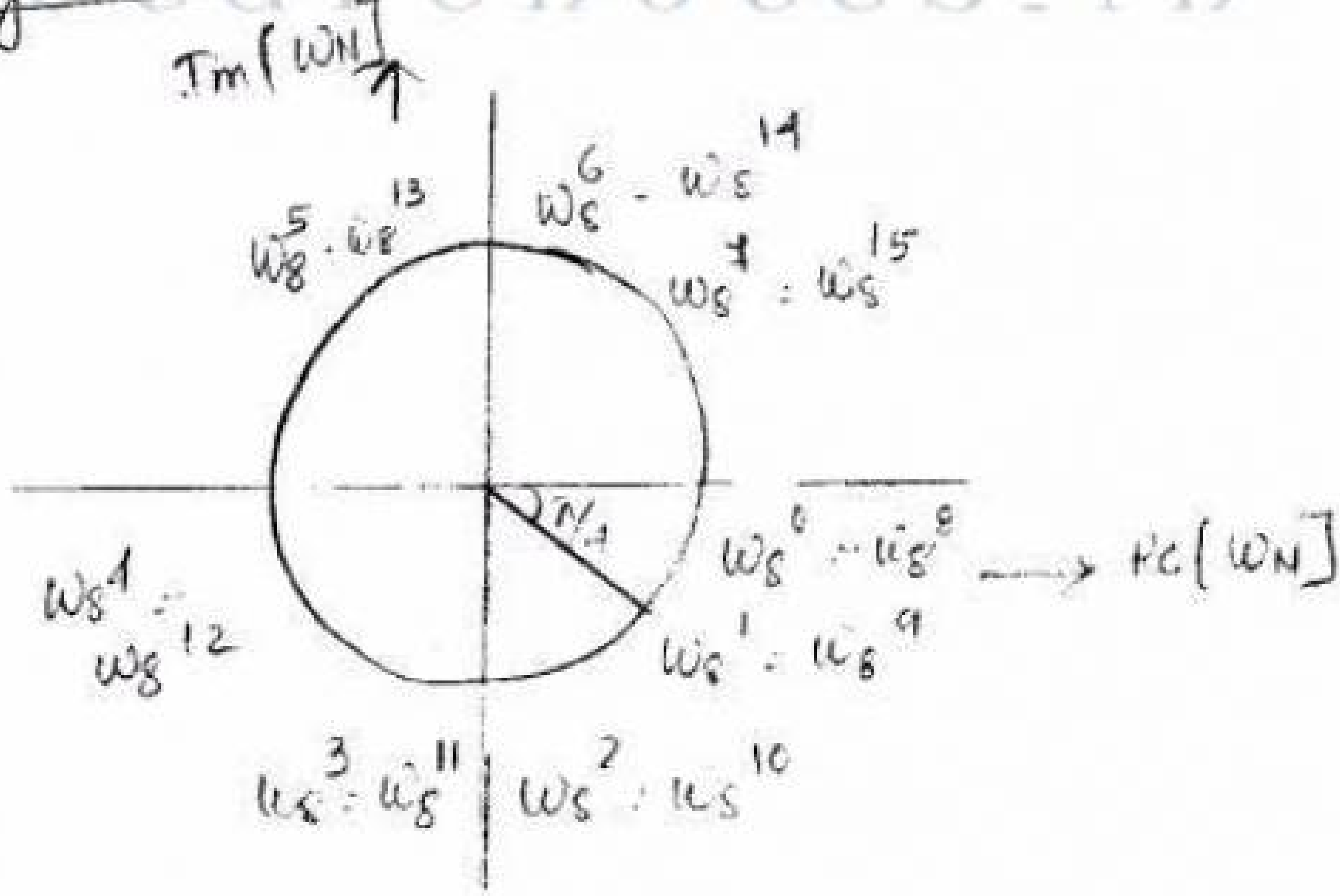
$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & J & -1 & -J \\ 1 & -1 & 1 & -1 \\ 1 & -J & -1 & J \end{bmatrix}\begin{bmatrix} 2 \\ 1-J \\ 0 \\ 1+J \end{bmatrix}$$

$$x(0) = \frac{1}{4}\left[2 + 1 - \cancel{J} + 0 + 1 + 1 + \cancel{J}\right]$$

$$= \frac{1}{4} \times 4 = 1$$

$\cancel{x(1) =}$

Plotting of W_N :- Let $W_N = W_8$

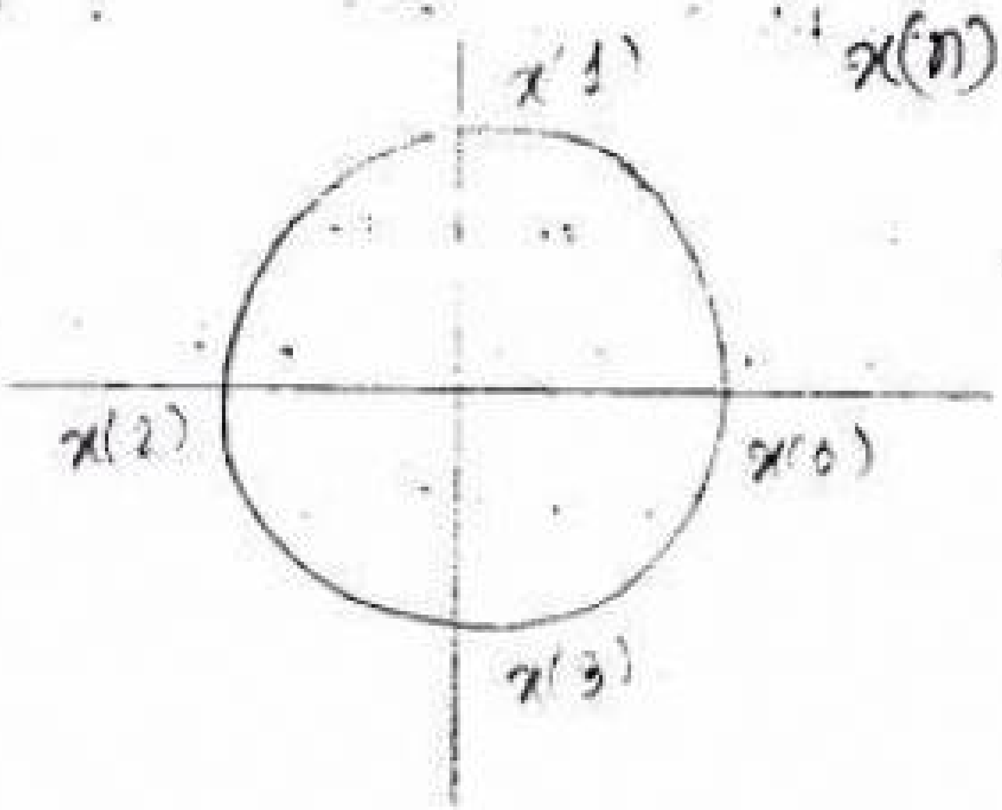


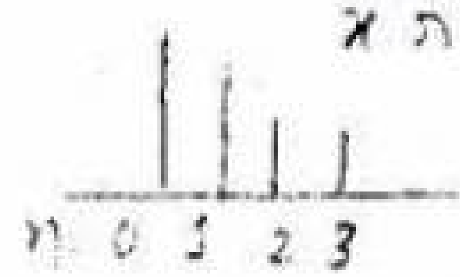
→ The periodic extension of any sequence $x(n)$ can be written as

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

One way of visualising the periodic sequence is wrapping the total sequence around the circle in counterclockwise direction which is selected as the +ve direction.

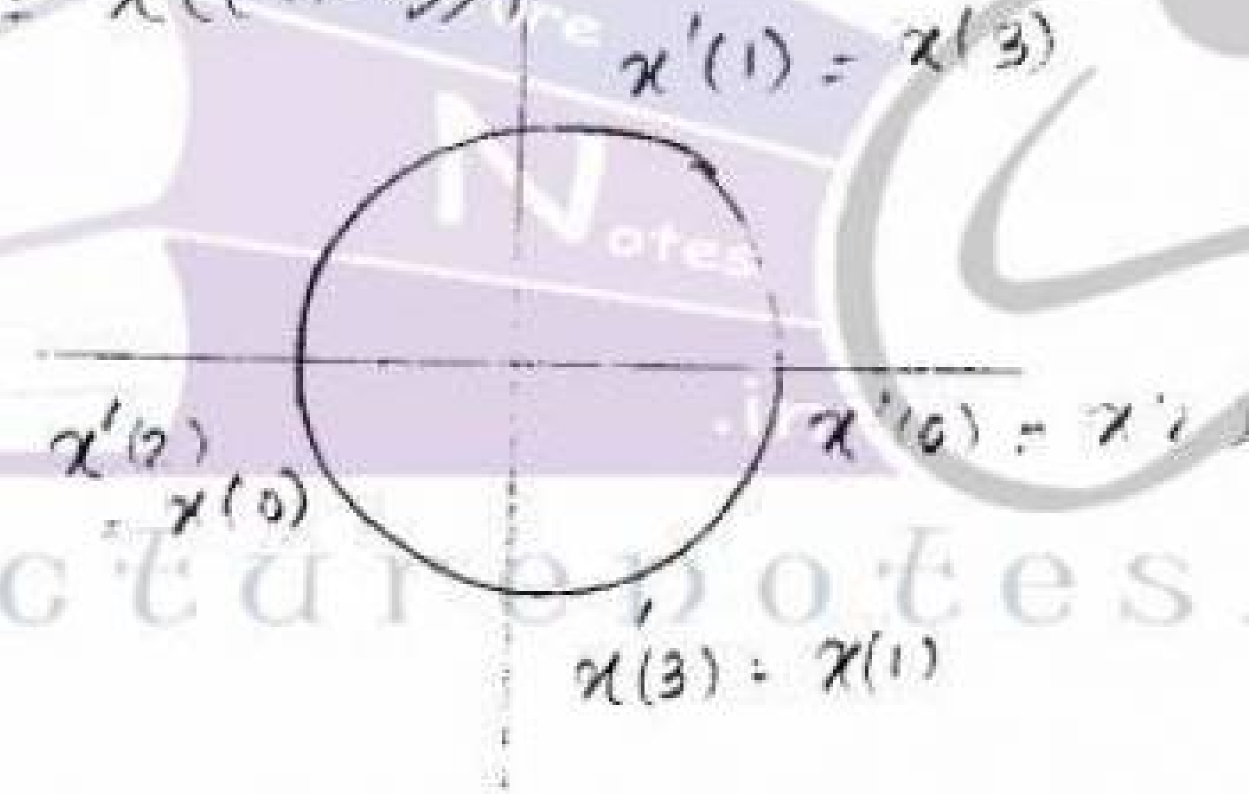
→ As ~~in~~ the we travels the circumference, the total finite sequence





$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$
$$= x(n \text{ modulo } N)$$
$$= x((n))_N$$

$$x'(n) = x((n-2))_4$$



$$x(n) = x(0),\ x(1),\ x(2) \ldots\ldots\ x(N-1)$$
$$x((N-1))_N = x(N-1),\ x(0),\ x(1) \ldots\ x(N-2)$$
$$x((n-2))_N = x(N-2),\ x(N-1),\ x(0),\ \ldots\ x(N-3)$$
$$x((n-k))_N = x(N-k),\ x(N-k+1) \ldots\ x(N-k-1)$$
$$x((n-N))_N = x(0),\ x(1) \ldots\ x(N-1)$$

If the argument ~~M~~ is written 0 to N-1

→ If the argument n is other than we subtract or add multiples of N with n until the result is 0 to N − 1 .

$$x(-3 \text{ modulo } 4)$$

$$= x(-3+4) = x(1)$$

$x(-11) =$

Properties of DFT

(1) Linearity

$$x_1(n) \xrightarrow{DFT} X_1(K)$$

$$x_2(n) \longleftrightarrow X_2(K)$$

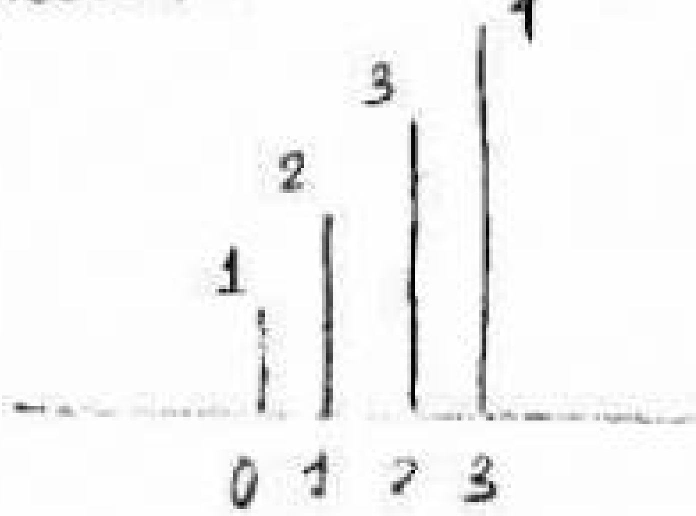
$$a_1 x_1(n) + b_1 x_2(n) \xleftrightarrow{DFT} a_1 X_1(K) + b_1 X_2(K)$$

(2) ~~Periodicity Peridic~~ Periodicity

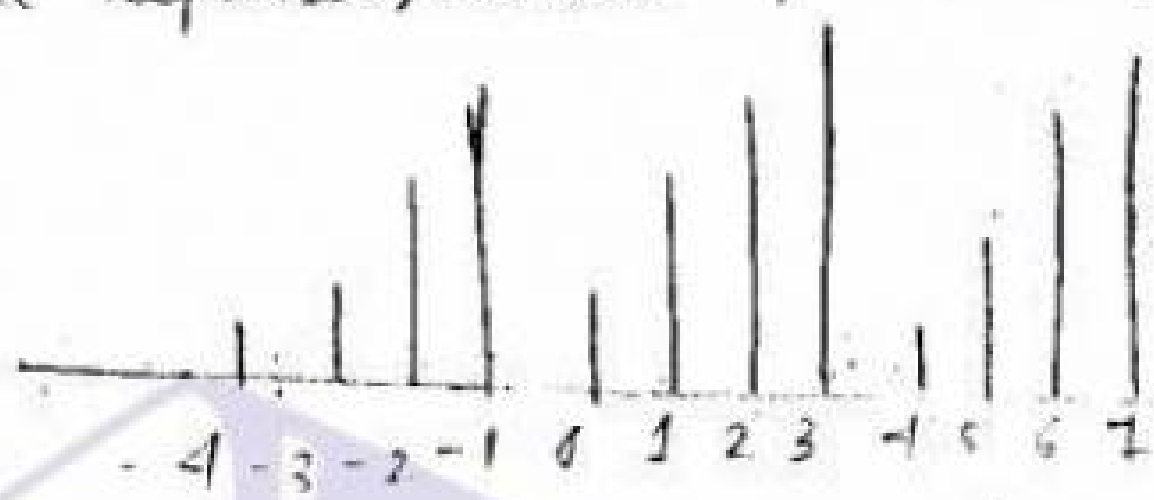
(3) Time shifting.

DT-05.03.11

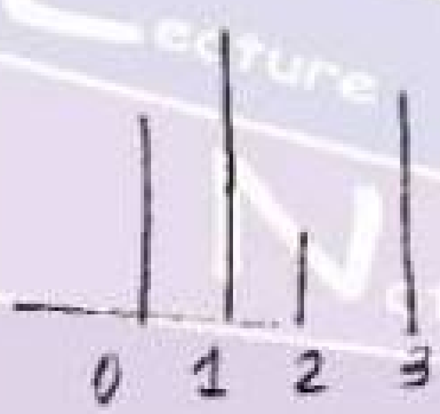
Periodic representation of $x(n)$

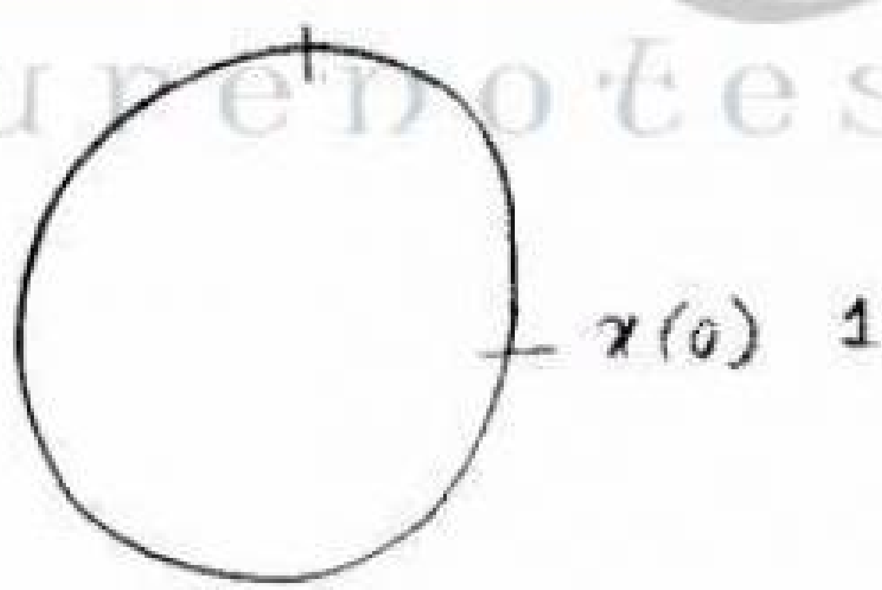


Periodic representation of $x(n)$ ~~$x_1(n) \rightarrow x(n-2)$~~



$x(n-2) \xrightarrow[N]{} x_1(n)$

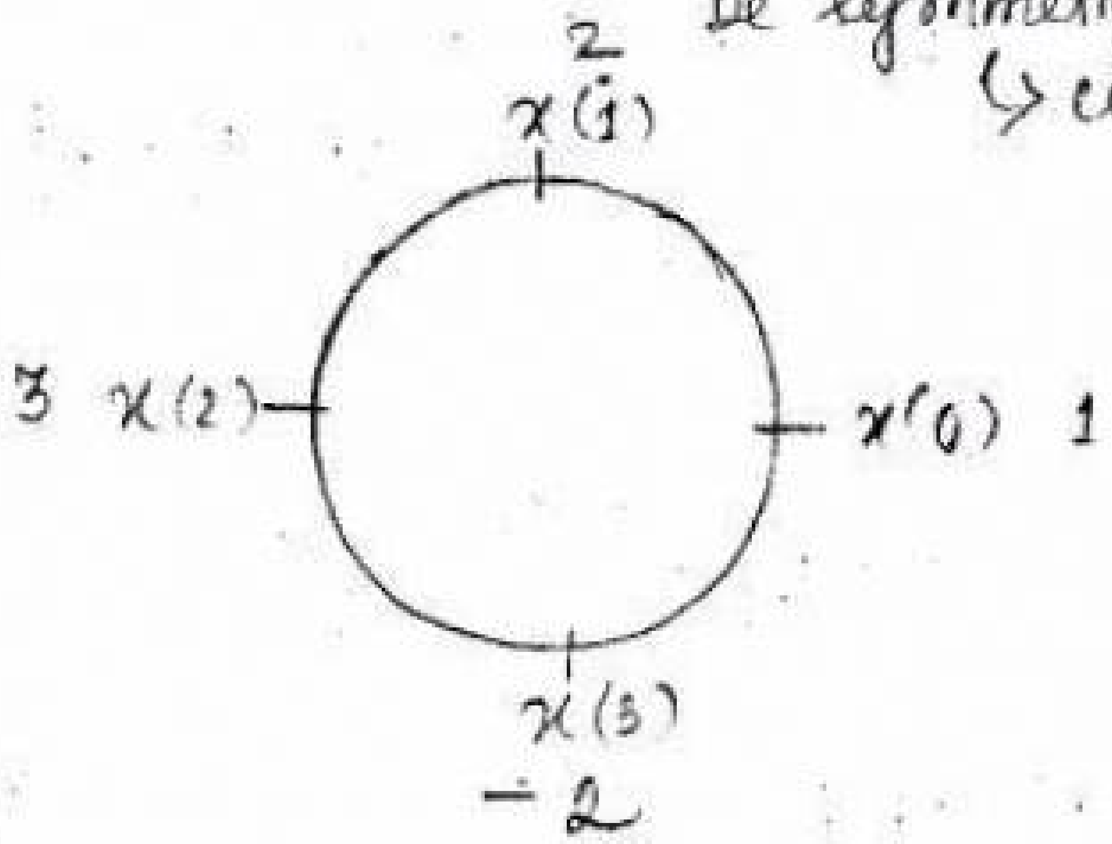




Circularly even and odd sequence :-

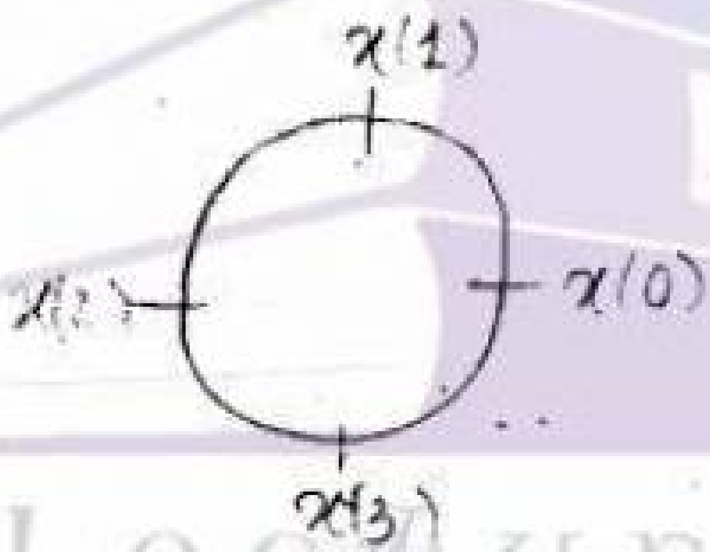
$x(N-n) = x(n) \dashrightarrow$ Around a point, it has to be symmetric.

↳ circularly even sequence



$x(N-n) = -x(n) \rightarrow$ circularly odd sequence

Time Reversal :-



Properties of DFT :-

$$x(n) = x_R(n) + j\,x_I(n)$$

If the sequence is real and even.

↳ only cosine side (cosine t...

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} (x_R(n) + j x_I(n))\left(\cos\frac{2\pi kn}{N} - j\sin\frac{2\pi kn}{N}\right)$$

$$= \sum_{n=0}^{N-1} x_R(n)\cos\frac{2\pi kn}{N} - j\,x_R(n)\cdot\sin\frac{2\pi kn}{N}$$

$$+ j\,x_I(n)\cos\frac{2\pi kn}{N} + x_I(n)\sin\frac{2\pi kn}{N}$$

$$X_R(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\frac{2\pi kn}{N} + x_I(n)\sin\frac{2\pi kn}{N}\right]$$

$$X_I(k) = \sum_{n=0}^{N-1}\left[x_R(n)\sin\frac{2\pi kn}{N} - x_I(n)\cos\frac{2\pi kn}{N}\right]$$

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_R(k)\cos\frac{2\pi kn}{N} - X_I(k)\sin\frac{2\pi kn}{N}$$

$$x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\sin\frac{2\pi kn}{N} + X_I(k)\cos\frac{2\pi kn}{N}\right]$$

Case-1 : If the input is real value.

$$x(n) = [1 \quad 1 \quad 2 \quad 1]$$

$$\xrightarrow{DFT} X(k) = [\,6 \quad -2+2j \quad -2 \quad -2-2j\,] \quad \} N=4$$

$$x(4-1) = x^*(1)$$

$$x(3) = x^*(1)$$

$$X(N-k) = X^*(k) = X(-k)$$

$$|X(N-k)| = |X(k)|$$

$$\angle X(N-k) = -\angle X(k)$$

Case-2 If the input is real and even

$$x(n) = x(N-n)$$

$$X_I(k) = 0$$

$$X(k) = \sum_{n=0}^{N-1} x(n) \cos \frac{2\pi kn}{N} \quad \text{(DFT)}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cos \frac{2\pi kn}{N} \quad \text{(IDFT)}$$

Case-3 If the input is real and odd

$$X_R(k) = 0$$

$$X(k) = -j \sum_{n=0}^{N-1} x(n) \sin \frac{2\pi kn}{N}$$

In case of IDFT.

$$x(n) = j \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sin \frac{2\pi kn}{N}$$

Case-4 If the input is purely imaginary

odd $X_R(k) = \sum_{n=0}^{N-1} x_I(n) \sin \frac{2\pi kn}{N}$

even $X_I(k) = \sum_{n=0}^{N-1} x_I(n) \cos \frac{2\pi kn}{N}$

Derivation

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N}$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi kn/N}$$

$$X_3(k) = X_1(k) \cdot X_2(k)$$

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k) e^{j2\pi km/N}$$

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N}\right] \left[\sum_{l=0}^{N-1} x_2(l) e^{-j2\pi kl}\right] e^{j2\pi km}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{j2\pi (m-n-l)/N}\right]$$

$$= \sum_{k=0}^{N-1} a^k = \begin{cases} N, & a = 1 \\ \dfrac{1-a^N}{1-a}, & a \neq 1 \end{cases}$$

$(m-n-l)$ is a multiple of N.

$$N((m-n-l))_N$$

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n))_N$$

↳ modulo N

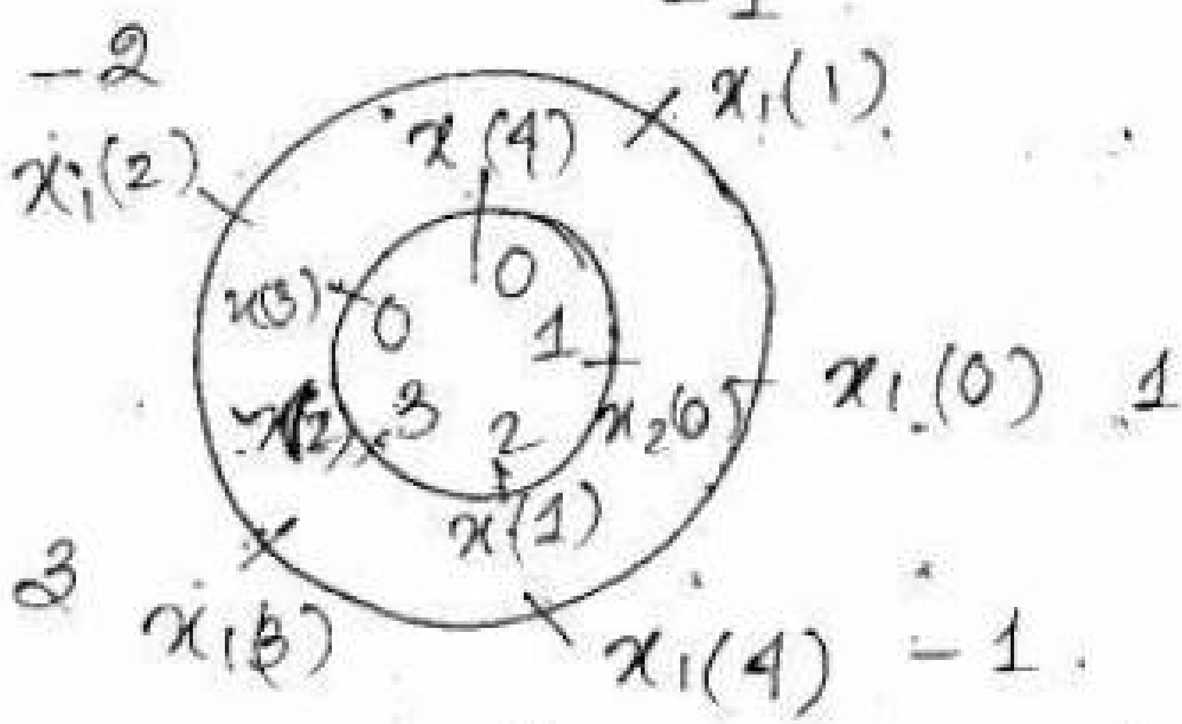
$$y(m) = \sum_{n=-\infty}^{\infty} x(n) h(m-n)$$

↳ x_2 value limits upto the value N.

Q) Circular convulution

$x_1(n) = \{1, -1, -2, 3, -1\}$

$x_2(n) = \{1, 2, 3\}$



$x_3(0) = 1 - 2 + 9 = 8$

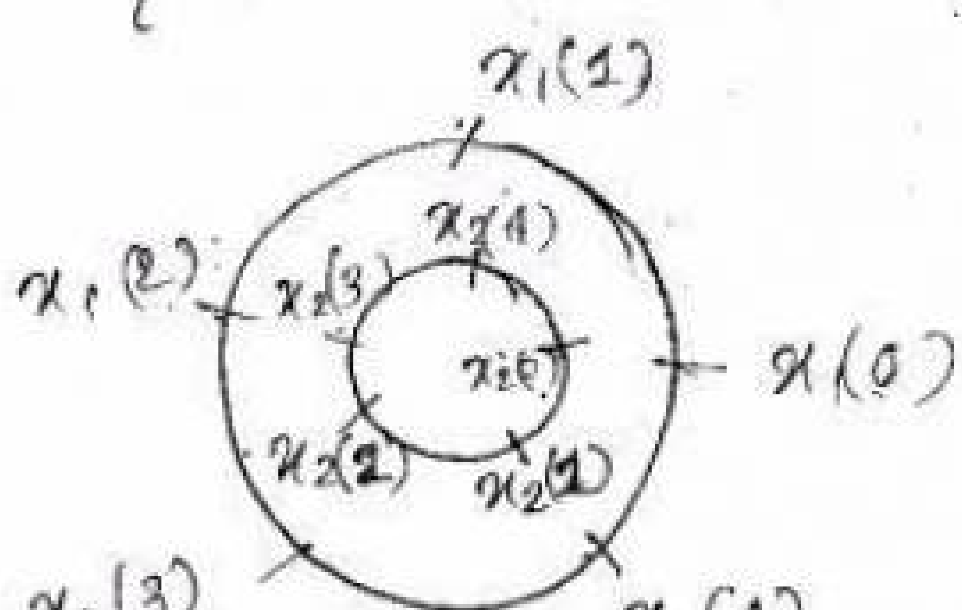
$x_3(1) =$

DT- 07-03-11

Q) Find its circular convulution

$x_1(n) = \{1, 2, 3, 2, 1\}$

$x_2(n) = \{1, 1, 2,\}$



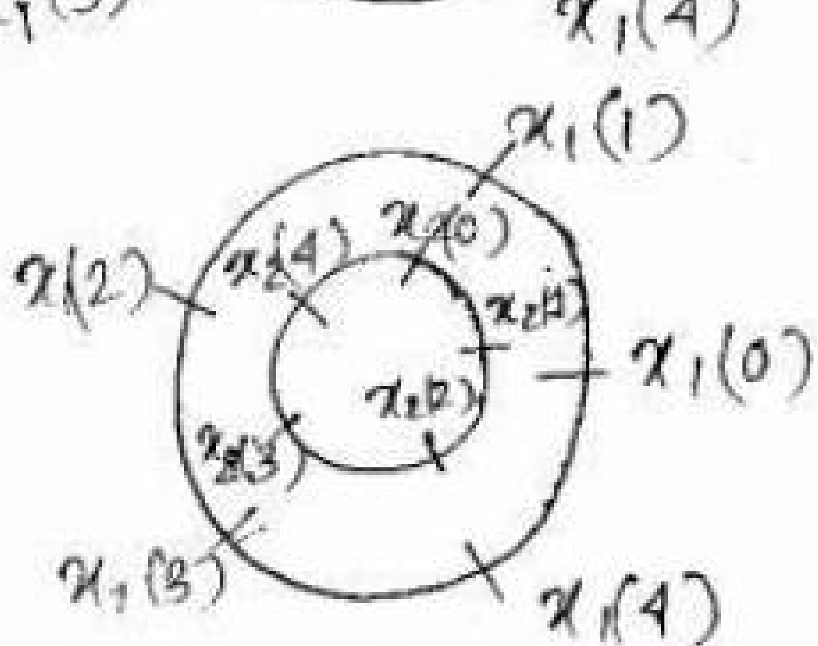
$x_3(0) = 6$

$x_3(1) = 5$

$x_3(2) = 7$

$x_3(3) = 9$

$x_3(4) = 9$



$$\begin{bmatrix} x_2(0) & x_2(N-1) & x_2(N-2) & \cdots & x_2(1) \\ x_2(1) & x_2(0) & x_2(N-1) & \cdots & x_2(2) \\ \vdots & & & & \vdots \\ x_2(N-1) & x_2(N-2) & x_2(N-3) & \cdots & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ \vdots \\ x_1(N) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 & 2 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 5 \\ 7 \\ 9 \\ 9 \end{bmatrix}$$

Circular Correlation:-

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N$$

Circular time shift :-

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $x((n-l))_N \xleftrightarrow[N]{DFT} X(k)\, e^{-j2\pi kl/N}$.

Filtering of Long duration data sequence :-

Linear convulution through circular convulution

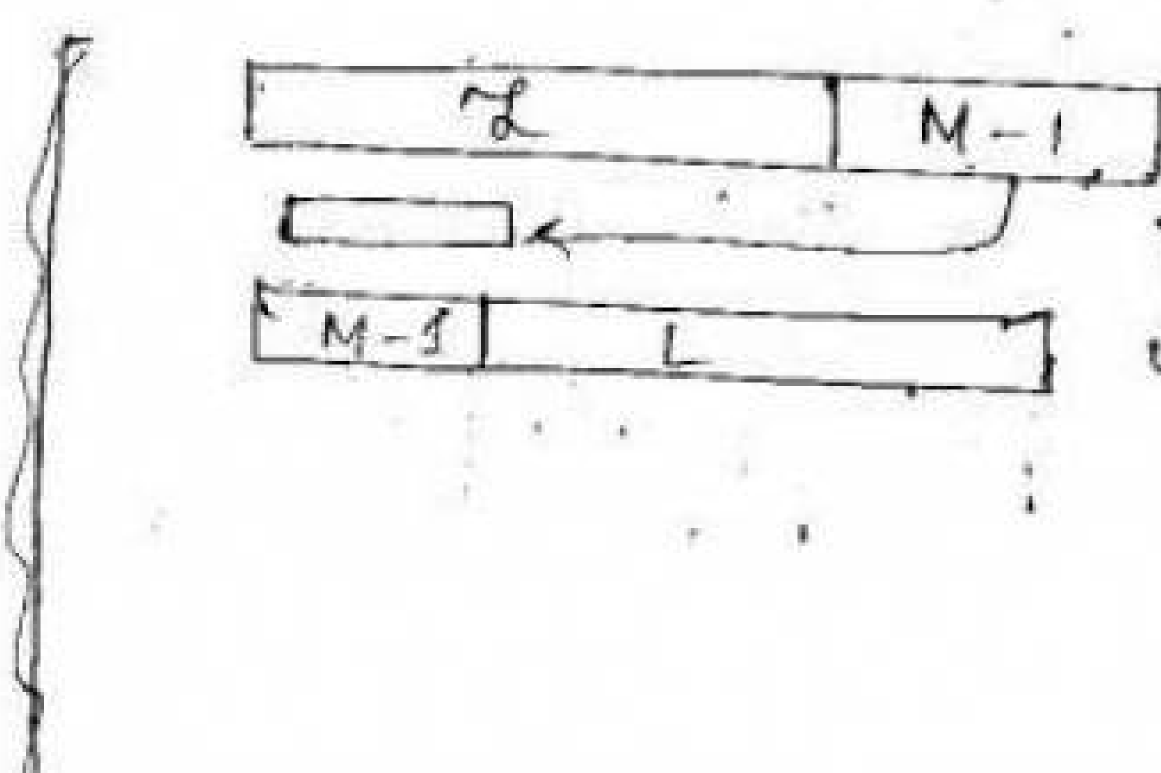
$x(n) = \{1, 2, 3, 4\}$

$h(n) = \{1, 2, 1\} = \{1, 2, 1, 0\}$

	1	2	3	4
1	1	2	3	4
2	2	4	6	8
1	1	2	3	4

$x(n) * h(n) = \{1, 4, 8, 12, 11, 4\}$ Linear convulution

$x(n) \, N \, h(n) = \{12, 8, 8, 8, 12\}$ ← Circular convulution



Linear convuluti

circular

L	M-1

→ Linear (M-1 zero padding)

M	L-1

→ Circular

$x(n) = \{1, 2, 3, 4, 5, 0\}$ $\quad M-1 = 3-1 = 2$

$h(n) = \{1, 2, 3, 0, 0, 0\}$

1. Overlap Save method :

$h(n) =$

1	1	2

$M = 3$

$x(n)$

1	2	1	1	1	1	2	3	1	2	1	4

0	0	1	2	1	1

$L = 4$

$M-1 = 3-1 = 2$ no. zero padding

1	1	1	1	2	3

$L = 4$

2	3	1	2	1	4

3	2	1	3	5	6

• Discard first M-1 elements

8	8	4	4	5	7

8	13	8	9	5	9	

1	3	5	6	4	4	5	7	8	9	5	9

Q) Find $y(n)$ for a filter with $h(n) = \{1, 1, 1\}$

& $x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\}$

$L = 3$. $M = 3$, $M - 1 = 2$

$x_1(n) = \{3, -1, 0\} = \{0, 0, 3, -1, 0\}$

$x_2(n) = \{1, 3, 2\} = \{-1, 0, 1, 3, 2\}$

$x_3(n) = \{0, 1, 2\} = \{3, 2, 0, 1, 2\}$

$x_4(n) = \{1, 0, 0\} = \{1, 2, 1, 0, 0\}$

$x_1(n) \; N \; h(n) = \{-1, 0, 3, 2, 2\}$

$x_2(n) \; N \; h(n) = \{4, 1, 0, 4, 6\}$

$x_3(n) \; N \; h(n) = \{6, 7, 5, 3, 3\}$

$x_4(n) \; N \; h(n) = \{1, 3, 4, 3, 1\}$

$y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 3, 4, 3, 1\}$

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Overlap Add Method :-

$x(n)$: $\underbrace{\boxed{1}\boxed{2}\boxed{-1}}\ \underbrace{\boxed{2}\boxed{3}\boxed{-2}}\ \underbrace{\boxed{3}\boxed{-1}\boxed{1}}\ \underbrace{\boxed{1}\boxed{2}\boxed{-1}}$

$h(n) = \{1, 2\} \rightarrow M = 2, \quad M-1 = 1$

$\boxed{1}\boxed{2}\boxed{-1}\boxed{0}$

$\boxed{2}\boxed{3}\boxed{-2}\boxed{0}$

$\boxed{3}\boxed{-1}\boxed{1}\boxed{0}$

$\boxed{1}\boxed{2}\boxed{-1}\boxed{0}$

$x_1(n) \; (N) \; h(n) = [\cancel{1, 4, 3}] \; [1 \quad 4 \quad 3 \quad -2]$

$x_2(n) \; (N) \; h(n) = [2 \quad 7 \quad 4 \quad -4]$

$x_3(n) \; (N) \; h(n) = [3 \quad 5 \quad -1 \quad 2]$

$x_4(n) \; (N) \; h(n) = [1 \quad 4 \quad 3 \quad -2]$

$y_1(n)$ $\boxed{1}\boxed{4}\boxed{3}\boxed{-2}$

+

$y_2(n)$ $\boxed{2}\boxed{7}\boxed{4}\boxed{-4}$

+

$\boxed{3}\boxed{5}\boxed{-1}\boxed{2}$

+

$\boxed{1}\boxed{4}\boxed{3}\boxed{-}$

$y(n) = \{1, 4, 3, 0, 7, 4, -1, 5, \ldots -2\}$

$x(n) = \{\underbrace{3, -1, 0}, \underbrace{1, 3, 2}, \underbrace{0, 1, 2}, \underbrace{1}\}$

$h(n) = \{1, 1, 1\}$. $M = 3$, $M - 1 = 2$

3	-1	0	0	0

1	3	2	0	0

0	1	2	0	0

1	0	0	0	0

$x_1(n)\ N\ h(n) = [3, 2, 2, -1, 0]$

$x_2(n)\ N\ h(n) = [1, 4, 6, 5, 2]$

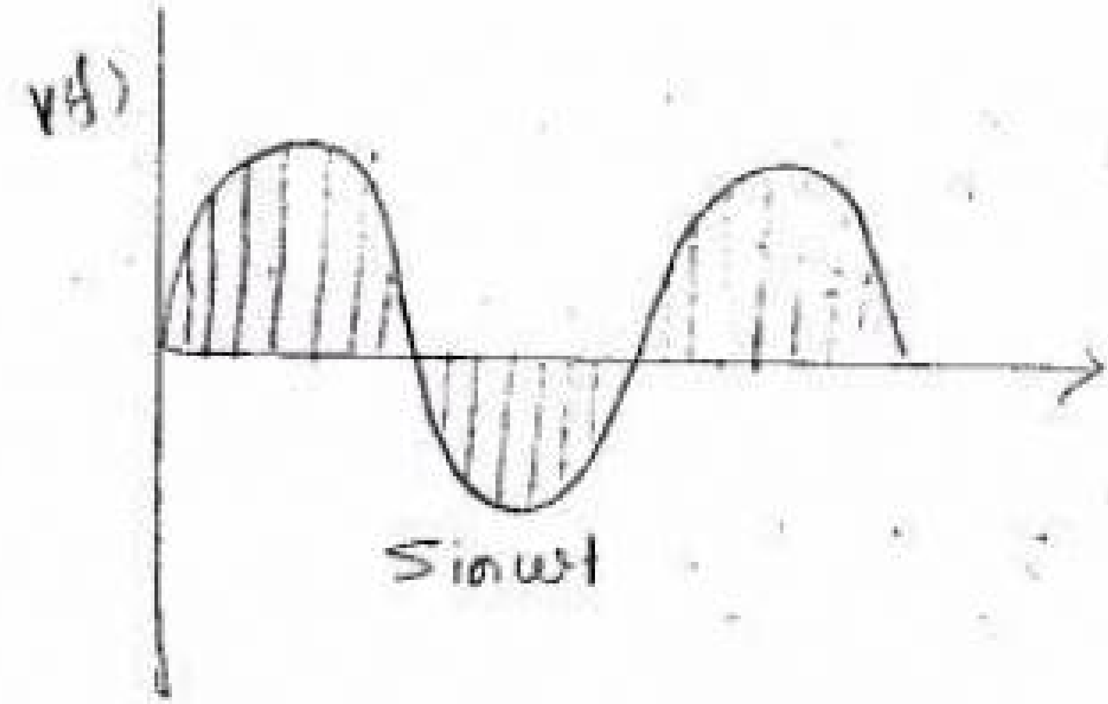
$x_3(n)\ N\ h(n) = [0, 1, 3, 3, 2]$

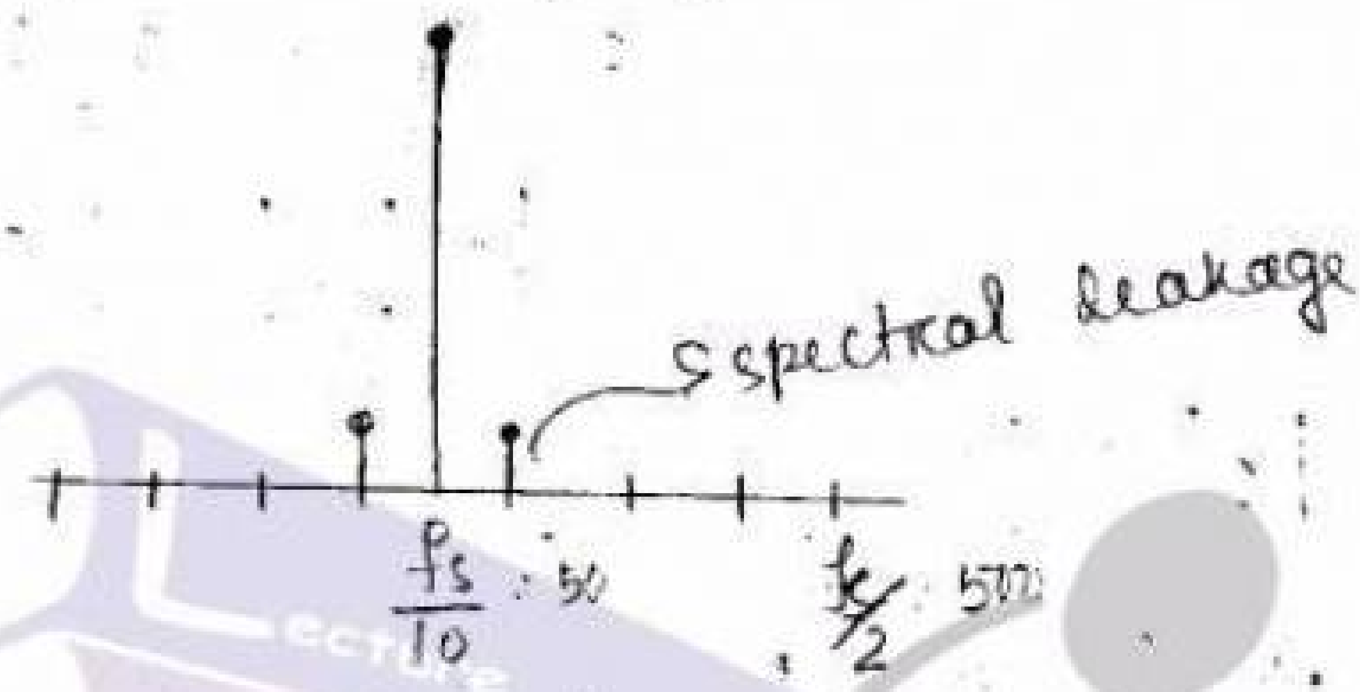
$x_4(n)\ N\ h(n) = [1, 1, 1, 0, 0]$

$\{\cancel{3, 2, 2, -1, 1, 4, 6, 5}$

$y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 3, 4, 3, 1$

Frequency Analysis of ~~Sing~~ Signals using DFT

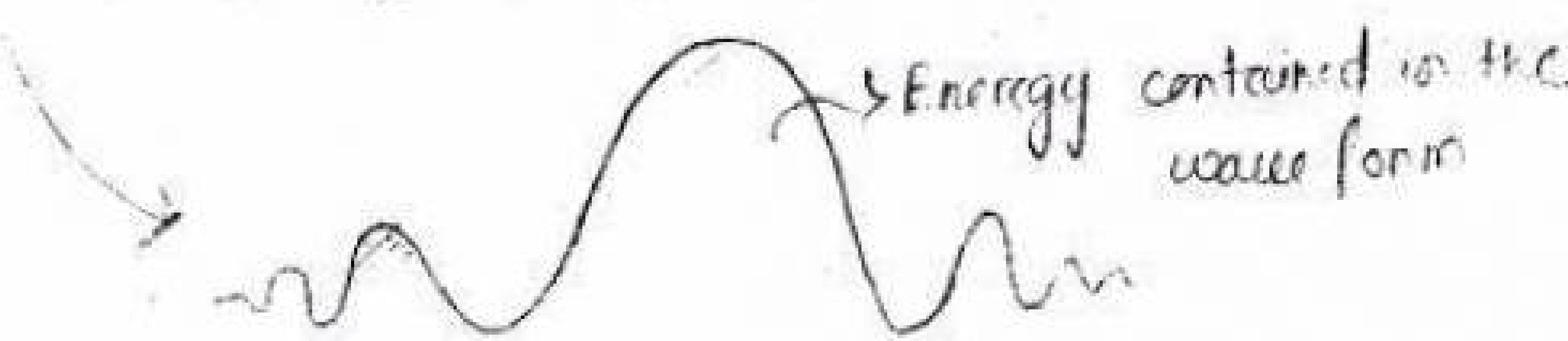




$$\downarrow F_0 \quad \downarrow F_s$$

$$\hat{x}(n) = x(n) \times \omega(n)$$

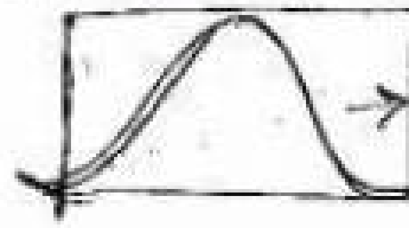
$$\hat{X}(\omega) = \frac{1}{2}\left[W(\omega - \omega_0) + W(\omega + \omega_0) \right].$$



Energy doesn't stick to ~~the~~ one spectrum it spreads ~~to~~ throughout the whole spectrum.

→ spectral ~~lin~~ ~~likage~~ leakage (Imp)

→ Spectral leakage can be minimised by using a gaussian window i.e



→ gaussian function

Hanning Window :-

$$w(n) = \begin{cases} \frac{1}{2}\left(1 - \cos\frac{2\pi}{L-1} n\right), & 0 < n \leq L-1 \\ 0 & \text{else} \end{cases}$$

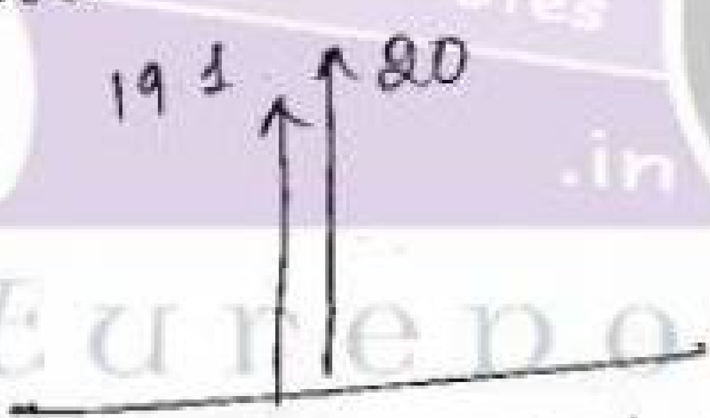
$$\omega_1 - \omega_2 > \frac{2\pi}{L} \leftarrow \text{Length of the window}$$

frequency components of the spectrum

→ It is very useful in FIR filters.

Need of spectral characteristics :(for frequency Resolution)

→ In industries, if ~~one~~ two motors are rotating & if in one motor, ~~there~~ if the nut or bolt is loosed, then vibration occurs
↓
the frequency changes from 20 to 19.1 Hz, there is a big difference



which can be eliminated ~~from~~ by frequency resolution.

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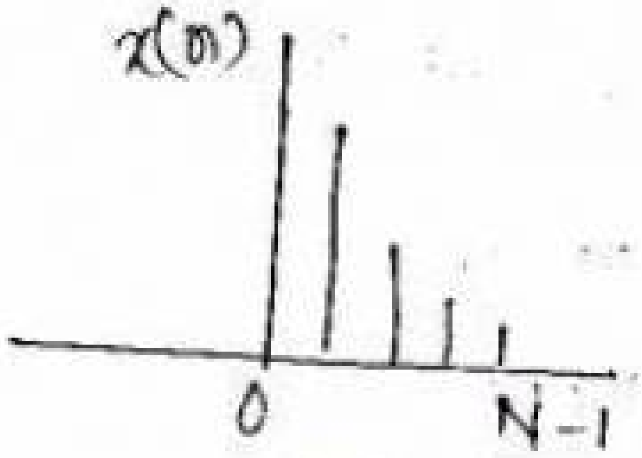
DISCRETE COSINE TRANSFORM (DCT)

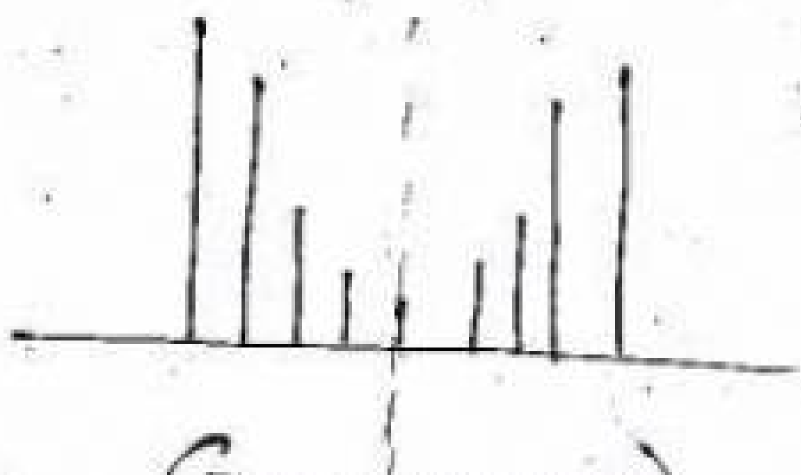
$x(n)$ = sequence

$x(n) = x(N-n)$ (integer to integer transform) – CCD - charge coupled device

real & even data

$x(N-k) = x(k)$ → The sequence is even sequence





(Even sequence)
(Mirror image)

Let a sequence is $x(n)$

Let even extension of $x(n)$ be $s(n)$

$$s(n) = \begin{cases} x(n), & 0 \le n \le N-1 \\ x(2N-n-1), & N \le n \le 2N-1 \end{cases}$$

Lets evaluate the 2N point DFT of $s(n)$

$$S(k) = \sum_{n=0}^{2N-1} s(n) e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{2N-1} s(n)\, W_{2N}^{kn}$$

$$S(k) = \sum_{n=0}^{N-1} x(n)\, W_{2N}^{kn} + \sum_{n=N}^{2N-1} x(2N-n-1)\, W_{2N}^{nk}$$

Let $m = 2N-1-n \Rightarrow n = 2N-1-m$

$$= \sum_{n=0}^{N-1} x(n)\, W_{2N}^{kn} + \sum_{m=N-1}^{0} x(m)\, W_{2N}^{(2N-1-m)k}$$

$$= \sum_{n=0}^{N-1} x(n)\, W_{2N}^{kn} + \sum_{m=N-1}^{0} x(m)\, W_{2N}^{2Nk} \cdot W_{2N}^{-k}\, W_{2N}^{-mk}$$

$$= \sum_{n=0}^{N-1} x(n)\, \omega_{2N}^{kn} + \sum_{m=N-1}^{0} x(m) \times 1 \times \omega_{2N}^{-k}\, \omega_{2N}^{-mk}$$

By factorising $W_{2N}^{-k/2}$.

$$= W_{2N}^{-k/2} \left[\sum_{n=0}^{N-1} x(n)\, W_{2N}^{kn} \cdot \omega_{2N}^{k/2} + \sum_{m=N-1}^{0} x(m)\, \omega_{2N}^{-k}\, W_{2N}^{-mk}\, \omega_{2N}^{k/2} \right]$$

$$= W_{2N}^{-k/2} \left[\sum_{n=0}^{N-1} x(n)\, W_{2N}^{kn} \cdot \omega_{2N}^{k/2} + \sum_{n=0}^{N-1} x(n)\, W_{2N}^{-k/2}\, \omega_{2N}^{-nk} \right]$$

$$= W_{2N}^{-k/2} \left[\sum_{n=0}^{N-1} x(n) \left[W_{2N}^{kn} \cdot W_{2N}^{k/2} + W_{2N}^{-k/2}\, W_{2N}^{-nk} \right] \right]$$

$$\left[\because W_N = e^{-j2\pi/N} = \cos 2\pi/N - j \sin 2\pi/N \right]$$

$$= \cancel{W_{2N}^{-k/2}} \Big\{$$

$$= W_{2N}^{-k/2} \left[\sum_{n=0}^{N-1} x(n) \cos \frac{\pi}{N} \left(n+\frac{1}{2}\right) k \right]$$

$$= W_{2N}^{-k/2}\; 2\, \text{Real} \left[W_{2N}^{k/2} \sum_{n=0}^{N-1} x(n)\, W_{2N}^{kn} \right]$$

If the forward DCT is defined as

$$x(2N-k) = x(k)$$

$$s(2N-1-n) = s(n)$$

$$v(k) = 2 \sum_{n=0}^{N-1} x(n) \cos \left[\frac{\pi}{N} \left(n - \frac{1}{2}\right) k \right]$$

$$0 \le k \le N-1$$

then $v(k) = W_{2N}^{k/2}\, s(k)$

$$\text{DCT of } x(n) = V(k) = 2\,\text{Real}\left[W_{2N}^{k/2} \sum_{n=0}^{N-1} x(n)\, W_{2N}^{kn} \right]$$

Inverse DCT

$$s(n) = \frac{1}{2N} \sum_{k=0}^{2N-1} S(k)\, W_{2N}^{-nk}$$

$S(2N-k) = S^{*}(k)$ as $S(k)$ is hermitian symmetric

$$s(n) = \frac{1}{2N} \sum_{k=0}^{N-1} S(k)\, W_{2N}^{-kn} + \frac{1}{2N} \sum_{k=N}^{2N-1} S(k)\, W_{2N}^{-kn}$$

$$= \frac{1}{2N} \sum_{k=0}^{N-1} S(k)\, W_{2N}^{-kn} + \frac{1}{2N} \sum_{m=1}^{N} S(2N-m)\, W_{2N}^{-(2N-m)n}$$

$$= \frac{1}{2N} S(0) + \frac{1}{2N} \sum_{k=1}^{N-1} S(k)\, W_{2N}^{-nk} + \frac{1}{2N} \sum_{k=1}^{N-1} S^{*}(k)\, W_{2N}^{kn}$$

$$\Rightarrow s(n) = \frac{1}{N}\,\text{Real}\left[\frac{S(0)}{2} + \sum_{k=1}^{N-1} S(k)\, W_{2N}^{-kn} \right]$$

$$S(k) = W_{2N}^{-k/2}\, V(k)$$

$$\boxed{x(n) = \frac{1}{N}\left\{ \frac{V(0)}{2} + \sum_{k=1}^{N-1} V(k) \cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)k\right] \right\}}$$

DCT as an orthogonal transform :-

Q) $x(n) = \{1, 2, 3, 4\}$

Find $V(k)$

Ans - $N = 4$

$$V(k) = 2 \sum_{n=0}^{N-1} x(n) \cos\left[\frac{\pi}{N}\left(n-\frac{1}{2}\right)k\right]$$

$$= 2 \sum_{n=0}^{3} x(n) \cos\left[\frac{\pi}{4}\left(n-\frac{1}{2}\right)k\right]$$

$$= 2\left\{ 1 \cos\left[\frac{\pi}{4} \times -\frac{1}{2} k\right] + 2 \cos\left(\frac{\pi}{4} \times \frac{1}{2}\right)k \right.$$
$$\left. + 3 \cos\left[\frac{\pi}{4}\left(\frac{3}{2}\right)k\right] + 4 \cos\left[\frac{\pi}{4} \times \left(\frac{5}{2}\right)k\right] \right.$$

$2-\frac{1}{2}$

3

$3-\frac{1}{2}=$

$$= 2\left\{ \cos\left(-\tfrac{\pi}{8}k\right) + 2\cos \tfrac{\pi}{8}k + 3\cos \tfrac{3\pi}{8}k + 4\cos\left[\tfrac{5\pi}{8}k\right]\right.$$

$$= 19.95$$

$$V(0) = 2\left[1\times\cos \tfrac{\pi}{4}\times -\tfrac{1}{2}\times 0 + 2\right.$$

$$= 2\left[1+2+3+4\right] = 20.$$

$$V(1) = 2\left[1\cos\tfrac{\pi}{4}\times -\tfrac{1}{2}\times 1 + 2\cos\tfrac{\pi}{4}\times\tfrac{1}{2}\times 1 + 3\cos\tfrac{\pi}{4}\times\tfrac{3}{2}\times 1 + 4\cos\tfrac{\pi}{4}\times\tfrac{5}{2}\right]$$

$$= 19.9.$$

DCT

$$C(k) = \alpha(k) \sum_{n=0}^{N-1} x(n) \cos\left[\frac{\pi(2n+1)k}{2N}\right]$$

IDCT

$$x(n) = \sum_{k=0}^{N-1} \alpha(k)\, C(k) \cos\left[\frac{\pi(2n+1)k}{2N}\right]$$

where $\alpha(0) = \sqrt{1/N}$. $\alpha(k) = \sqrt{2/N}$ $1 \le k \le N-1$

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$$

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k)\, W_N^{-nk}$$

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DCT as an orthogonal transform

* forward DCT.

$$V(k) = 2 \sum_{n=0}^{N-1} x(n) \cos\left[\frac{\pi}{N}\left(n+\tfrac{1}{2}\right)k\right], \quad 0 \le k \le N-1$$

* Inverse DCT

$$x(n) = \frac{1}{N}\left\{\frac{V(0)}{2} + \sum_{k=1}^{N-1} V(k) \cos\left[\frac{\pi}{N}\left(n+\tfrac{1}{2}\right)k\right]\right., \quad 0 \le n \le N-1$$

* Forward DFT

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$$

(Forward DFT & reverse DFT used in linear metering

* Inverse DFT

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x(k)\, W_N^{-nk}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(3) \end{bmatrix} = \begin{array}{c|c} & \begin{array}{cccc} k=0 & k=1 & k=2 & k=3 \end{array} \\ \begin{array}{c} x=0 \\ \\ x=2 \\ x=3 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \end{array} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

4x4 twiddle factor matrix.

* Forward DCT

$$V(k) = \alpha(k) \sum_{n=0}^{N-1} x(n) \cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)k\right]$$

$$0 \le k \le N-1$$

* Inverse DCT

$$x(n) = \sum_{k=0}^{N-1} \alpha(k)\, V(k) \cos\left[\frac{\pi}{N} \frac{(2n+1)k}{2N}\right]$$

$$\alpha(k) = \frac{1}{\sqrt{N}}$$

$$C_{kn} = \begin{cases} \frac{1}{\sqrt{N}}, & k=0 \ \& \ 0 \le n \le N-1 \\ \sqrt{\frac{2}{N}} \cos\frac{\pi(2n+1)k}{2N}, & 1 \le k \le N-1 \\ & \& \ 0 \le n \le N-1 \end{cases}$$

$$C = \begin{bmatrix} C_{01} & C_{02} & \cdots & \\ C_{11} & & & \\ \vdots & \vdots & & \\ C_{N-1,1} & & & \end{bmatrix}$$

$$\begin{bmatrix} V(1) \\ V(2) \\ \vdots \\ V(k) \end{bmatrix} = \begin{bmatrix} & & \\ & C & \\ & & \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Q) If $x(n) = \{1, 2, 3, 4\}$.

$$\underbrace{\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}}_{V_N} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} \\ \sqrt{\frac{2}{N}}\cos\frac{\pi}{2N} & \sqrt{\frac{2}{N}}\cos\frac{3\pi}{2N} & \sqrt{\frac{2}{N}}\cos\frac{5\pi}{2N} & \sqrt{\frac{2}{N}}\cos\frac{7\pi}{2N} \\ \sqrt{\frac{2}{N}}\cos\frac{\pi}{N} & \sqrt{\frac{2}{N}}\cos\frac{6\pi}{2N} & \sqrt{\frac{2}{N}}\cos\frac{10\pi}{2N} & \sqrt{\frac{2}{N}}\cos\frac{14\pi}{2N} \\ \sqrt{\frac{2}{N}}\cos\frac{3\pi}{2N} & \sqrt{\frac{2}{N}}\cos\frac{9\pi}{2N} & \sqrt{\frac{2}{N}}\cos\frac{15\pi}{2N} & \sqrt{\frac{2}{N}}\cos\frac{21\pi}{2N} \end{bmatrix}}_{C_N} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$\boxed{V_N = C_N X_N}$ DCT

$\boxed{X_N = C_N^{T} V_N}$ IDCT

FAST FOURIER TRANSFORM (FFT) Algorithm :-

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$(a+ib)(c+id)$

$\underbrace{ac}_{1} + \underbrace{iad}_{2} + \underbrace{ibc}_{3} - \underbrace{bd}_{4}$

N^2 complex multiplications

$N(N-1)$ complex additions

Computational complexity of DFT

Divide and Conquer Approach :-

$x(n) = \{ \underbrace{1, 2, 3, 4}_{16}, \underbrace{5, 6, 7, 8}_{16} \}$

8 nos

$8^2 = 64$ complex multiplication

Result : $\{p_1, p_2, p_3, p_4\}$ $\{q_1, q_2, q_3, q_4\}$

$q_1 \quad q_2 \quad q_3 \quad q_4$

2 point DFT

$16 + 16 + 16 = 48$

$8 = 4 \times 2$

$= 2 \times 2 \times 2$ ← Radix

$N = r^L$

↑ radix

Decimation in time (DIT) radix-2 FFT algorithm

Let a sequence be $x(n)$.

even index $f_1(m) = x(2m) \quad m = 0, 1, \ldots \left(\frac{N}{2} - 1\right)$

odd index $f_2(m) = x(2m+1) \quad m = 0, 1 \ldots \frac{N}{2} - 1$

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} \quad (\text{DFT of } x(n)).$$

$$= \underbrace{\sum_{m=0}^{N/2-1} x(2m) W_N^{2km}}_{\text{even}} + \underbrace{\sum_{m=0}^{N/2-1} x(2m+1) W_N^{k(2m+1)}}_{\text{odd}}$$

$m = 0 \ldots \frac{N}{2} - 1$

$$= \sum_{m=0}^{N/2-1} f_1(m) \left(W_N^2\right)^{km} + \sum_{m=0}^{N/2-1} f_2(m) \left(W_N^2\right)^{km} \cdot W_N^k$$

$$W_N^2 = e^{-j4\pi/N} = e^{-j2\pi/\frac{N}{2}}$$

$$= W_{N/2}^{1}$$

$$= \underbrace{\sum_{m=0}^{N/2-1} f_1(m) W_{N/2}^{km}} + \sum_{m=0}^{N/2-1} f_2(m) W_{N/2}^{km} \cdot W_N^k$$

$$= F_1(k) + F_2(k) \cdot W_N^k$$

where $f_1(m) \xleftrightarrow[N/2]{\text{DFT}} F_1(k)$

$f_2(m) \xleftrightarrow[N/2]{\text{DFT}} F_2(k)$

$$X(k) = F_1(k) + F_2(k) W_N^k \quad k = 0, \ldots \frac{N}{2} - 1$$

$$X\left(k + \frac{N}{2}\right) = F_1\left(k + \frac{N}{2}\right) + F_2\left(k + \frac{N}{2}\right) W_N^{k + N/2}$$

$$= F_1(k) - F_2(k)$$

Property of twiddle factor

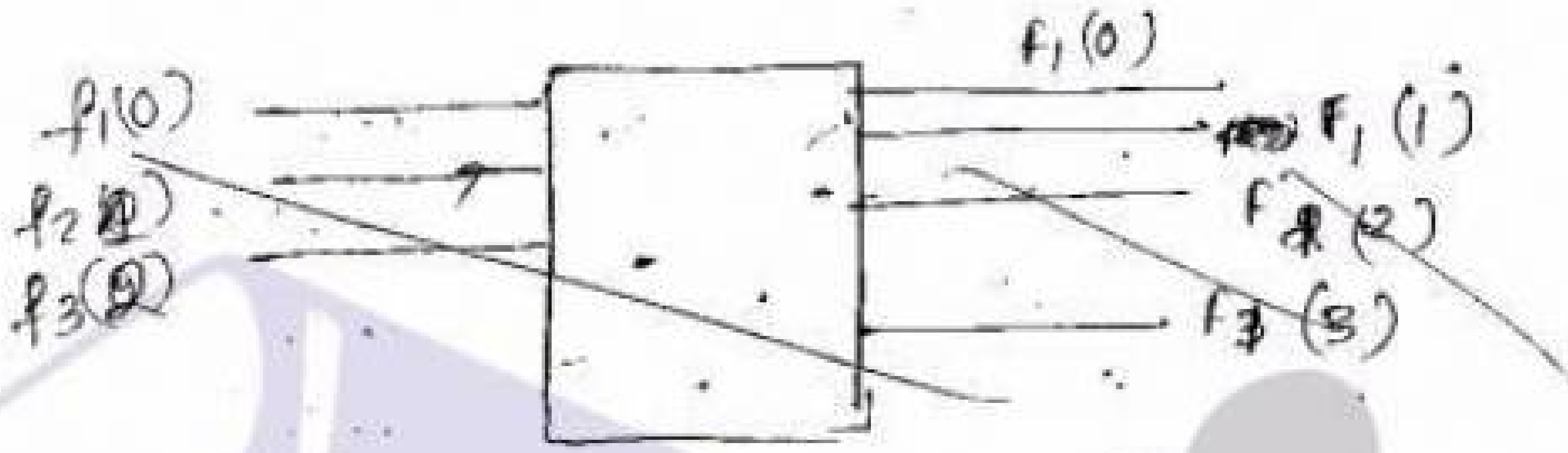
$$W_N^{k+N} = W_N^{k}$$

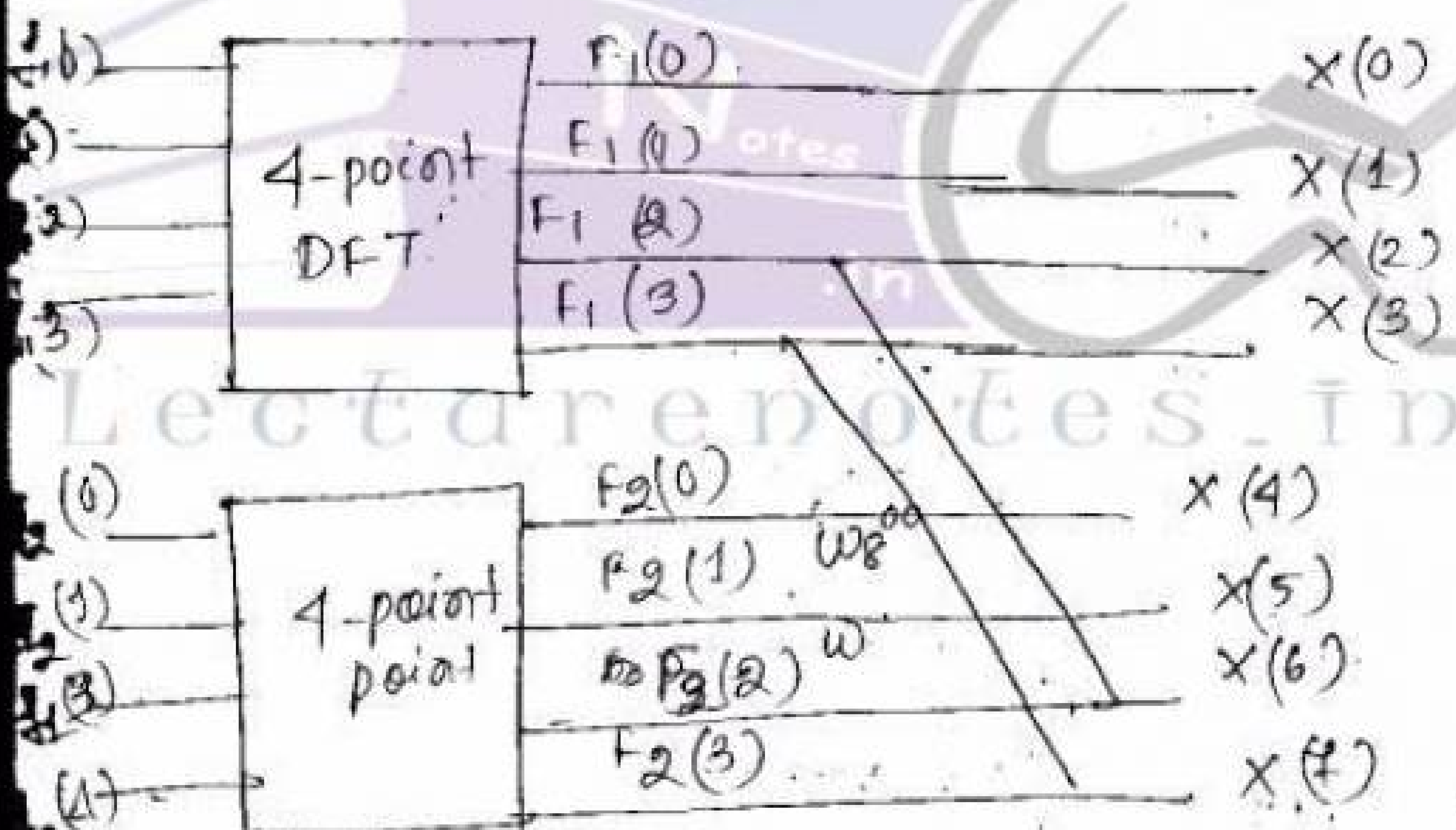
$$W_N^{k+N/2} = -W_N^{k} \text{ (symmetric)}.$$

$x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5) \ldots x(7)\}$

$f_1(n) = \{x(0), x(2), x(4), x(6)\}$ even indic

$f_2(m) = \{x(1), x(3), x(5)\ x(7)\}$.





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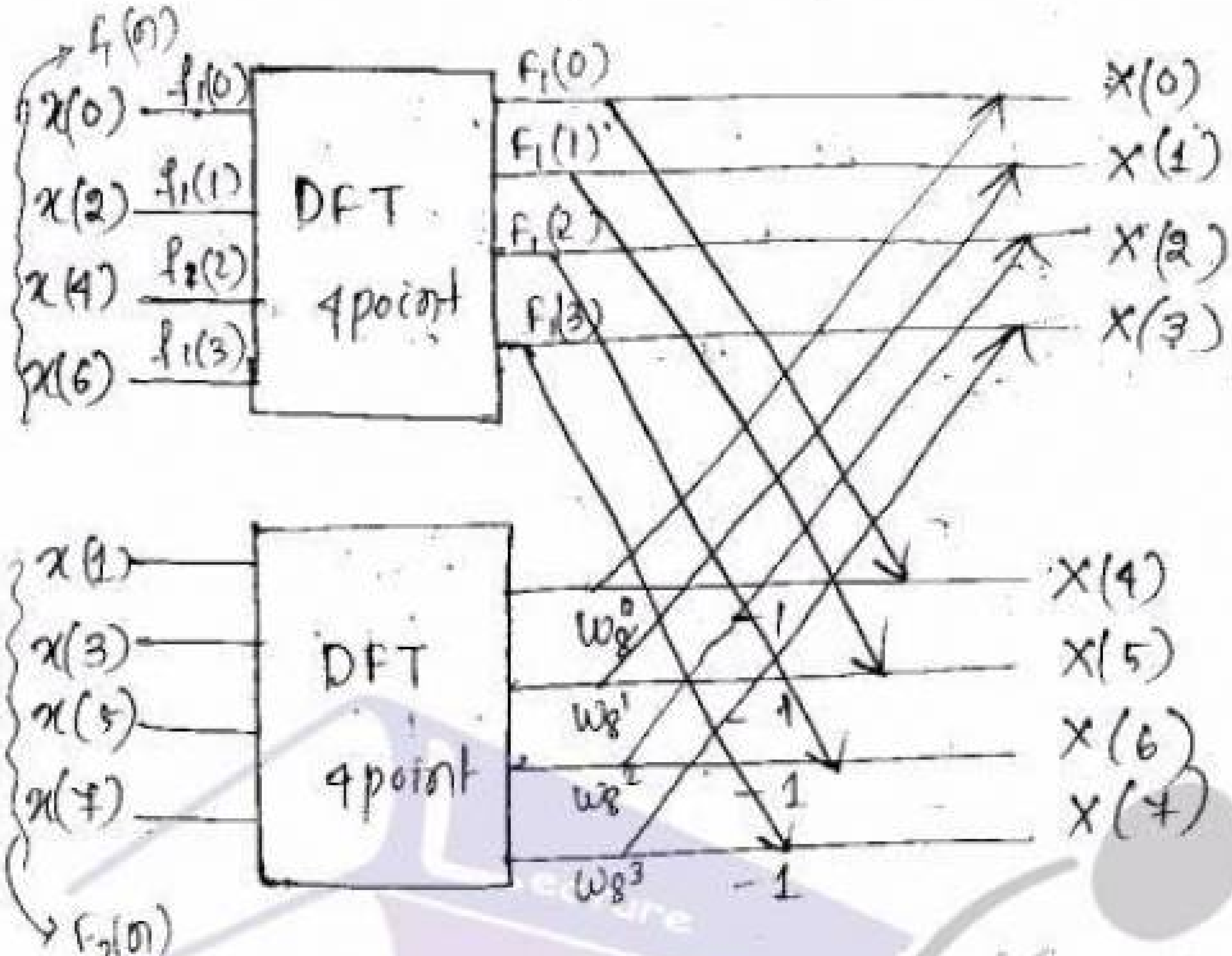
DIT FFT



$$f_1(n) = \begin{cases} V_{11}(m) = f_1(2m) & \text{even} \\ V_{12}(m) = f_1(2m+1) & \text{odd} \end{cases} \quad m = 0 \ldots \frac{N}{4}$$

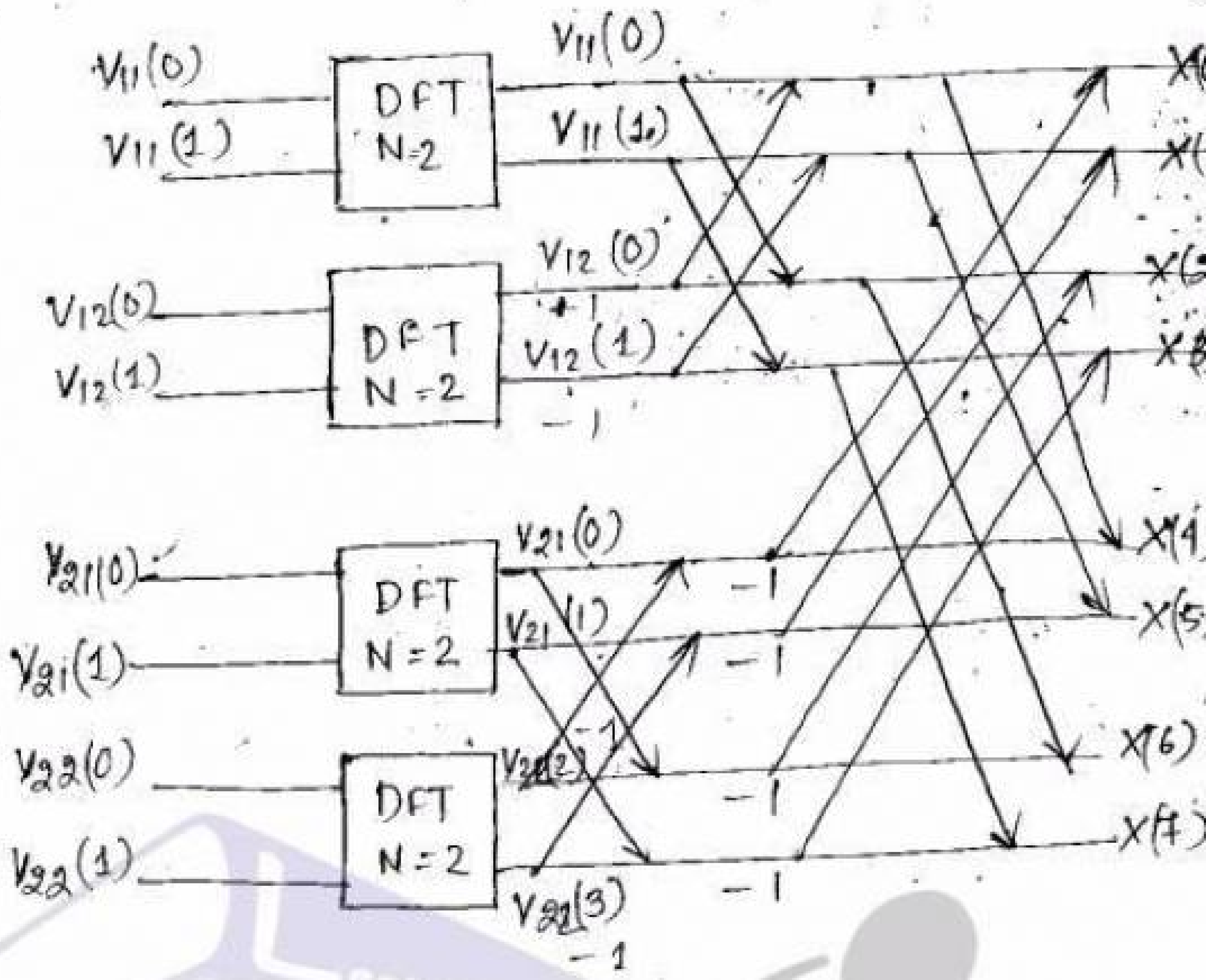
$$f_2(n) = \begin{cases} V_{21}(m) = f_2(2m) \\ V_{22}(m) = f_2(2m+1) \end{cases}$$

$$V_{11}(m) \xleftrightarrow[N/4]{DFT} V_{11}(K)$$

$$V_{12}(m) \xleftrightarrow[N/4]{DFT} V_{12}(K)$$

$$V_{21}(m) \xleftrightarrow[N/4]{DFT} V_{21}(K)$$

$$V_{22}(m) \xleftrightarrow[N/4]{DFT} V_{22}(K)$$



$K = 0, 1, \ldots \frac{N}{4} - 1$

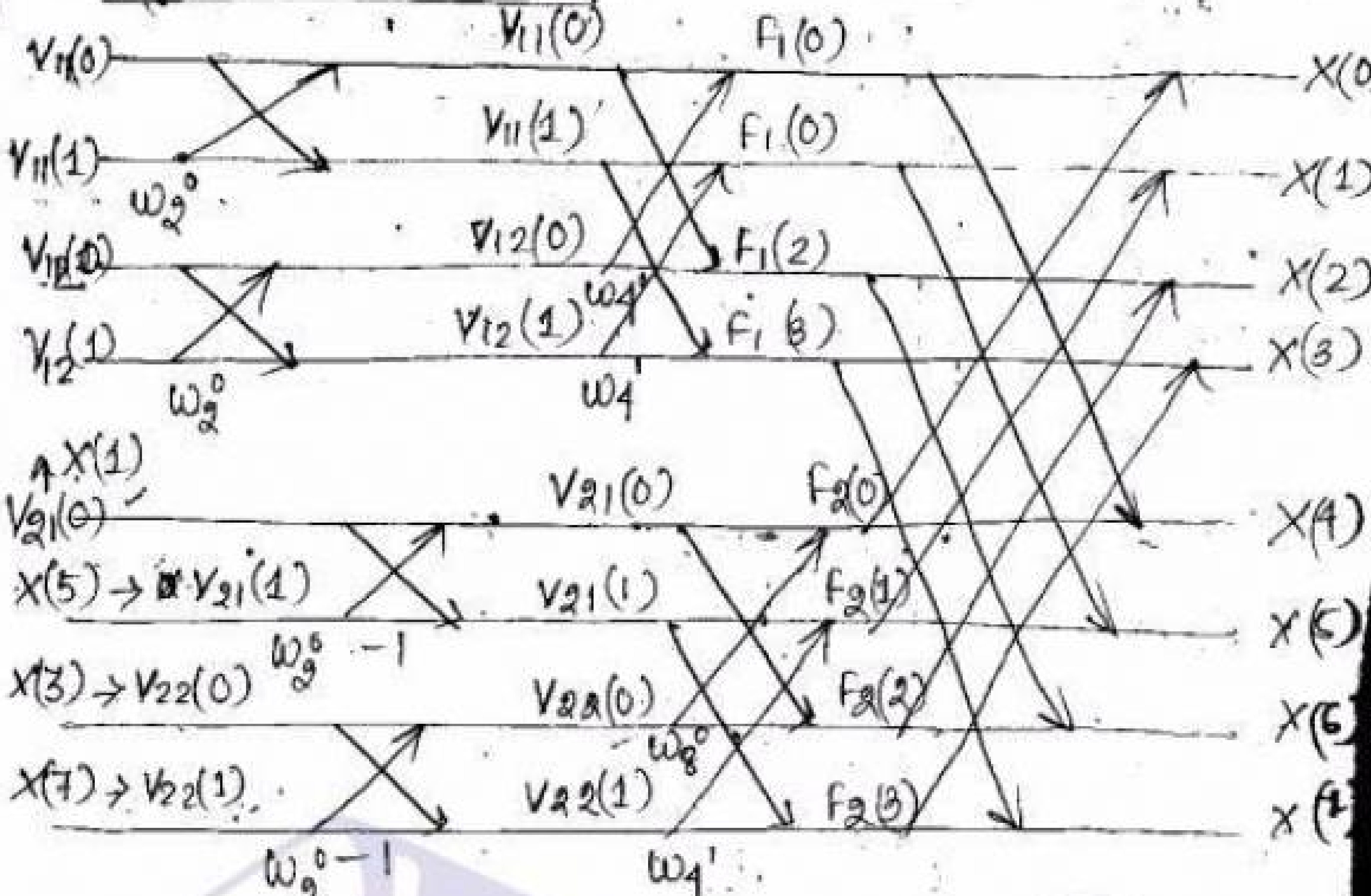
$F_1(K) = V_{11}(K) + W_{N/4}^{K} V_{12}(K)$

$F_1(K + N/2) = V_{11}(K) - W_{N/4}^{K} V_{12}(K)$

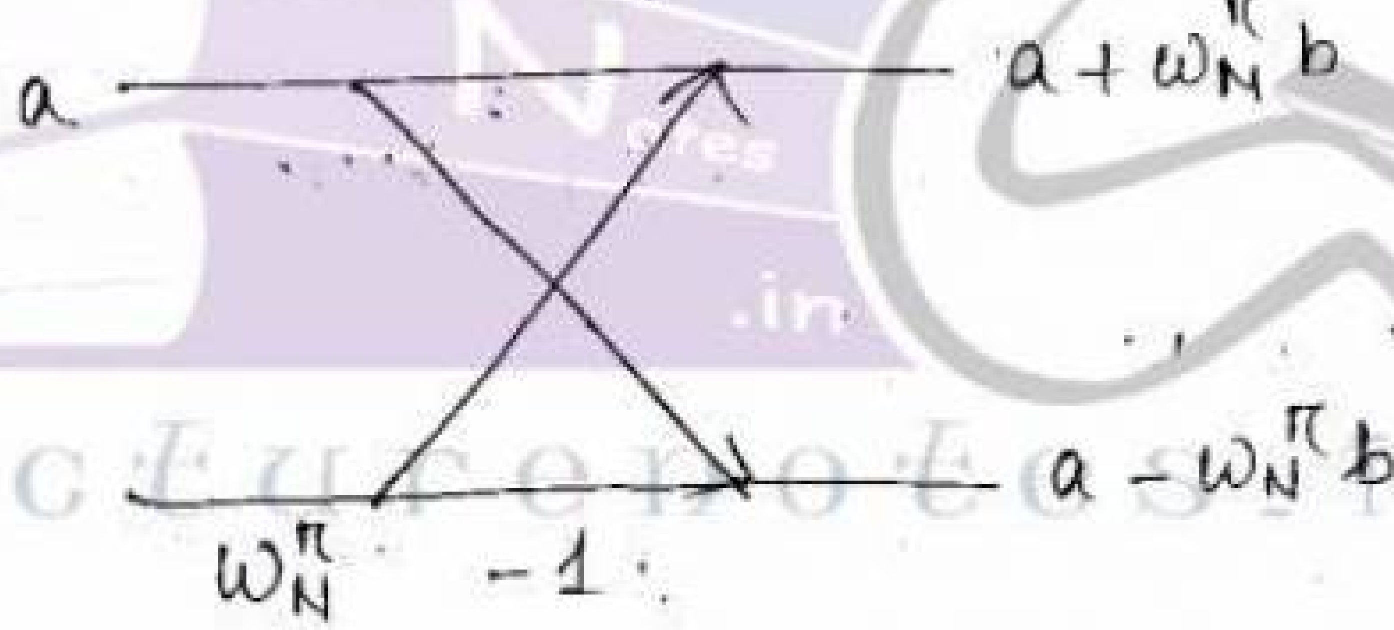
$F_2(K) = V_{21}(K) + W_{N/4}^{K} V_{22}(K)$

$F_2(K + N/2) = V_{21}(K) - W_{N/4}^{K} V_{22}(K)$

Signal Flow Graph :- (Implemented to)



Butterfly Structure



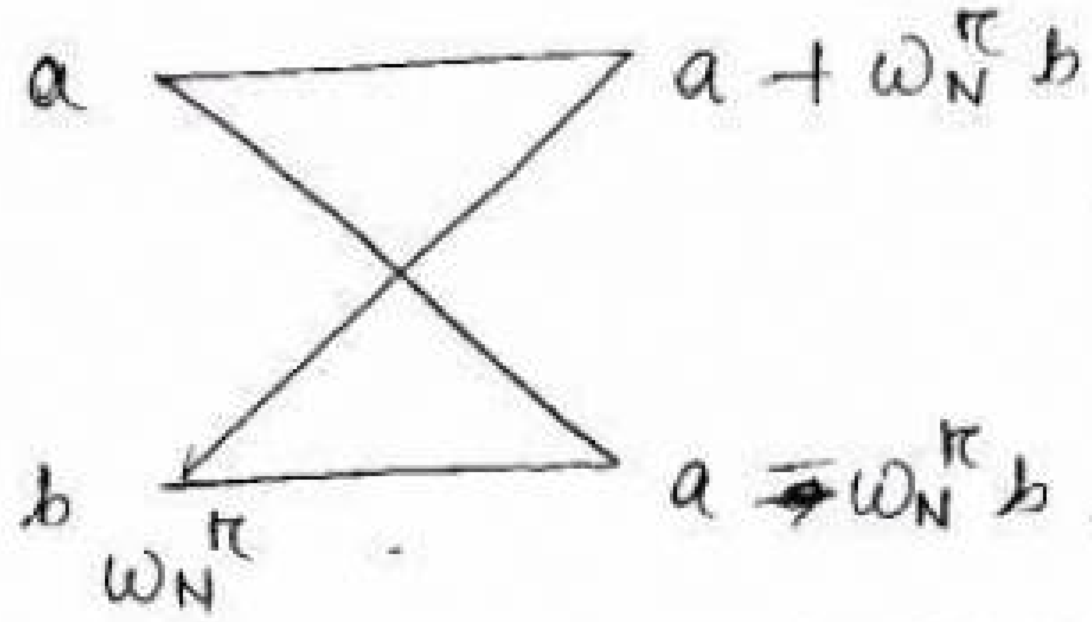
Bit Reversal :-

K		n	
0 = 000	⟷	000	— 0
1 = 001	⟷	100	— 4
2 = 010	⟷	010	— 2
3 = 011	⟷	110	— 6
4 = 100	⟷	001	— 1
5 = 101	⟷	101	— 5
6 = 110	⟷	011	— 3

$\underline{K}$ $\underline{M}$

$7 = 111 \longleftrightarrow 111 \text{ — } 7$

Computational complexity :-



$1 \longleftarrow$ complex Multiplication ..
$2 \longleftarrow$ Complex addition .

$N/2$ butterflies in each stage .

$N = 2^5$ Total complex multiplication

$S = \log_2 N$ $= N/2 \log_2 N$.

$2 \times N/2 \log_2 N$ = Total complex additio

$= N \log_2 N$

$N = 8$.

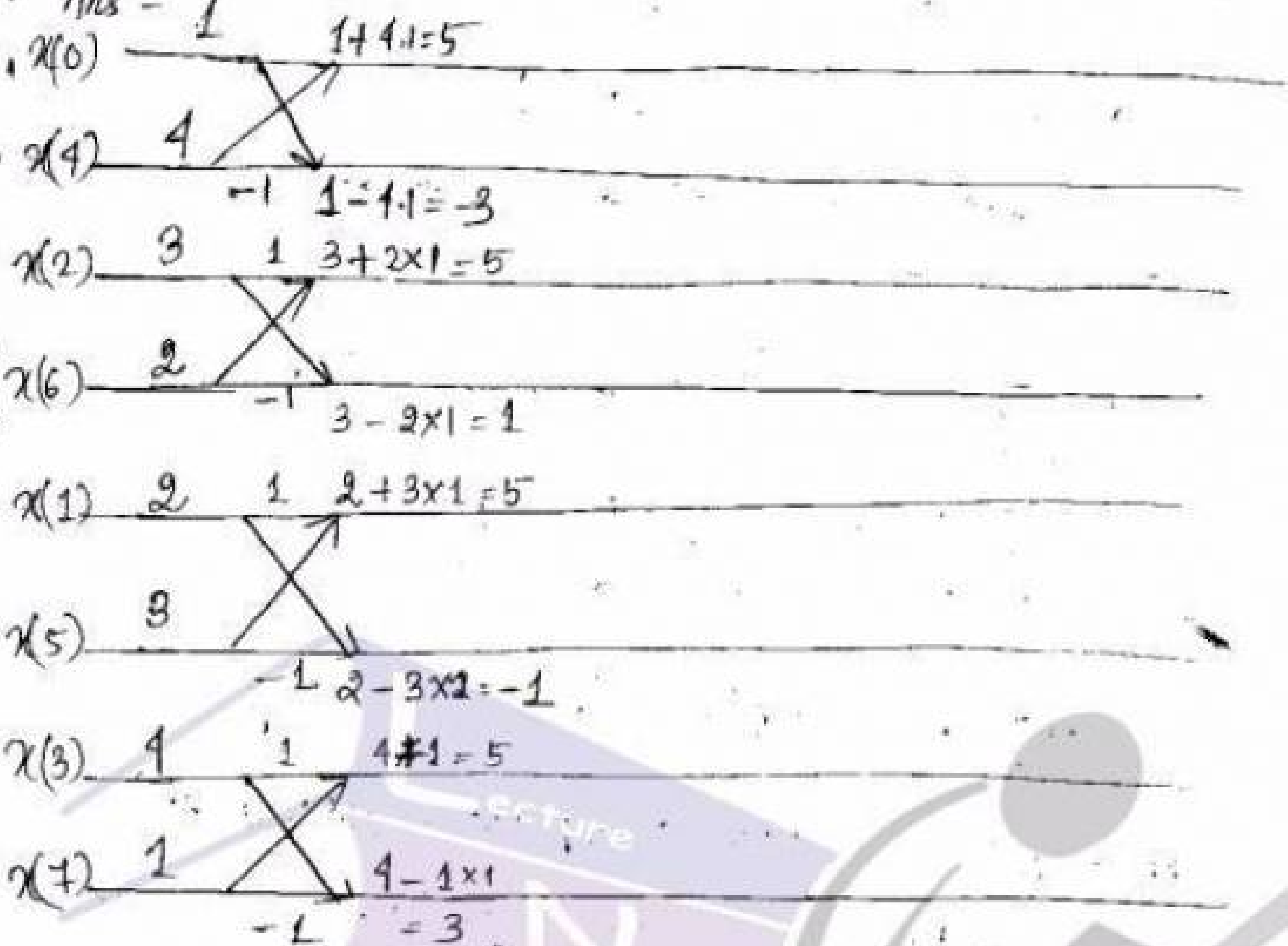
$\left.\begin{array}{l} 64 \longrightarrow 12 \\ 56 \longrightarrow 24 \end{array}\right\}$ complex addition

(DFT)   (FFT)

Q) $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$

Find DFT through radix-2 DIT.

Ans -



$$W_N = e^{-j2\pi/N}$$

$$W_2^0 = e^{-j2\pi/2 \times 0} = 1$$

$$W_8^1 = e^{-j2\pi/8 \times 1} = e^{-j\pi/4 \times 1}$$

T-21.03.11

Internal Question

$$X(k) = \sum_{n=0}^{N-1} \delta(n-n_0)\, e^{-j2\pi nk/N}$$

$$= e^{-j2\pi n_0 k/N}$$

Decimation in frequency :-

DIF FFT :- Radix-2

$x(n) \longleftarrow$ N-point (always a multiple of 2)

~~for~~ ~~Dir~~ DFT of $x(n)$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N/2-1} x(n)\, W_N^{kn} + \sum_{n=N/2}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + \sum_{n=0}^{N/2-1} x(n+N/2)\, W_N^{k(n+N/2)}$$

$$= \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + W_N^{kN/2} \sum_{n=0}^{N/2-1} x(n+N/2)\, W_N^{kn}$$

$$W_N^{kN/2} = e^{-j2\pi kN/2N} = (e^{-j\pi})^k = (-1)^k$$

$$= \sum_{n=0}^{N/2-1} x(n)\, W_N^{kn} + (-1)^k \sum_{n=0}^{N/2-1} x(n+N/2)\, W_N^{kn}$$

Lets divide $X(k)$ into even and odd indexed sequence

$X(2m)$ or $X(2k)$

$$= \sum_{n=0}^{N/2-1} x(n)\, W_N^{2kn} + \sum_{n=0}^{N/2-1} x(n+N/2)\, W_N^{2kn}$$

or $$= \sum_{n=0}^{N/2-1} x(n)\, W_{N/2}^{kn} + \sum_{n=0}^{N/2-1} x(n+N/2)\, W_{N/2}^{kn}$$

$$X(2K+1) = \sum_{n=0}^{N/2-1} x(n)\, W_N^{(2k+1)n} - \sum_{n=0}^{N/2-1} x(n+N/2)\, W_N$$

$$= \sum_{n=0}^{N/2-1} x(n)\, W_N^{2kn} \cdot W_N^{+n} - \sum_{n=0}^{N/2-1} x(n+N/2)\, W_N^{2kn} \cdot W_N$$

$$\cancel{= \sum_{n=0}^{N/2-1} x(n)\, W_{N/2}^{kn} - \sum_{n=0}^{N/2-1} x(x+N/2)\, W_{N/2}^{kn}}$$

$$= \left[\sum_{n=0}^{N/2-1} x(n)\, W_{N/2}^{kn}\, W_N^{+n} - \sum x(n+N/2)\, W_{N/2}^{kn}\, W_N^{+n} \right.$$

$G(k)$

$$X(2k) = \sum_{n=0}^{N/2-1} \left[x(n) + x(n+N/2) \right] W_{N/2}^{kn}$$

$G_2(k)$

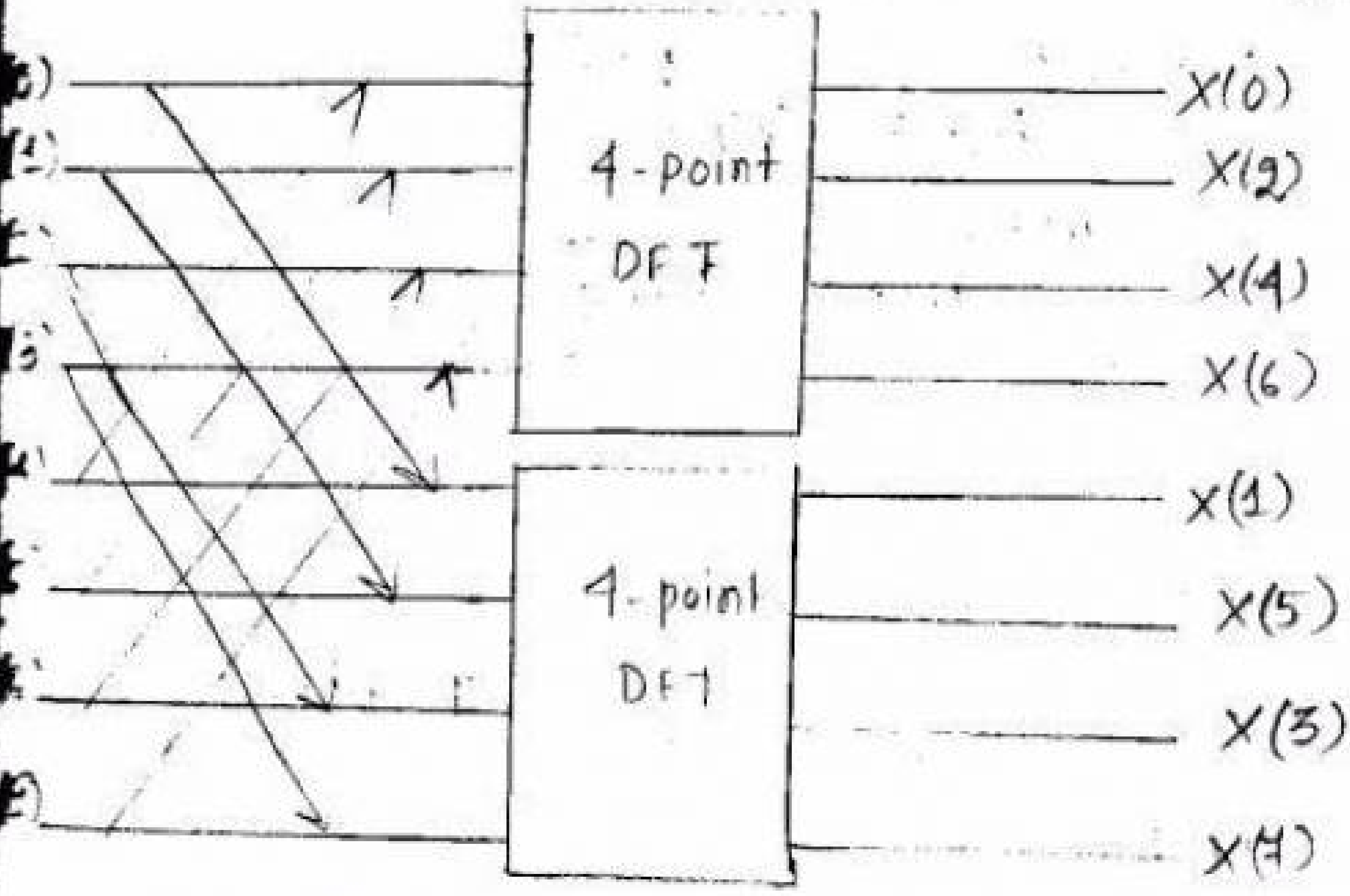
$$= X(2K+1) = \sum_{n=0}^{N/2-1} \left[x(n) - x(n+N/2) \right] W_N^{n} \cdot W_{N/2}^{kn}$$

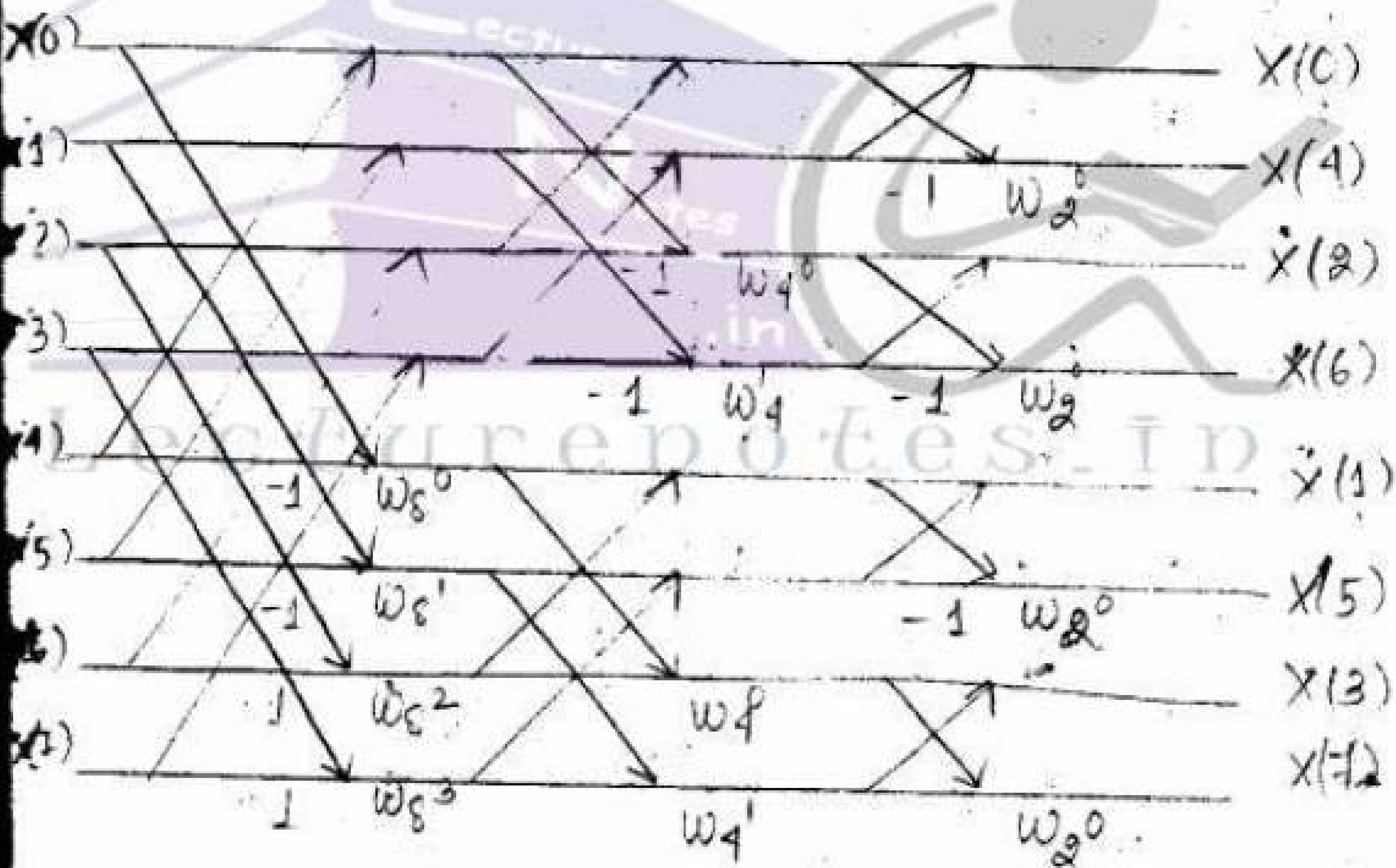
Let $g_1(n) = x(n) + x(n+N/2)$

$g_2(n) = \left[x(n) - x(n+N/2) \right] W_N^{n}$

$$X(2k) = \sum_{n=0}^{N/2-1} g_1(n)\, W_{N/2}^{kn}$$

$$X(2K+1) = \sum_{n=0}^{N/2-1} g_2(n)\, W_{N/2}^{kn}$$


4-point DFT
4-point DFT
X(0)
X(2)
X(4)
X(6)
X(1)
X(5)
X(3)
X(7)


X(0)
X(4)
X(2)
X(6)
X(1)
X(5)
X(3)
-1
W_8^0
W_8^1
W_8^2
W_8^3
W_4^0
W_4^1
W_2^0

Second stage decimation :-

$$G_1(2k) = \sum_{n=0}^{N/4-1} h_{11}(n)\, W_{N/4}^{kn}$$

$$G_1(2k+1) = \sum_{n=0}^{N/4-1} h_{12}(n)\, W_{N/4}^{kn}$$

where $h_{11}(n) = g_1(n) + g_1(n+N/4)$

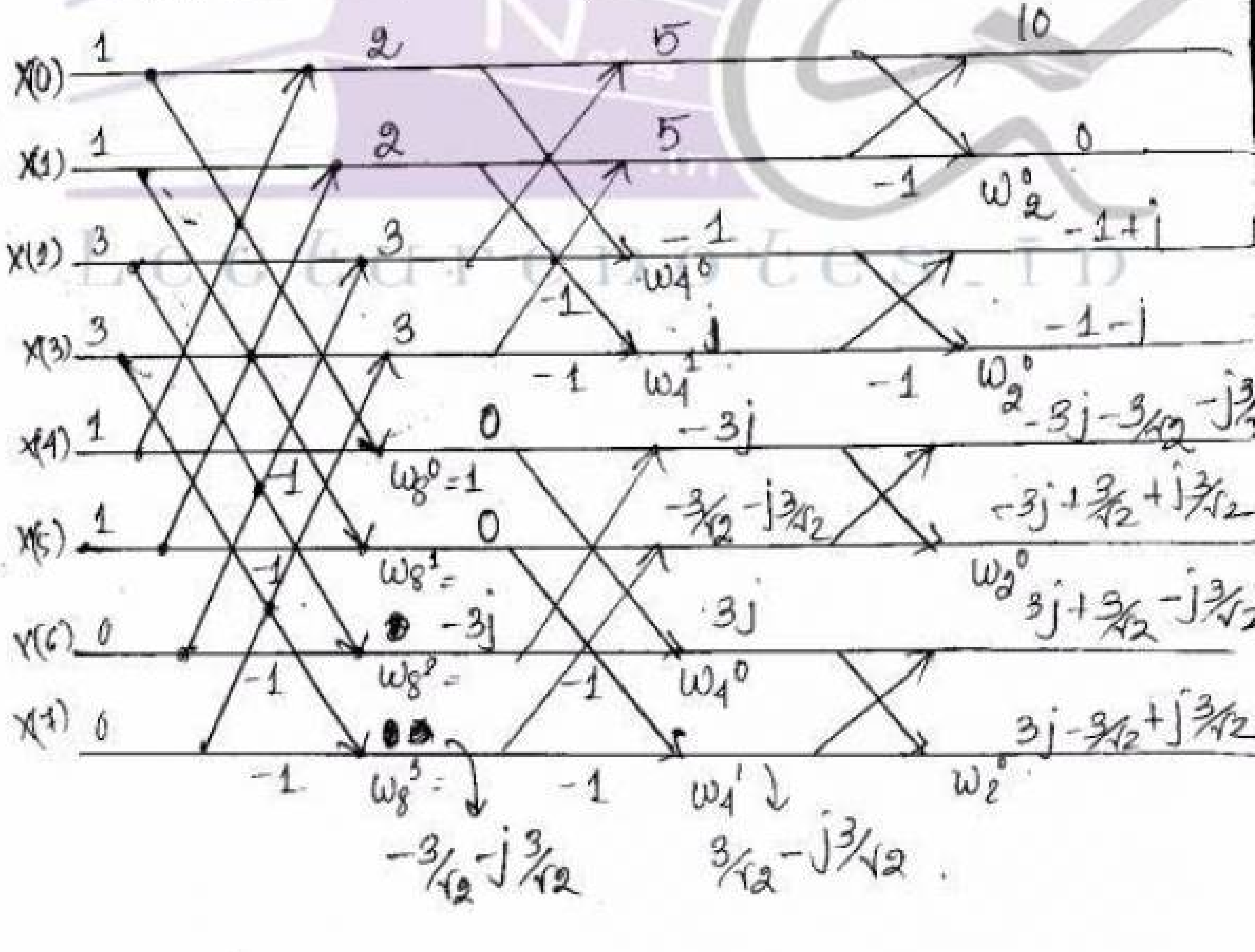
$$h_{12}(n) = \{g_1(n) - g_1(n+N/4)\}\, W_{N/2}^{n}$$

Computational complexity of DIF FFT is same as that of DIT-FFT.

DT-25.03.11

Q) $x(n) = \{1, 1, 3, 3, 1, 1, 0, 0\}$.

Find its DFT through DIF-FFT



$$W_8^1 = e^{-j\pi/4} = \cos \pi/4 - j \sin \pi/4 .$$

$$= \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^2 = \left(e^{-j\pi/2}\right) = -j$$

$$W_{\cancel{8}}^8 = e^{-j\,\pi \times 3/4} = -1/\sqrt{2} - j\,1/\sqrt{2}$$

$\cancel{W_8^3 - 3 - 0}$

$$1 - 1\cdot\left(1/\sqrt{2} - j\,1/\sqrt{2}\right)$$

$$= 1 - 1/\sqrt{2} + j\,1/\sqrt{2}$$

$$= 0.293 + j\,0.707 .$$

$$\begin{array}{r} 1.000 \\ 0.707 \\ \hline 0.293 \end{array}$$

Applications of FFT :-

(1) Efficient computation of DFT ~~or~~ of two real sequences

Let there be two real sequences

$$x_1(n) \;\&\; x_2(n) \quad 0 \le n < N.$$

$$x(n) = x_1(n) + j\, x_2(n)$$

$$X(k) = X_1(k) + j\, X_2(k)$$

$$x_1(n) = \frac{x(n) + x^*(n)}{2}$$

$$x_2(n) = \frac{x(n) - x^*(n)}{2j}$$

$$x(n) \xrightarrow{DFT} X(k)$$

$$X_1(k) = \frac{X(k) + X^*(N-k)}{2}$$

$$X_2(k) = \frac{X(k) - X^*(N-k)}{2j}$$

(3) Efficient computation of DFT of 2N point real sequences , $n = 0, 1 \ldots 2N-1$

DFT of $g(n) = G(k)$.

$$x_1(n) = g(2n) \quad n = 0, 1, \ldots N-1$$

$$x_2(n) = g(2n+1), \quad n = 0, 1 \ldots N-1$$

$$x(n) = x_1(n) + j\,x_2(n)$$

$$X(k) = X_1(k) + j\,X_2(k)$$

$$G(k) = \sum_{n=0}^{2N-1} g(n)\, W_N^{nk}$$

$$= \sum_{n=0}^{N-1} g(2n)\, W_N^{2nk} + \sum_{n=0}^{N-1} g(2n+1)\, W_N^{(2n+1)k}$$

$$= \sum_{n=0}^{N-1} x_1(n)\, W_N^{nk} + W_{2N}^{k} \sum_{n=0}^{N-1} x_2(n)\, W_N^{nk}$$

$$k = 0 \ldots N-1$$

$$\boxed{G(k) = X_1(k) + W_{2N}^{k} X_2(k)}$$

$$G(k+N) = X_1(k) - W_{2N}^{k} X_2(k)$$

for finding out FFT, we ~~we~~ use DIF.

" " " IFFT, we use DIT.

$$\text{IFFT}\,\{\underbrace{\underset{\uparrow}{X_1(k)}^{x_1(n)} \times H(k)^{h(n)}}_{\text{is in bit reversal order}}\}.$$

T-26.03.11

Application of FFT in filtering :-

Overlap save ~~and add method~~ method :-

- No. of complexities = $N \log_2 2N$.
- No. of complex multiplications = $N \log_2 N + N$.
- For each step, $N \log_2 2N$ complex multiplication
- No. of complex multiplications required if implemented through fft.

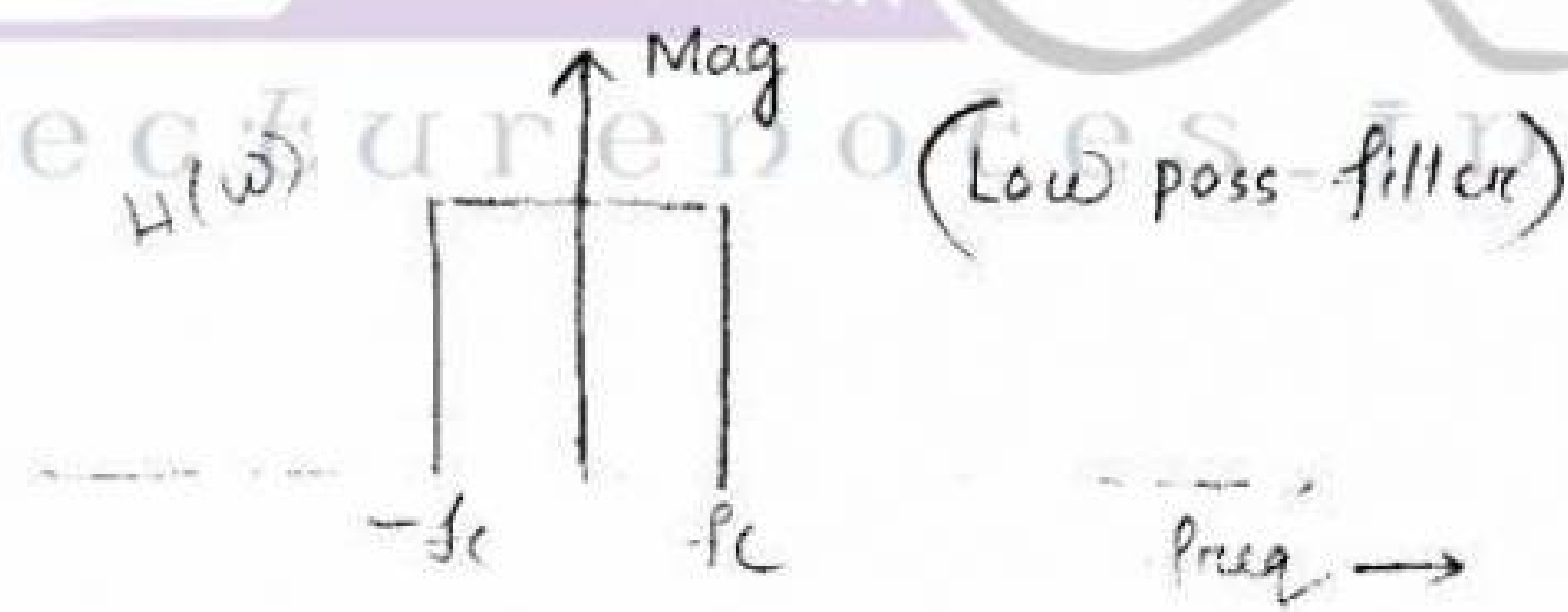
$$\boxed{\frac{N \log_2 2N}{L}}$$ → Overlap save method

$N = 50 \qquad L = 5$

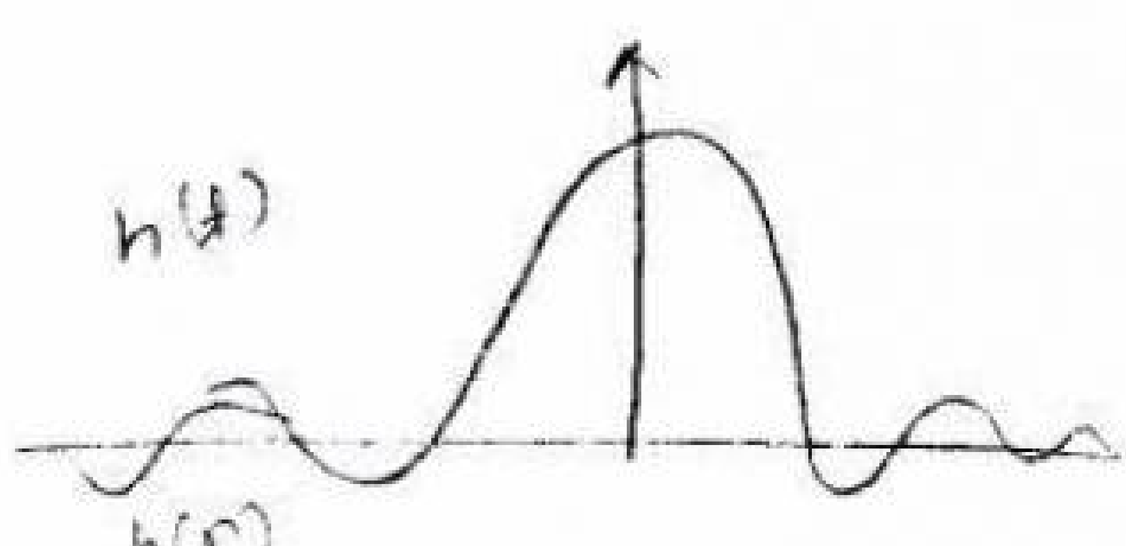
$N^2 \qquad \dfrac{N \log_2 N}{L}$

$250 \qquad = 60$

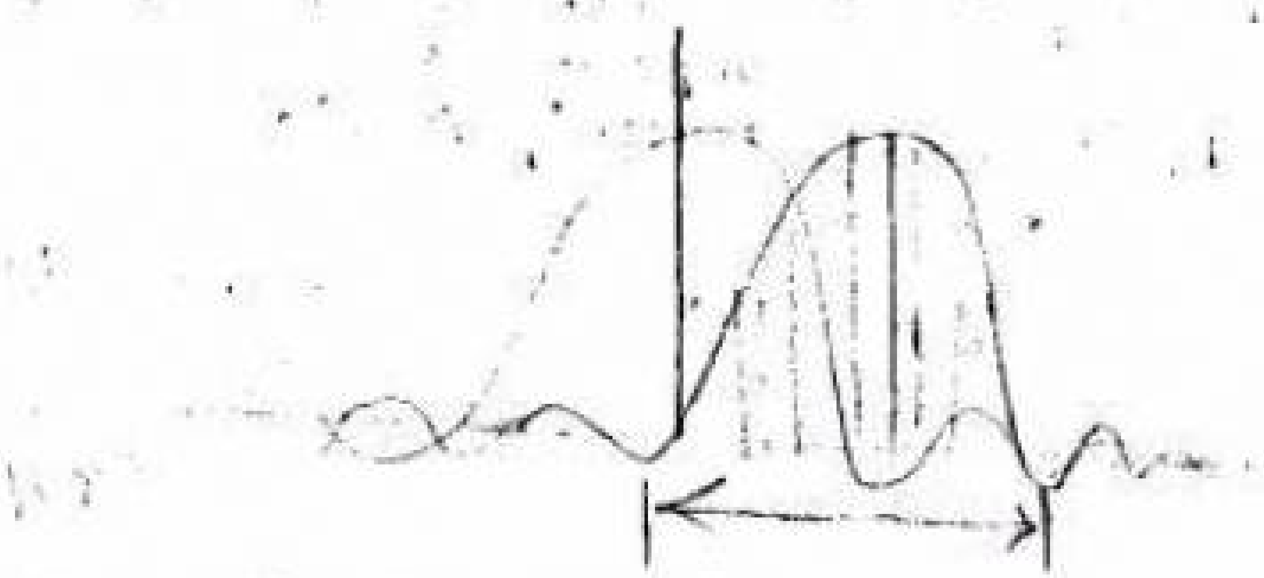
→

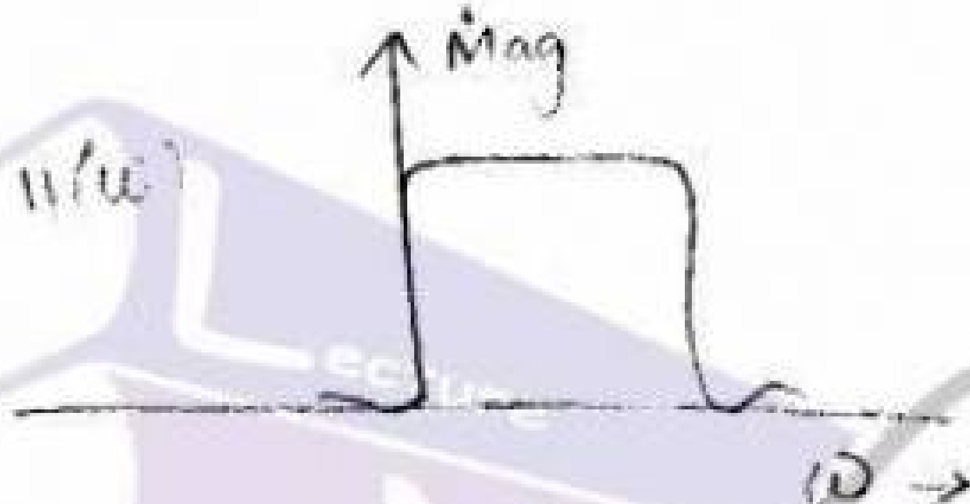


(Low pass filter)



(Non-causal)





T-04.04.11

FIR filter Design :-

FIR filters have

-> Linear phase (const phase delay and group delay)
-> Are stable (no poles) (Inherent stability)
-> Simpler design method
-> Efficient hardware implementation
-> Filter ~~stable~~ start up transients are less

A FIR filter has linear phase if

$$h(n) = \pm h(M-1-n)$$

Let the FIR filter has a freq. response

$$H(\omega) = \sum_{n=0}^{M-1} h(n)\, e^{-j\omega n}$$

Magnitude response :-

$$|H(\omega)| = \sqrt{Re\{H(\omega)\}^2 + Im\{H(\omega)\}^2}$$

Phase response :-

$$\theta(\omega) = \tan^{-1} \frac{Im\{H(\omega)\}}{Re\{H(\omega)\}}$$

Phase delay, $\tau_p = \dfrac{-\theta(\omega)}{\omega}$

Group delay, $\tau_g = \dfrac{-d\theta(\omega)}{d\omega}$

$$\tau_p = \frac{-\theta(\omega)}{\omega} = \tau = \text{constant}$$

$$\Rightarrow -\theta(\omega) = \tau\omega$$

$$\Rightarrow -\tan^{-1}\frac{Im\{H(\omega)\}}{Re\{H(\omega)\}} = \tau\omega$$

$$\Rightarrow \frac{Im\{H(\omega)\}}{Re\{H(\omega)\}} = -\tan(\tau\omega)$$

$$\Rightarrow \frac{-\sum_{n=0}^{M-1} h(n)\sin\omega n}{\sum_{n=0}^{M-1} h(n)\cos\omega n} = \frac{-\sin(\tau\omega)}{\cos(\tau\omega)}$$

$$\Rightarrow \sum_{n=0}^{M-1} h(n)\sin\omega n\cos\tau\omega - \sum_{n=0}^{M-1} h(n)\cos\omega n\,\sin\tau\omega)$$

$$\Rightarrow \sum_{n=0}^{M-1} h(n)\sin(n-\tau) = 0.$$

$$\tau = \frac{M-1}{2}, \quad h(n) = h(M-1-n).$$

[symmetric]

If we consider constant group delay.

$$h(n) = -h(M-1-n) \quad \text{(antisymmetric)}$$

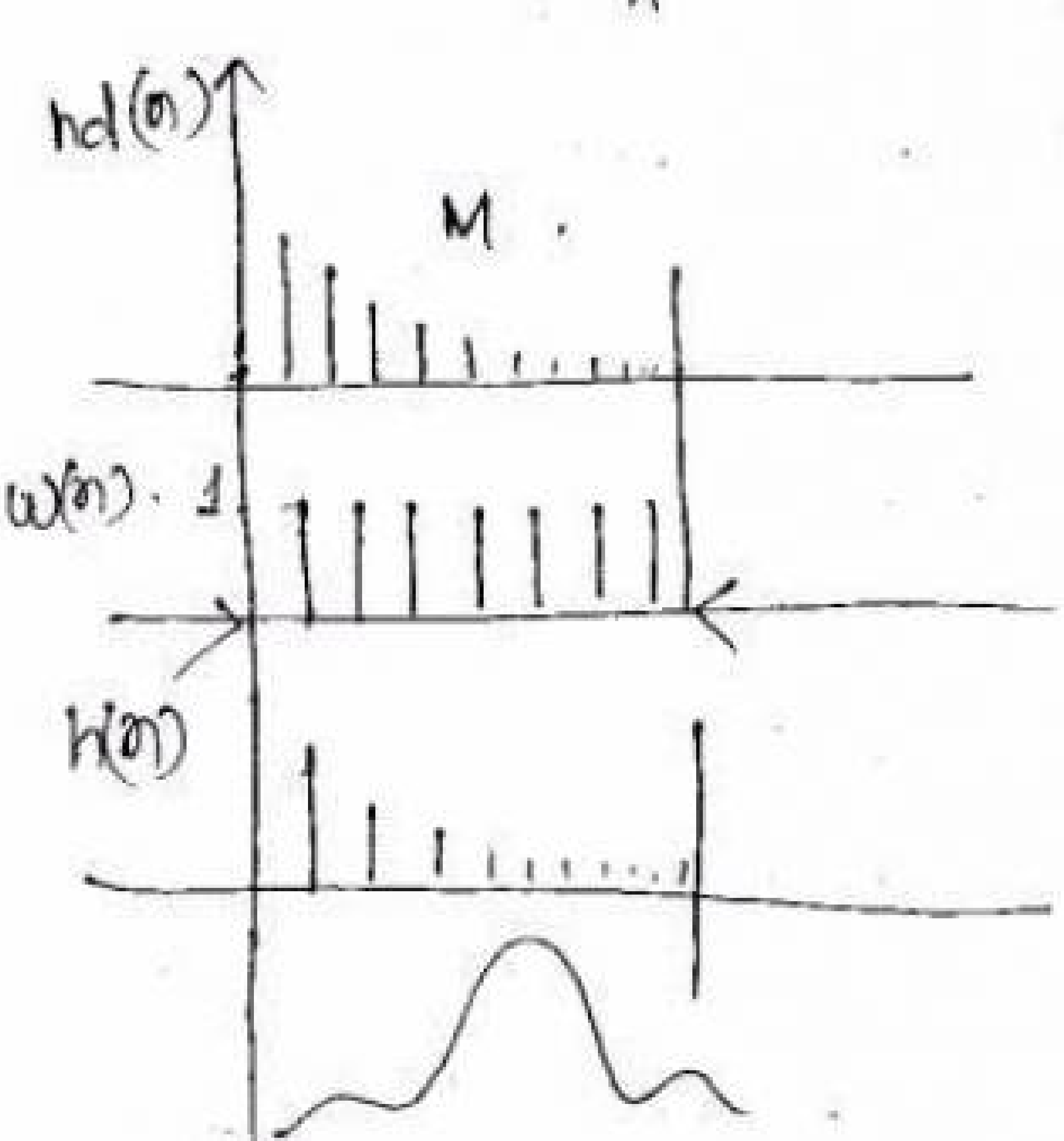
$$h(n) = \pm h(M-1-n).$$

Q) Design of FIR filters using windows.

Ans - Desired frequency response $= H_d(\omega)$

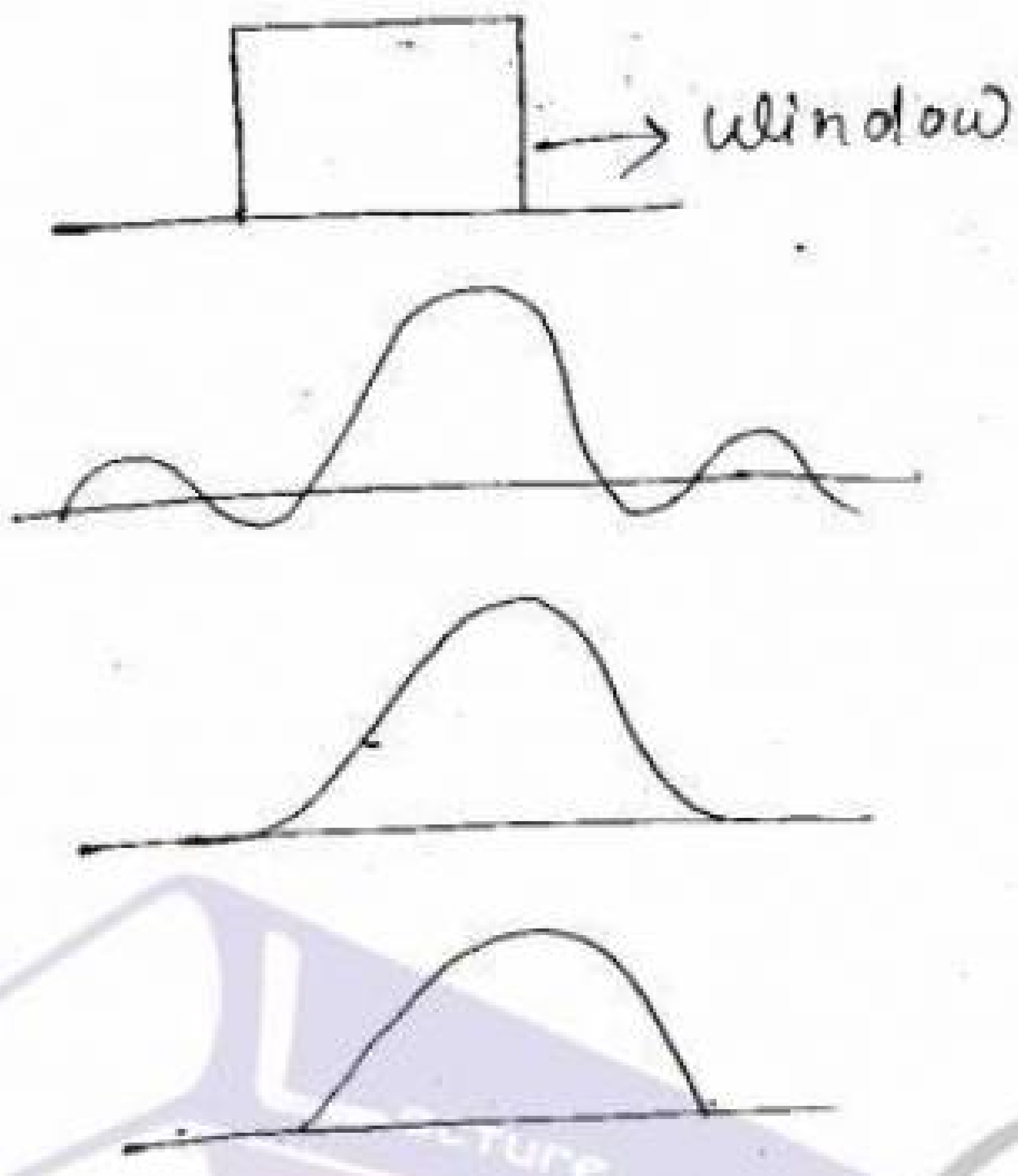
$$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n)e^{-j\omega n}$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\,e^{j\omega n}\,d\omega$$



$h(n) = h_d(n) * w(n)$

$\Rightarrow H(\omega) = H_d(\omega) * W(\omega)$



Types of windows

(1) Barlett window (triangular)

$$w(n) = 1 - \frac{2\left|n - \frac{M-1}{2}\right|}{M-1}$$

M - order of filter

(2) Blackman

$$w(n) = 0.42 - 0.5\cos\frac{2\pi n}{M-1} + 0.08\cos\frac{4\pi n}{M-1}$$

(3) Hamming

$$w(n) = 0.54 - 0.46\cos\frac{2\pi n}{M-1}$$

(4) Hanning

$$w(n) = \frac{1}{2}\left[1 - \cos\frac{2\pi n}{M-1}\right]$$

(5) Kaiser window.

$$w(n) = \frac{I_0\left[\alpha\sqrt{\left(\frac{M-1}{2}\right)^2 - \left(n - \frac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\frac{M-1}{2}\right)\right]}$$

Steps for solution :-

given $H_d(\omega)$, $w(n) \to$ window function of length M.

1. $h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$.
2. $w(n)$ for $n = 0 \ldots M-1$.
3. $h(n) = h_d(n) \times w(n)$
4. $H(\omega) = \sum_{n=0}^{M-1} h(n)\, e^{-j\omega n}$.

Q) Design a FIR filter with desired freq response

$$H_d(\omega) = \begin{cases} e^{-j2\omega} & -\pi/4 \le \omega \le \pi/4 \\ 0 & \pi/4 < |\omega| \le \pi \end{cases}$$

find $h(n)$ taking a rectangular window.

Ans - N = length = 5

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j2\omega}\, e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\omega(n-2)}\, d\omega.$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-2)}}{(n-2)}\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{\pi(n-2)} \left[\frac{e^{j\pi/4(n-2)} - e^{-j\pi/4(n-2)}}{2j} \right]$$

$$hd(n) = \frac{1}{\pi(n-2)} \sin \pi/4 (n-2)$$

$$hd(n) = \frac{\sin \pi/4 (n-2)}{\pi(n-2)} \quad n \neq 2$$

$$= \frac{\cancel{\pi}/4 \cos \pi/4 (n-2)}{\cancel{\pi}} \quad n = 2$$

$$= \frac{1}{4} \quad , n = 2$$

$$hd(0) = \frac{1}{2\pi}$$

$hd(1) =$

$hd(2) =$

$hd(3) =$

$hd(4) =$

$$h(n) = hd(n) \cdot \omega(n)$$

$$\omega(n) = \begin{cases} 1 & , n = 0 \ldots 4 \\ 0 & \text{elsewhere} \end{cases}$$

$h(0) = hd(0)$, $h(1) = hd(1)$, $h(2) = hd(2)$

$h(3) = hd(3)$, $h(4) = hd(5)$

DT-08.04.11

Q) A LPF has $H_d(\omega) = \begin{cases} e^{-j\omega}, & -\omega_c \le \omega \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$.

Find 'τ' such that $h_d(n) = h_d(-n)$.

Soln:- $h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\tau)} d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{\pi(n-\tau)} \times \sin\omega_c(n-\tau)$$

$$= \frac{\sin\omega_c(n-\tau)}{\pi(n-\tau)}, \quad n \neq \tau$$

$$= \frac{\omega_c}{\pi}, \quad n = \tau$$

$h_d(n) = h_d(-n)$

$$\Rightarrow \frac{\sin\omega_c(n-\tau)}{\pi(n-\tau)} = \frac{-\sin\omega_c(n+\tau)}{-\pi(n+\tau)}$$

$$\Rightarrow n-\tau = n+\tau$$

$$\boxed{\tau = 0}$$

~~Design a FI~~

1) Design a FIR filter with cut-off freq $f_c = 850$ at a sampling frequency or $f_s = 5000$ Hz. Choose a rectangular window of length '5'.

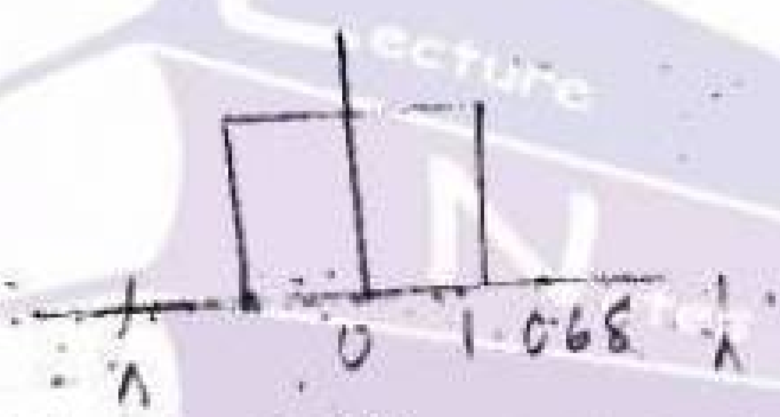
Ans. $f_c = 850$ Hz.

$$\omega_c = 2\pi \times f_c/f_s = \frac{2\pi \times 850}{5000}$$

$$= 1.068$$

$$H_d(\omega) = \begin{cases} 1 \; ; & 0 \le \omega \le 1.068 \\ & 1.068 < |\omega| < \pi \end{cases}$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$



$$= \frac{1}{2\pi} \int_{-1.068}^{1.068} e^{j\omega n} d\omega = \frac{\sin(1.068 n)}{\pi n} , \; n \neq 0$$

$$= \frac{1.068}{\pi} \; n = 0$$

$$n = 0, 1, 2, 3, 4.$$

$$h_d(0) = \frac{1.068}{\pi}, \quad h_d(1) = \frac{\sin(1.068)}{\pi}$$

FIR filter design by frequency sampling :-

Windowing Technique :- stop band & pass band are not precisely specified (disadvantages).

$H(\omega)$ is sampled to $H(k)$.

In frequency sampling method, ~~Hd(ω)~~ $H_d(\omega)$ is specified at equally spaced frequency

$$\omega_k = \frac{2\pi}{M}(k+\alpha)$$

where $k = 0, 1 \ldots \frac{M-1}{2}$.

and $\alpha = 0$ or $1/2$.

$$H_d(\omega) = \sum_{n=0}^{M-1} h(n)\, e^{-j\omega n}$$

$$H_d(\omega_k) = H_d(k+\alpha) = H_d\left(\frac{2\pi}{M}(k+\alpha)\right)$$

and $H_d(k+\alpha) = \sum_{n=0}^{M-1} h(n)\, e^{-j2\pi(k+\alpha)n/M}$

Multiplying both sides by $e^{j2\pi km/M}$ and summing over $k = 0, 1 \ldots M-1$.

$$\boxed{h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H_d(k+\alpha)\, e^{\frac{j2\pi(k+\alpha)n}{M}}}$$

$n = 0, 1 \ldots M-1$.

Simplification of computation by Symmetric pr[...]

The impulse response $h(n)$ is real & symmetric frequency samples $H(K+\alpha)$ satisfying the following conditions

$$H_d(K+\alpha) = H_d^*(M-k-\alpha)$$

This symmetric cond^n can be used to reduce the frequency ~~from~~ points. M to $M/2$ points when M is even

$$H_d(\omega) = H_r(\omega)\, e^{-j(-\omega(M-1)/2 + \pi/2)}$$

$$\Rightarrow H_d(K+\alpha) = H_r\left[\frac{2\pi}{M}(k+\alpha)\right] e^{j\left(\beta\pi/2 - \frac{2\pi(k+\alpha)(M-1)}{2M}\right)}$$

when $\beta = 0$, when $h(n)$ symmetric

$\beta = 1$, when $h(n)$ is antisymmetric

Lets define a set of real frequencies

$$G(K+\alpha) = (-1)^K H_r\left(\frac{2\pi}{M}(k+\alpha)\right)$$

then

$$H_d(K+\alpha) = G(k+\alpha)\, e^{j\pi k}\, e^{j\left[\frac{\beta\pi}{2} - \frac{2\pi(k+\alpha)(M-1)}{2M}\right]}$$

$$\alpha = 0, \tfrac{1}{2}, \quad \beta = 0, 1$$

~~Sym~~ Symmetric $\beta = 0$

$$\beta=0\begin{cases} M(k) = G(k)\, e^{j\pi k/M} \\ G(k) = (-1)^K H_r\left(\frac{2\pi k}{M}\right), \quad G(k) = G(M-k) \\ h(n) = \frac{1}{M}\left\{ G(0) + 2\sum_{k=1}^{U} G(k)\cos\frac{2\pi k}{M}\left(n+\frac{1}{2}\right)\right\} \end{cases}$$

$$U = \begin{cases} \frac{M-1}{2}, & \text{for } M\text{ - odd} \\ \frac{M}{2}-1 & \text{for } M\text{ - even} \end{cases}$$

DT- 09·04·11

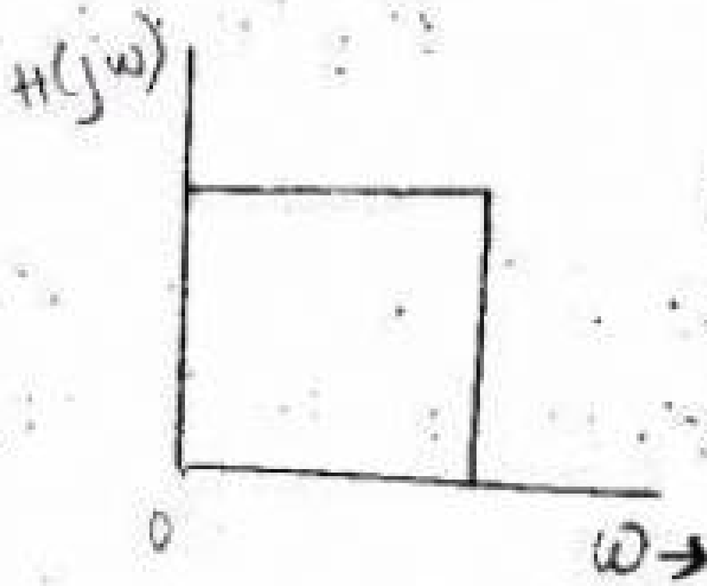
ANALOG FILTER DESIGN :-

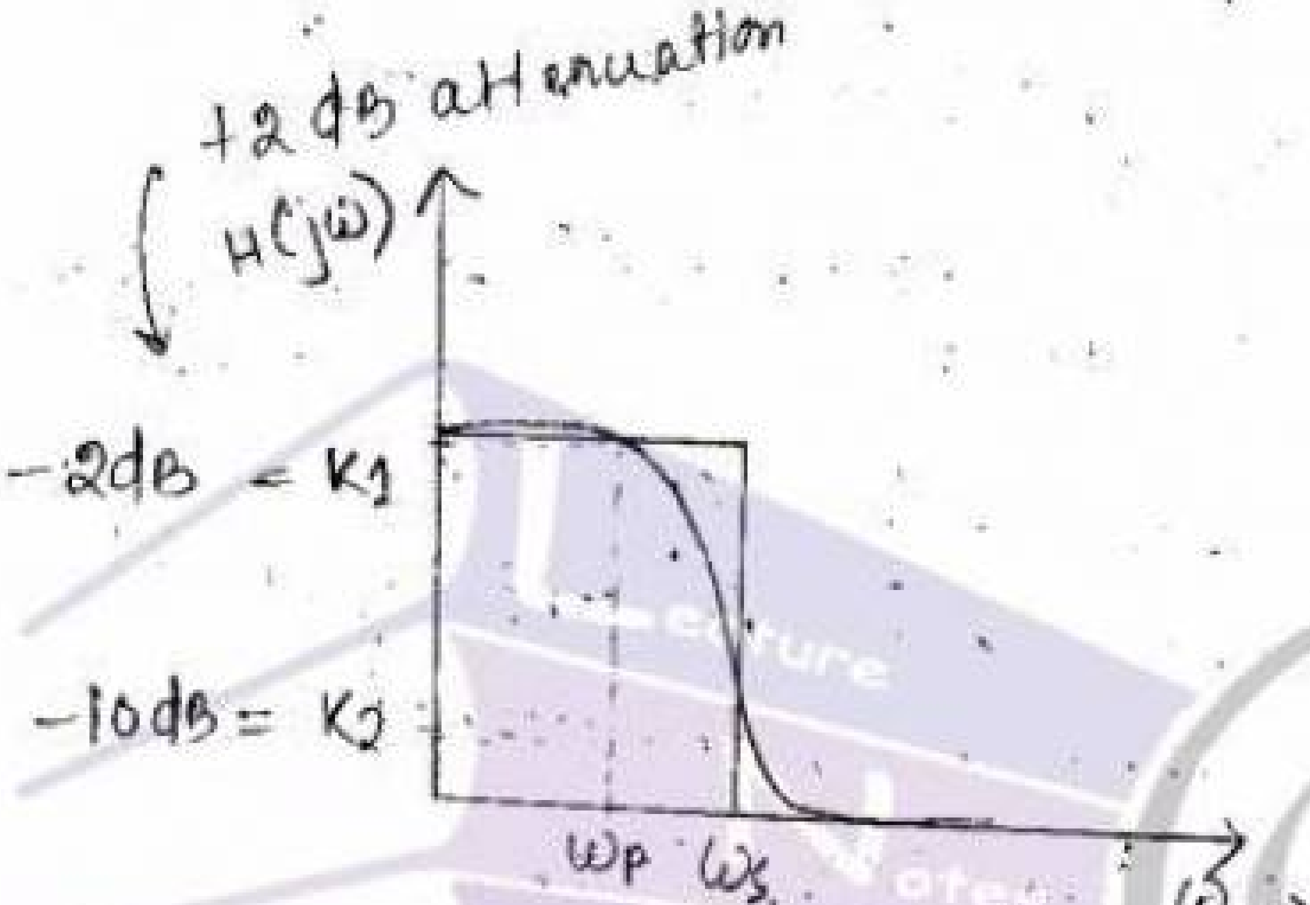
ButterWorth Filter :-

$$|H(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

n → order

Ideal Filter





Order of the filter

$$n = \frac{\log\left(\frac{10^{-K_1/10}-1}{10^{-K_2/10}-1}\right)}{2\log\left(\omega_p/\omega_s\right)}$$

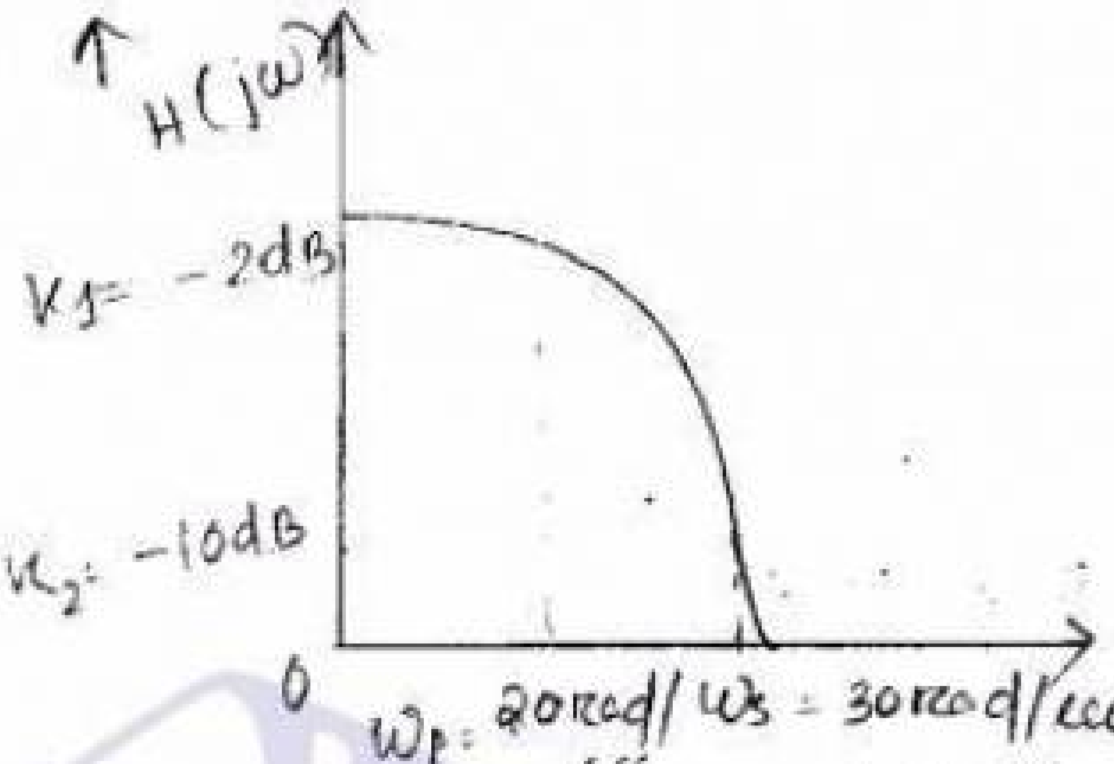
Butterworth polynomials of order n

Order		Polynomials
1	⟶	$s+1$
2	⟶	$s^2+\sqrt{2}s+1$
3	⟶	$(s^2+s+1)(s+1)$
4	⟶	$(s^2+0.76536s+1)(s^2+1.84776s+1)$
5	⟶	$(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)$

for eq. $H(S) = \frac{1}{S+1}$ for $n = 1$

5) Design a LPF analog Butterworth filter that has -2dB or better cut-off frequency at 20 rad/sec and at least -10 dB of attenuation at 30 rad/sec.

Ans -



$$\text{Order, } n = \frac{\log\left(\frac{10^{-(-2/10)} - 1}{10^{-(-10)/10} - 1}\right)}{20\log\left(\omega_p/\omega_s\right)}$$

$= 4.$

$$H(S) = \frac{1}{(S^2 + 0.76536S + 1)(S^2 + 1.84776S + 1)}$$

$S \to S/\omega_c$.

$$-2 = 10\log\left(\frac{1}{1 + (20/\omega_c)^4}\right) = -1.585$$

$$\omega_c = 21.3868 \text{ rad/sec}.$$

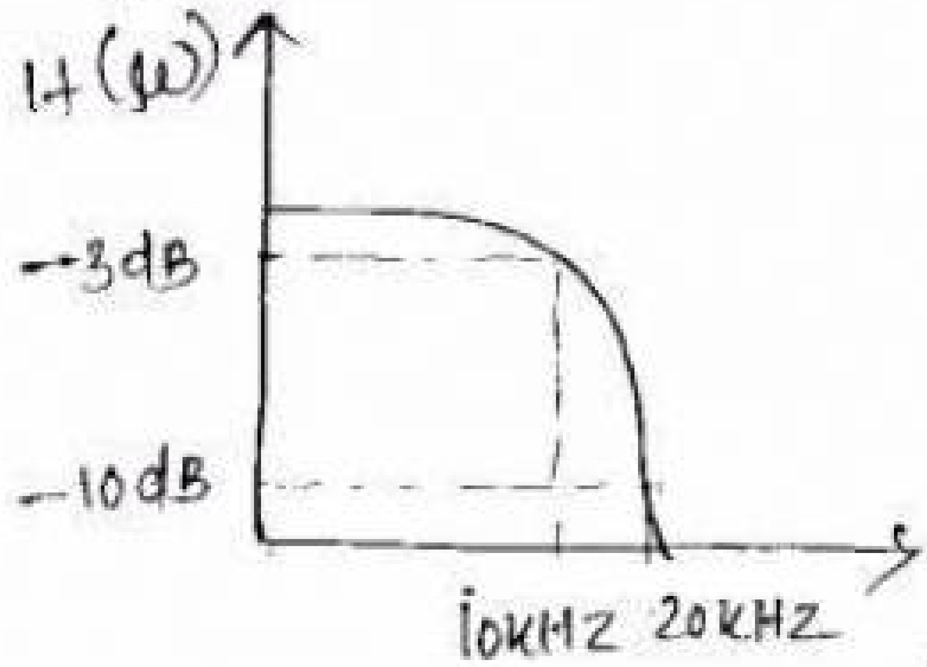
Q) Design an analog LPF Butterworth filter that has

$0 - 10\,KHz \leftarrow$ passband

$10 - 20\,KHz \leftarrow$ Transition band.

stopband attenuation is 10dB & starts at 20KHz

Ans -



Order = 2

$$H(s) = \frac{1}{s+\sqrt{2}s+1}$$

<u>Transformation from LPF to HPF :-</u>

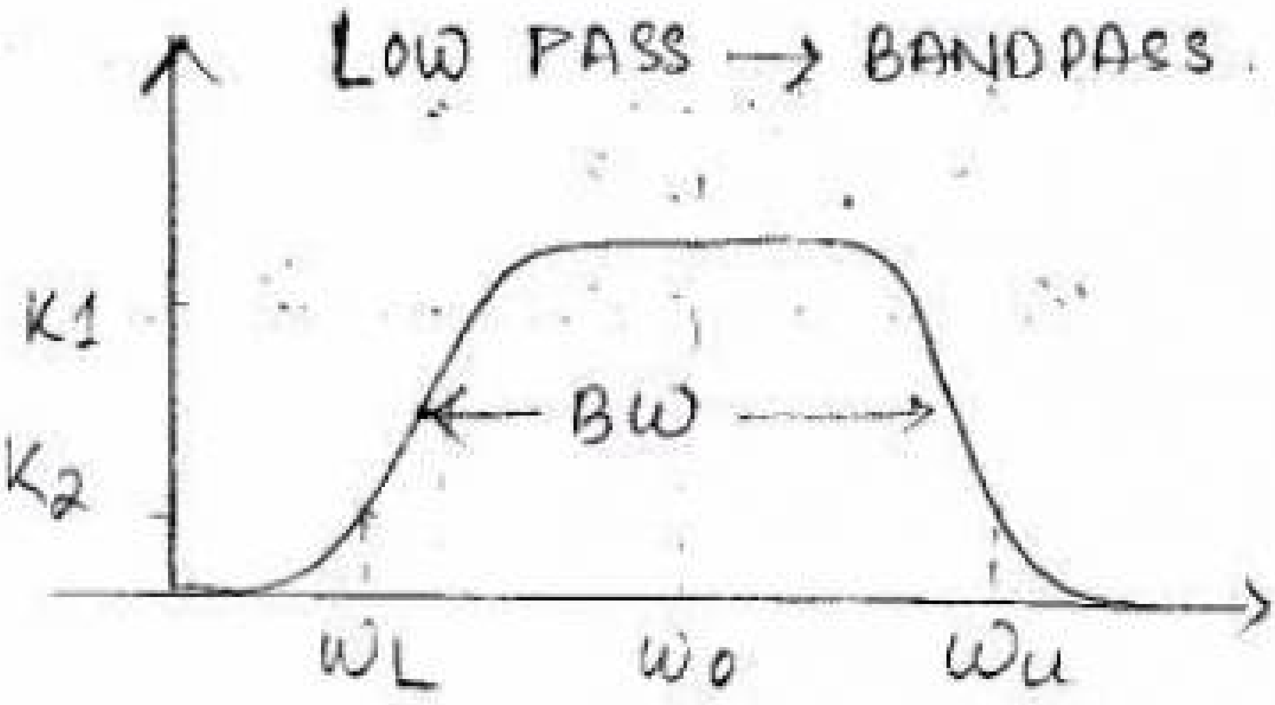
replace $s \rightarrow 1/s$

$s \rightarrow \omega_c/s$

$$H(s)\Big|_{LPF} = \frac{1}{s+1}$$

$$H(s)\Big|_{HPF} = \frac{s}{s+1}$$

LOW PASS $\rightarrow$ BANDPASS



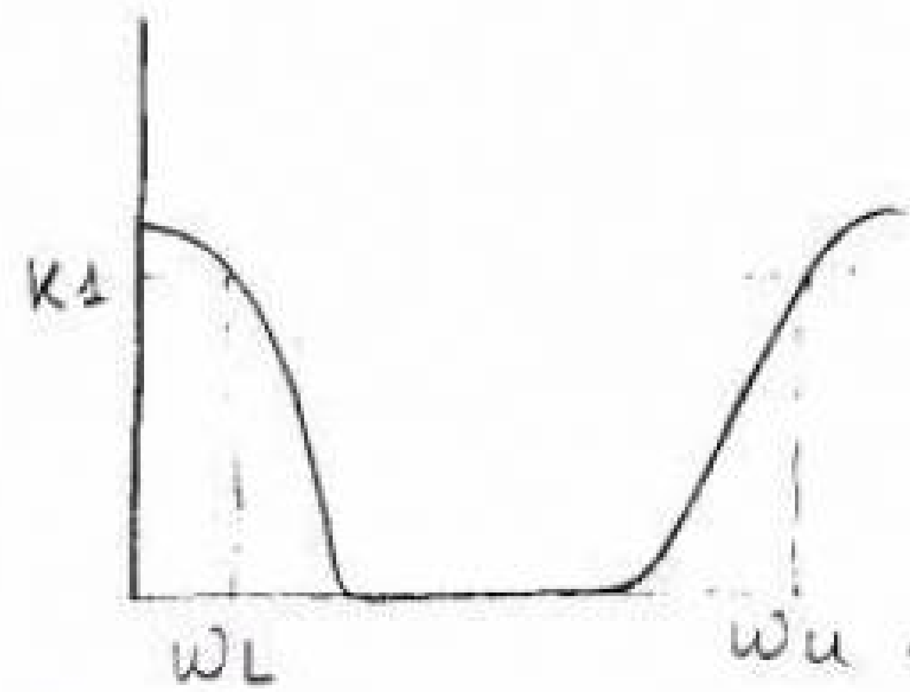
$$s \rightarrow \frac{s^2+\omega_u\omega_L}{s(\omega_u-\omega_L)}$$

$$s \rightarrow \frac{s^2 + \omega_0^2}{Bs}$$

$B \leftarrow$ Bandwidth

$\omega_0 \leftarrow$ Centre frequency

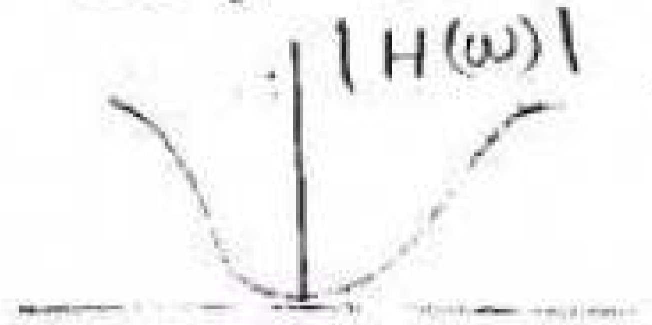
LOW PASS FILTER — BANDSTOP FILTER :—

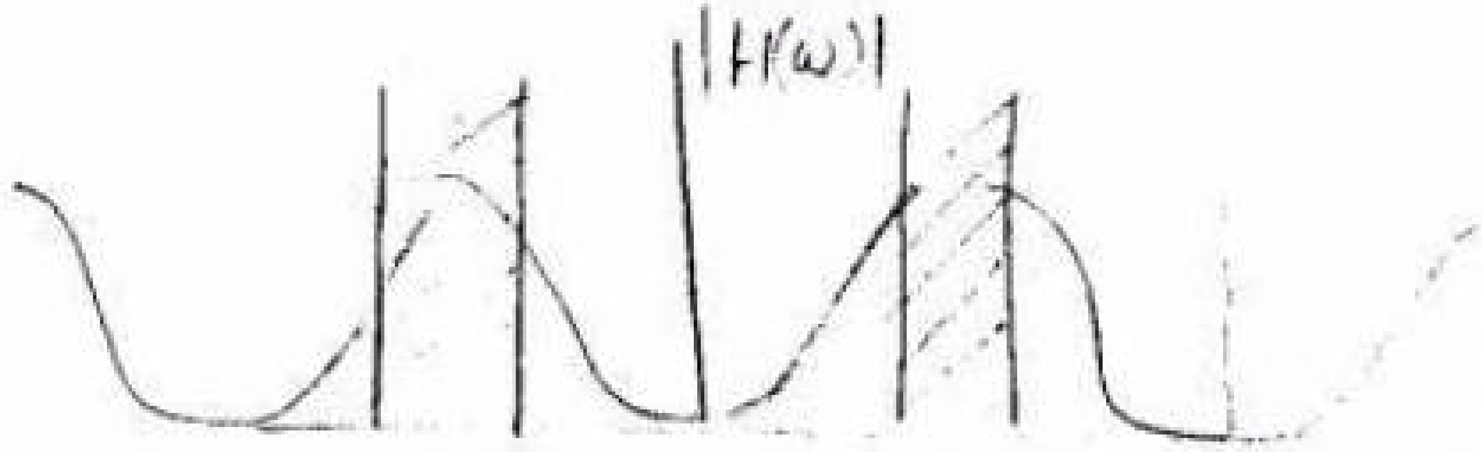


$$s \rightarrow \frac{s(\omega_u - \omega_L)}{s^2 + \omega_L \omega_u}$$

DT-15.04.11

IIR filter Design by Bilinear Transformation





(SAMPLING OUTPUT)

Bilinear Trans

→ It is a conformal mapping that transforms $j\Omega$ axis in laplace domain to the unit circle in z-domain only once & hence avoid aliasing

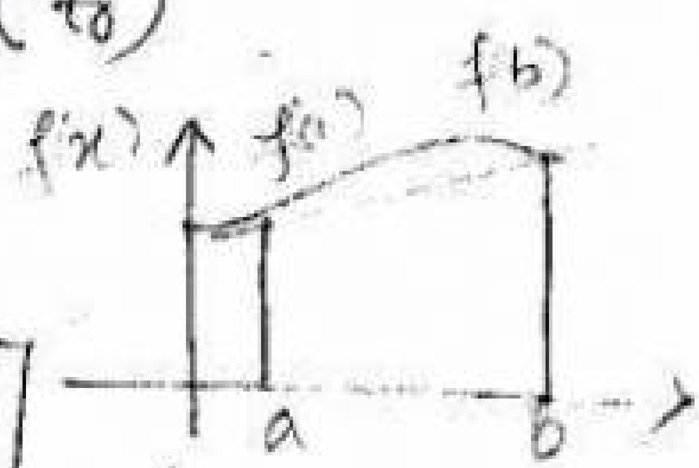
$$H(s) = \frac{b}{s+a}$$

One pole is there, Order = 1.

This system can be described by

$$\frac{dy(t)}{dt} + a y(t) = b x(t) \quad \text{--- (1)}$$

$$y(t) = \int_{t_0}^{t} y'(x)\,dx + y(t_0)$$



Let $t = nT$ & $t_0 = nT - T$.

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T)$$

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left(f(b) - f\right.$$

$$y'(nT) + ay(nT) = b\,x(nT)$$

$\Rightarrow$ ~~$y'(nT) + ay(nT) = bx(nT)$~~

$\Rightarrow$ ~~$y'(nT) + a\left\{\left[\frac{T}{2}\{y'(nT) + y'(nT-T)\right] + y(nT-T)\right\}$~~

$$= bx(nT)$$

$$\Rightarrow y'(nT) = -ay(nT) + bx(nT) \quad -(3)$$

Substituting eq-(3) in eq-(2)

$$y(nT) = \frac{T}{2}\left[-ay(nT) + bx(nT) - a\,y(nT-T) + bx(nT-T)\right] + y(nT-$$

$$\Rightarrow \left(1 + \frac{aT}{2}\right) y(n) - \left(1 - \frac{aT}{2}\right) y(n-1) = \frac{bT}{2}\left(x(n) + x(n$$

Taking the z transform

$$\left(1 + \frac{aT}{2}\right) Y(z) - \left(1 - \frac{aT}{2}\right) z^{-1} Y(z) = \frac{bT}{2}\left(X(z) + z^{-1}X(z)\right)$$

$$\frac{Y(z)}{X(z)} = \frac{\left(bT/2\right)\left(1 + z^{-1}\right)}{1 + \frac{aT}{2} - \left(1 - \frac{aT}{2}\right) z^{-1}}$$

$$\Rightarrow H(z) = \frac{b}{\frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + a}$$

$$H(s) = \frac{b}{s + a}$$

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

Relationship between 'ω' and 'Ω'

$$z = re^{j\omega}$$

$$s = \sigma + j\Omega$$

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\frac{(z-1)}{(z+1)}$$

$$= \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right)$$

$$s = \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right)$$

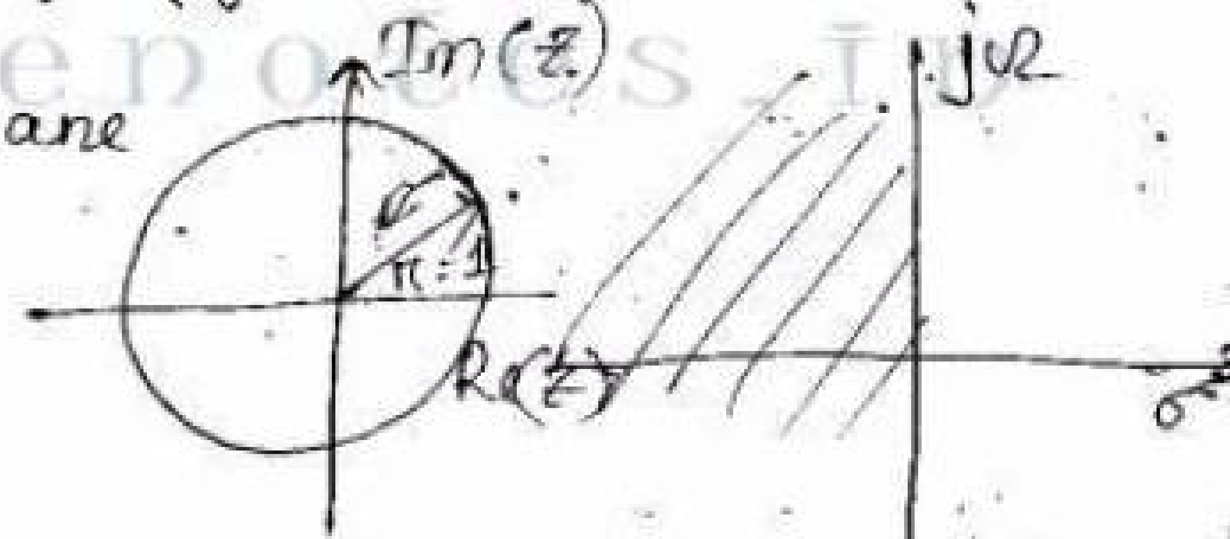
as $s = \sigma + j\Omega$

So, $$\boxed{\Omega = \frac{2}{T} \times \frac{2r\sin\omega}{1+r^2+2r\cos\omega}}$$

~~If r<1~~ $$\sigma = \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos\omega}\right)$$

If $r < 0$ $\sigma < 0$

→ Left hand of s-plane gets mapped onto inside of the unit of circle



If $r > 1$, $\sigma > 0$ ⇒ ~~Right~~ Left hand of s-plane gets mapped onto the exterior of unit circle.

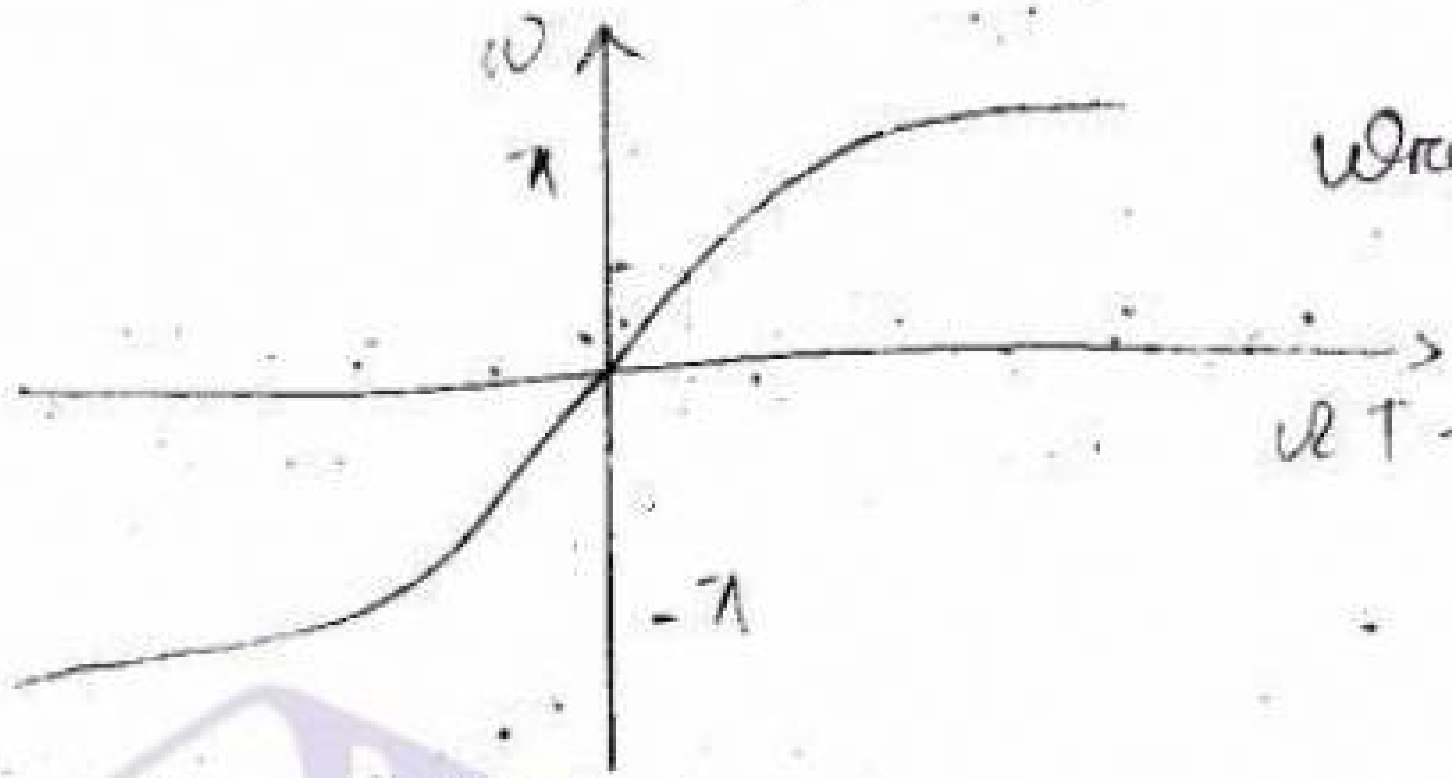
If $r = 1$, then $\sigma = 0$

$$\Omega = \frac{2}{T}\frac{\sin\omega}{1+\cos\omega} \quad [r = 1]$$

$$= \frac{2}{T}\frac{2\sin\omega/2\cos\omega/2}{\sin^2\omega/2 + \cos^2\omega/2 + \cos^2\omega/2 - \sin^2\omega/2}$$

$$\boxed{\Omega = \frac{2}{T}\tan\frac{\omega}{2}}$$

$$\boxed{\omega = 2\tan^{-1}\frac{\Omega T}{2}}$$



Wrapping effe

Wrapping Effect :-

In Bilinear transformation, the mapping betn analog frequency Ω & digital frequency ω ~~is a~~ linear at lower frequency but for higher frequency, the relationship betn Ω & ω is non-linear & distortion is introduced in the frequency scale of digital filters.

Steps for designing IIR filter by bilinear transformation :-

1) From given specification, we find out of analog frequency

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

↓ find out the corresponding digital frequency

2) Using Ω, find H(s) for analog filters.
3) Select Sampling time 'T'.
4) Substitute

$$s = \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})} \quad \text{in } H(s)$$

Find H(z).

Q) $H(s) = \frac{2}{(s+1)(s+2)}$. Find $H(z)$

Take $T = 1$ sec .

Ans – $s = \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}$

$$H(s) = \frac{2}{\left\{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right\}\left\{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right\}}$$

$$= \frac{2}{\left\{\frac{(2-2z^{-1}+1+z^{-1})}{(1+z^{-1})}\right\}\left\{\frac{2-\cancel{2z^{-1}}+2+\cancel{2z^{-1}}}{1+z^{-1}}\right\}}$$

$$H(z) = \frac{0.166\,(1+z^{-1})^2}{(1-0.33z^{-1})}$$

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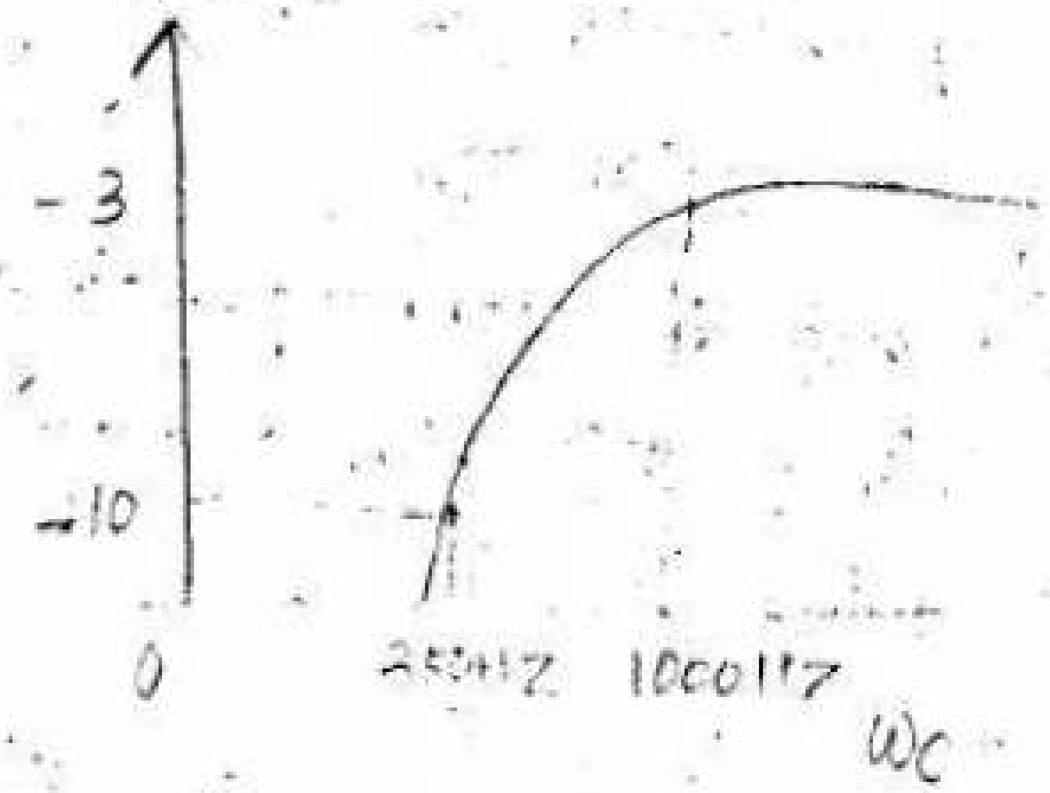
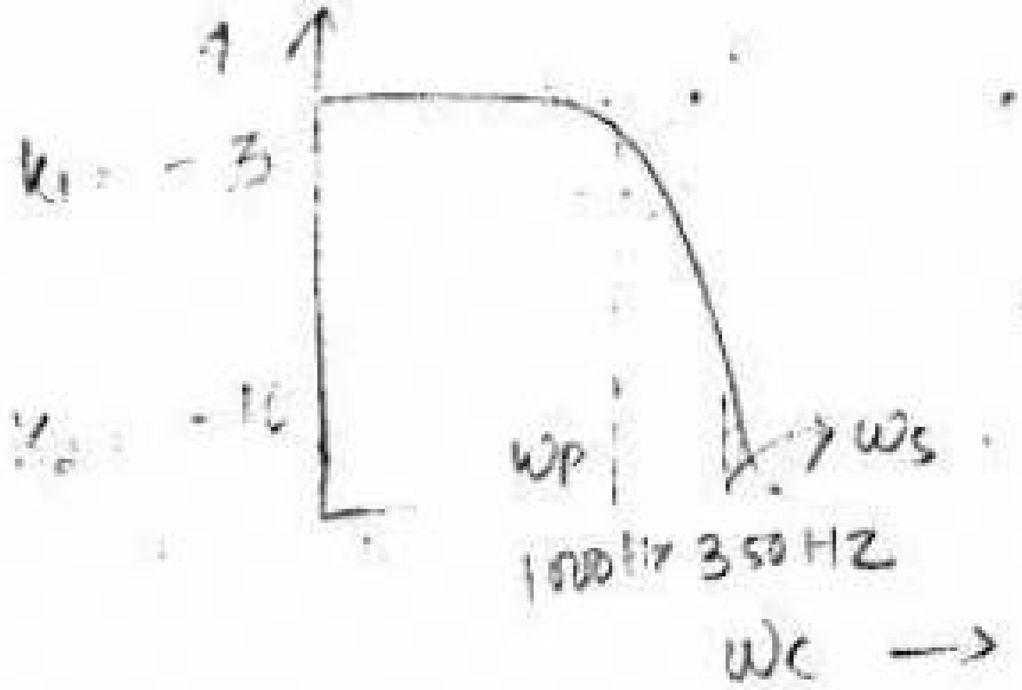
Q) Using Bilinear Transformation design a HPF monotonic in passband and cut-off frequency 1000Hz down to 10 dB at 350Hz . $f_s = 5000$ Hz





$$\cancel{\omega = \frac{2\pi f}{f_s}}$$

$$\cancel{\omega_p = \frac{2\pi \times 1000}{5000}}$$

$k_1 = -3,\ k_2 = -10$

$$\omega = 2\pi f$$

$$\omega_p = 2\pi \times 1000 = 2000\pi \rightarrow \text{Analog pass band frequency}$$

$$\omega_s = 2\pi \times 350 = 700\pi$$

$$\left.\begin{aligned} \Omega_p &= \frac{2}{T}\tan\frac{\omega_p T}{2} = 7265 \text{ rad/sec} \\ \Omega_s &= \frac{2}{T}\tan\frac{\omega_s T}{2} = 2235 \text{ rad/sec} \end{aligned}\right\} \rightarrow \text{Digital}$$

$$T = \frac{1}{f_s} = \frac{1}{5000} = 2\times 10^{-4} \text{ sec}$$

$$N = \frac{\log\left(\dfrac{10^{0.3/10} - 1}{10^{10/10} - 1}\right)}{2\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} = 0.932 = 1$$

$$H(s) = \frac{1}{1+s} \text{ for LPF}$$

For the HPF replace s by ω_p/s.

$$H(s) = \left.\frac{1}{s+1}\right|_{s = \frac{7265}{s}} = \frac{s}{s+7265}$$

$$H(z) = \frac{9}{s+7.265}\bigg|_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{0.5792\,(1-z^{-1})}{1-0.1584\,z^{-1}}$$

$$\Omega = \frac{2}{T}\tan\frac{\omega T}{2} = \frac{2}{T}\tan\frac{\omega/f_s}{2}$$

$$= \frac{2}{T}\tan\frac{\omega_p}{2}$$

Implementation of Discrete Time System :-

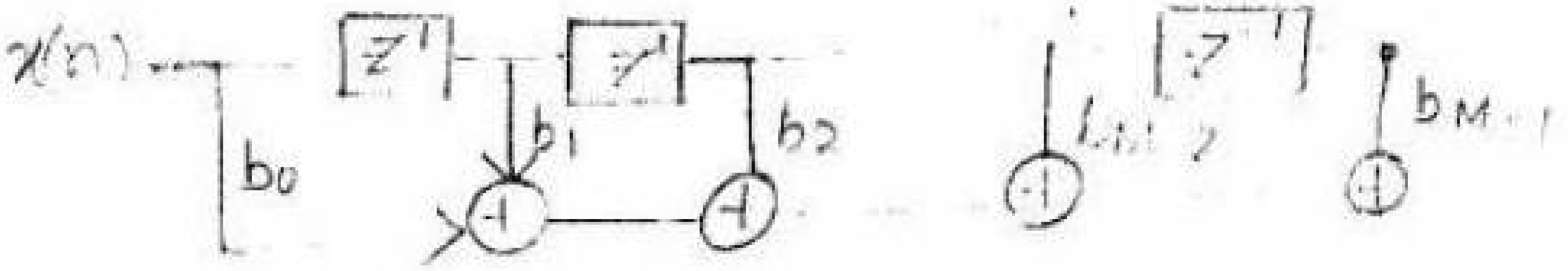
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$ → delay

Major factors that influence the choice of specific realisation :-

(1) Computational Complexity
(2) Memory requirement
(3) Finite wordlength effect.

FIR System Structure :-

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

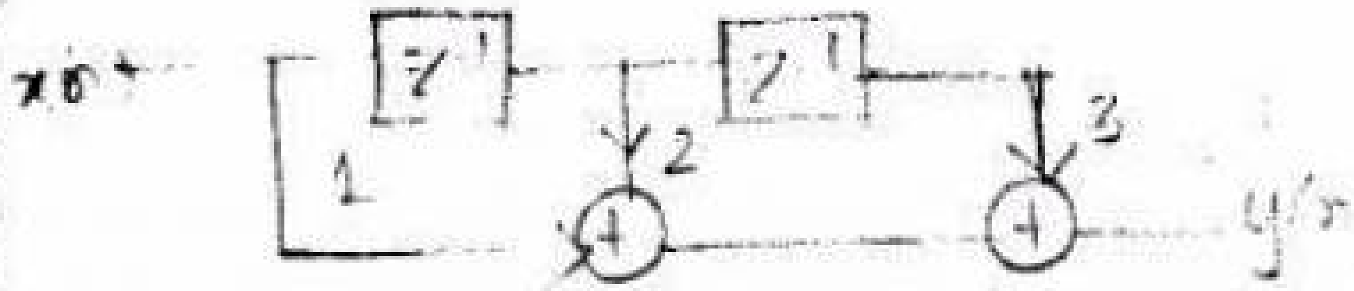


$$H(z) = 1 + 2z^{-1} + 3z^{-2}$$

$$\frac{Y(z)}{X(z)} = 1 + 2z^{-1} + 3z^{-2}$$

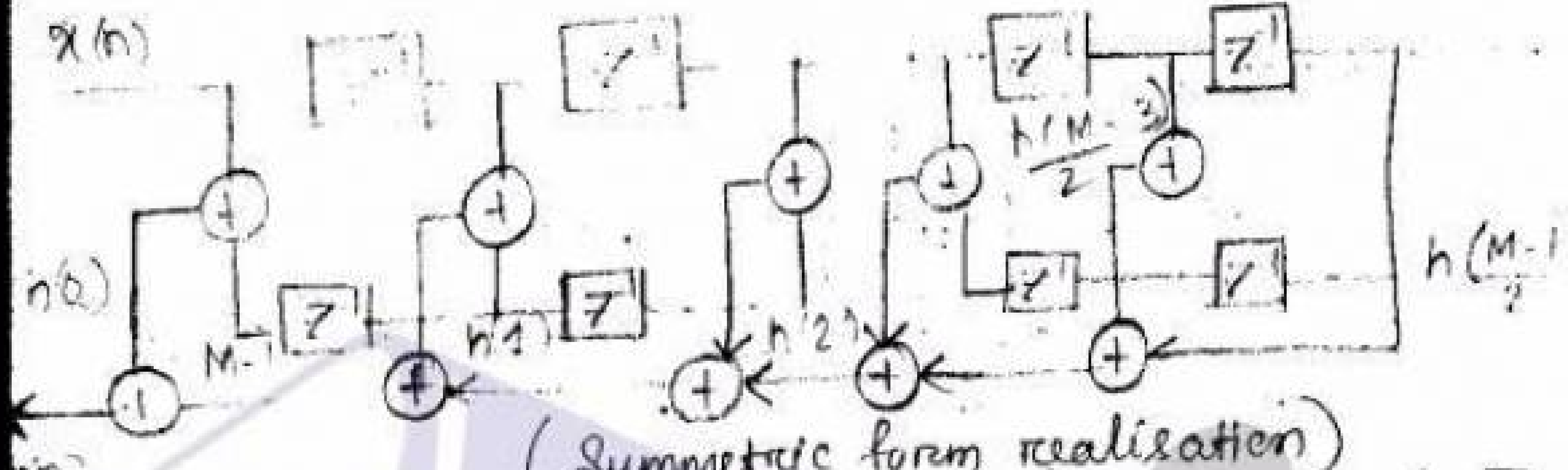
$$\Rightarrow Y(z) = X(z) + 2z^{-1}X(z) + 3z^{-2}X(z)$$

$$y(n) = x(n) + 2 \cdot x(n-1) + 3x(n-2)$$



$$h(n) = \pm h(M-1-n)$$

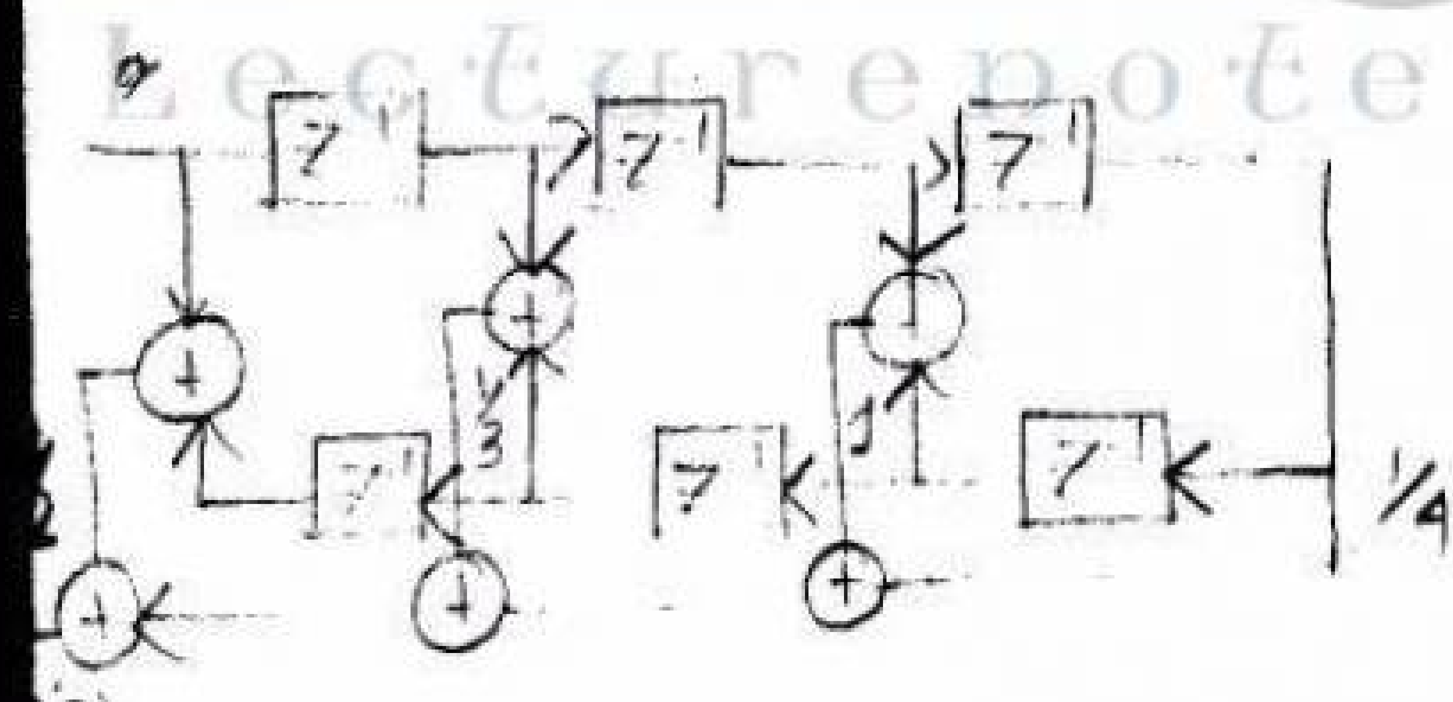
$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k}$$



(Symmetric form realisation)

$$H(z) = \frac{1}{2} + \frac{1}{3}z^{-1} + z^{-2} + \frac{1}{4}z^{-3} + z^{-4} + \frac{1}{3}z^{-5} + \frac{1}{2}z^{-6}$$

$$h(n) = \left\{\frac{1}{2}, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{3}, \frac{1}{2}\right\}$$



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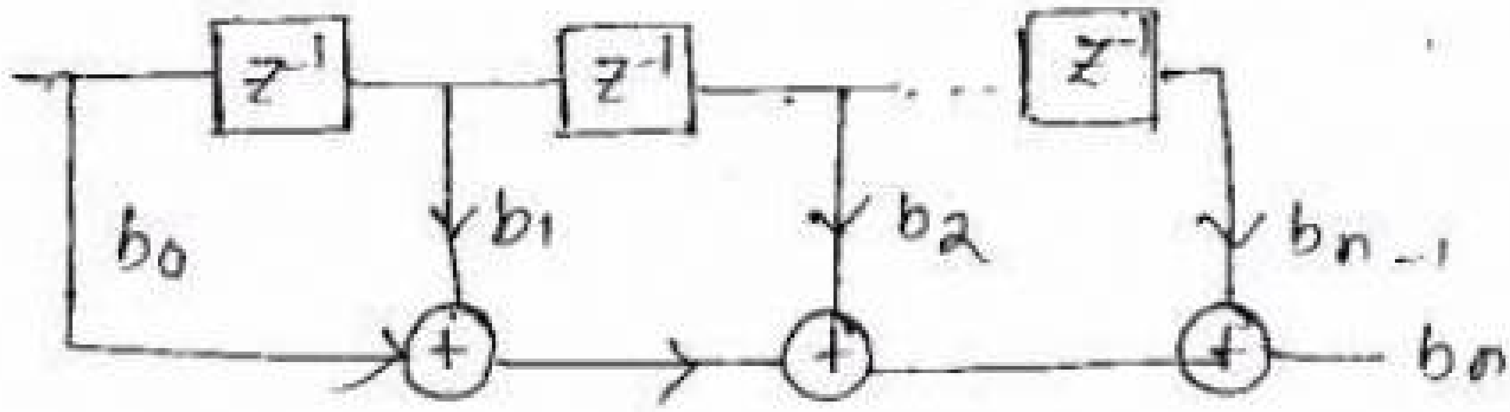
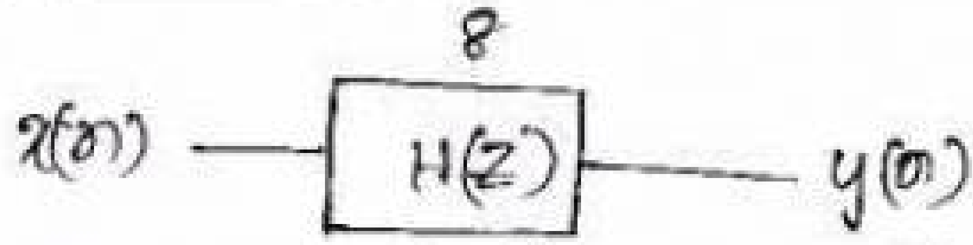
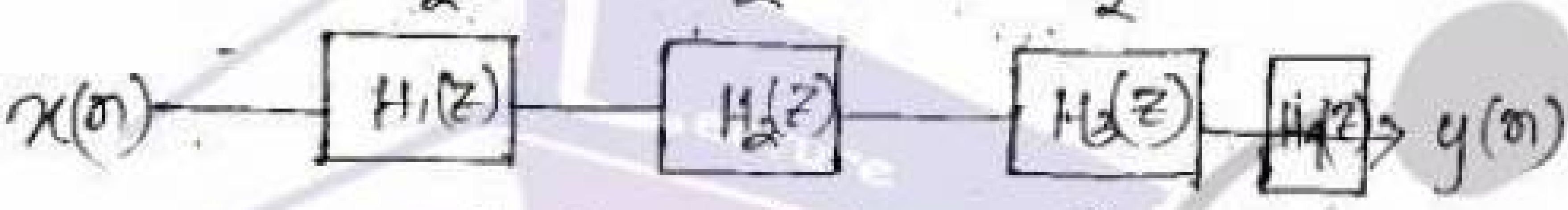
$$H(z) = \sum_{k=0}^{M-1} b_0 z^{-k}$$



Cascade form Structure



$z_1, z_1^*, z_2, z_2^*, z_3, z_3^*, z_4, z_4^*$



$$H_1(z) = (z - z_1)(z - z_1^*)$$

$$H(z) = \prod_{k=1}^{K} H_k(z)$$

where $H_k(z) = b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}$

→ Filter parameter b_0 ~~mainly~~ may be equally distributed into 'K' filter sections such that

$$b_0 = b_{10}\, b_{20} \ldots b_{K0}$$

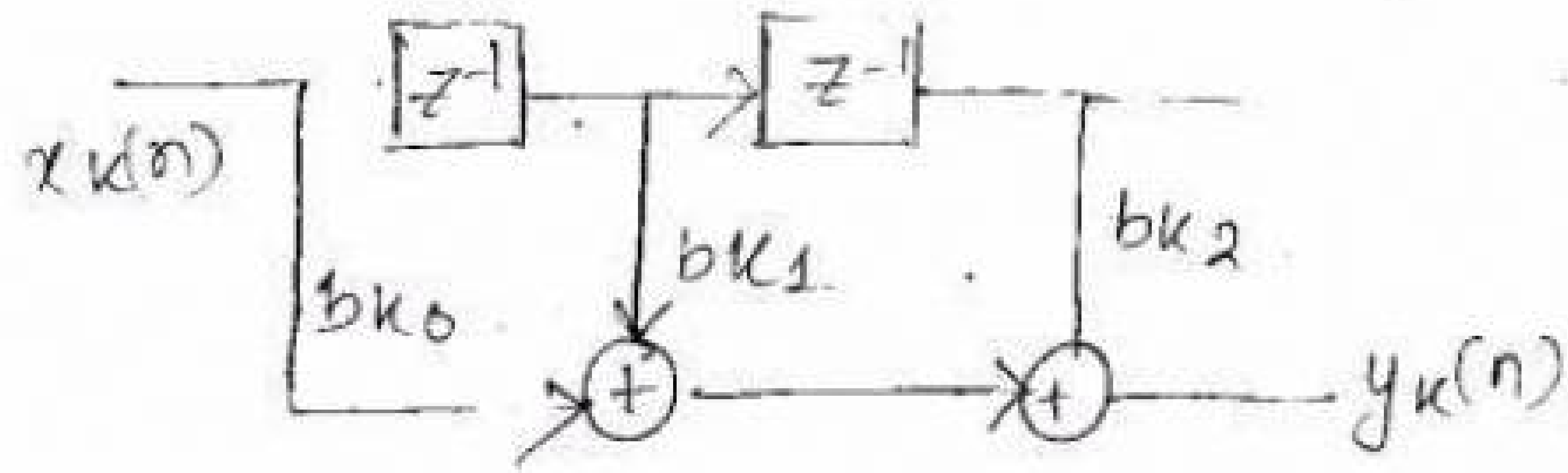
$$b_1 = b_{11}\, b_{21} \ldots b_{K1}$$

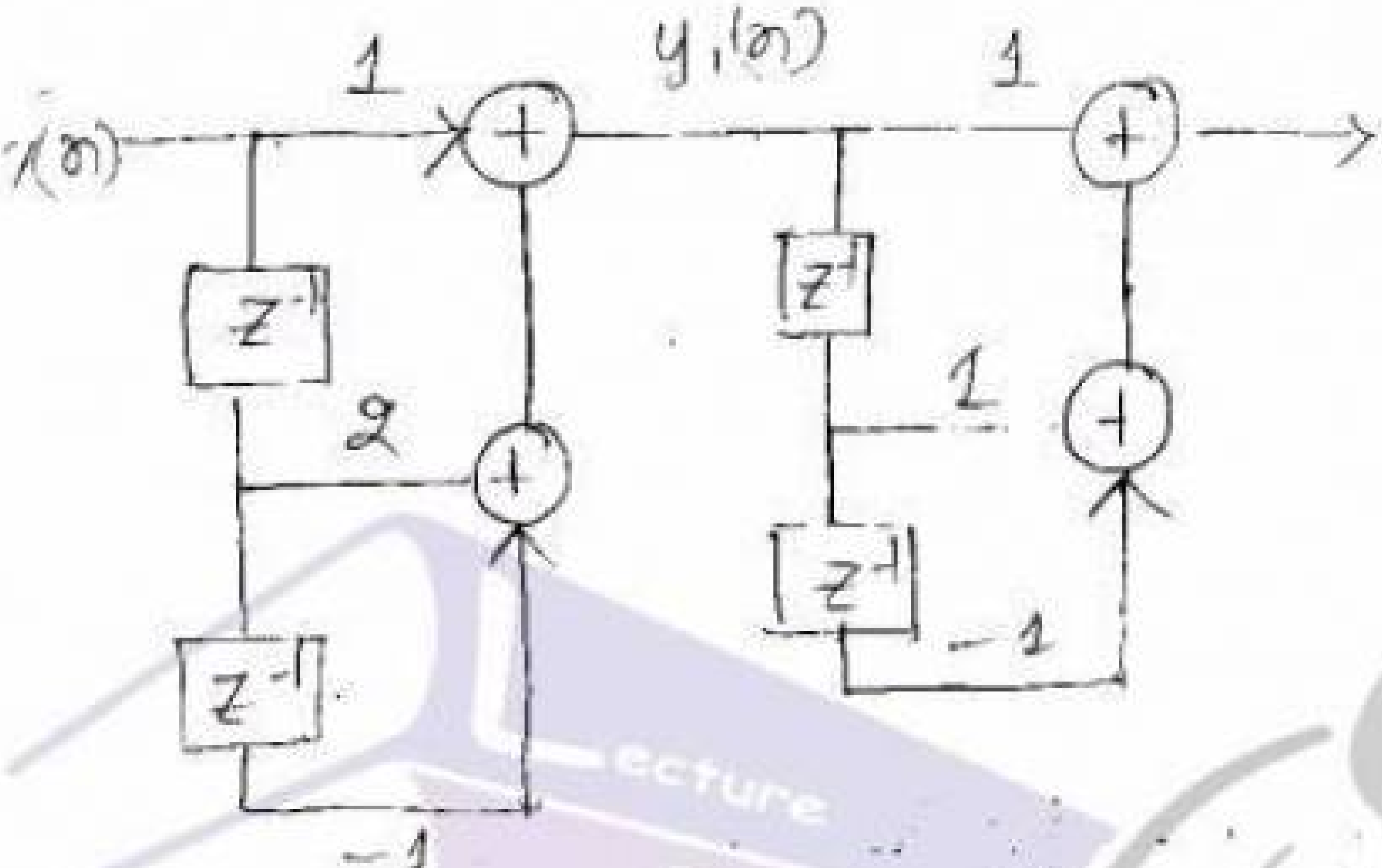
~~or it can be~~

→ Zeros of $H(z)$ are grouped in pairs to form 2nd order FIR system.

→ The complex conjugate roots are paired together so that $\{b_{ki}\}$ ~~be~~ has a real value

$H_k(z) = b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}$

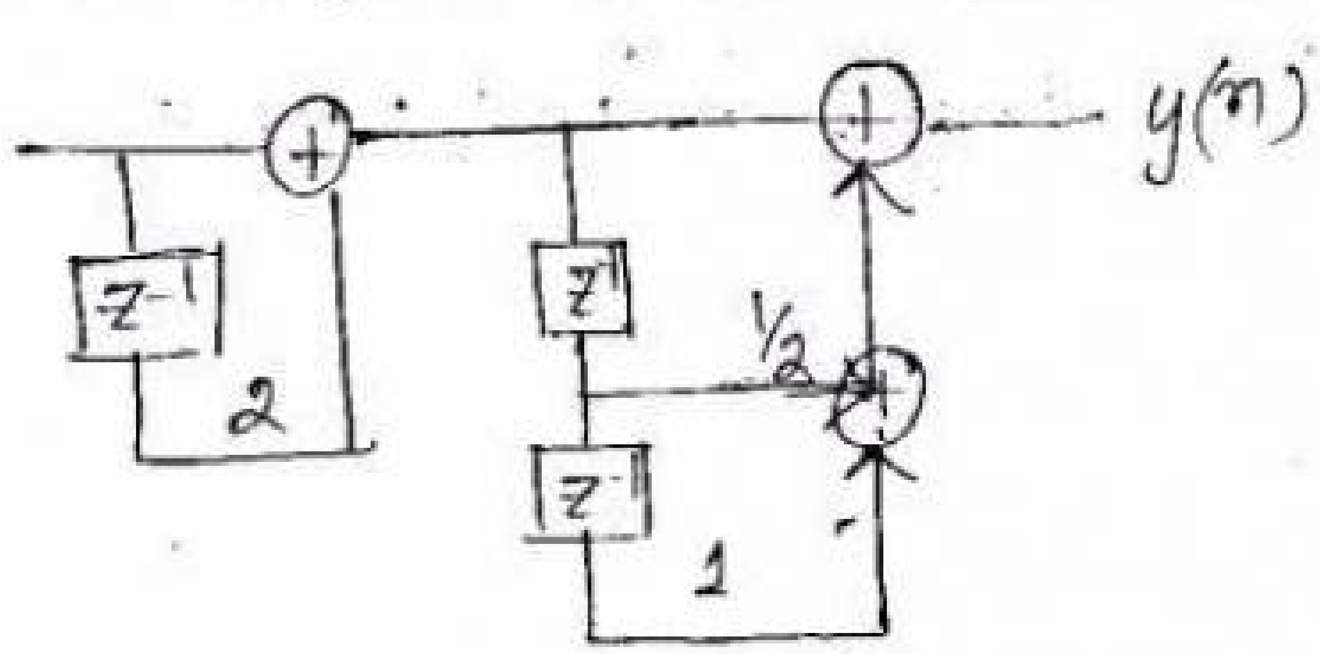




$H(z) = (1 + 2z^{-1} - z^{-2})(1 + z^{-1} - z^{-2})$

$H(z) = 1 + \frac{5}{2} z^{-1} + 2z^{-2} + 2z^{-3}$

b) $H(z) = (1 + 2z^{-1})(1 + \frac{1}{2} z^{-1} + z^{-2})$

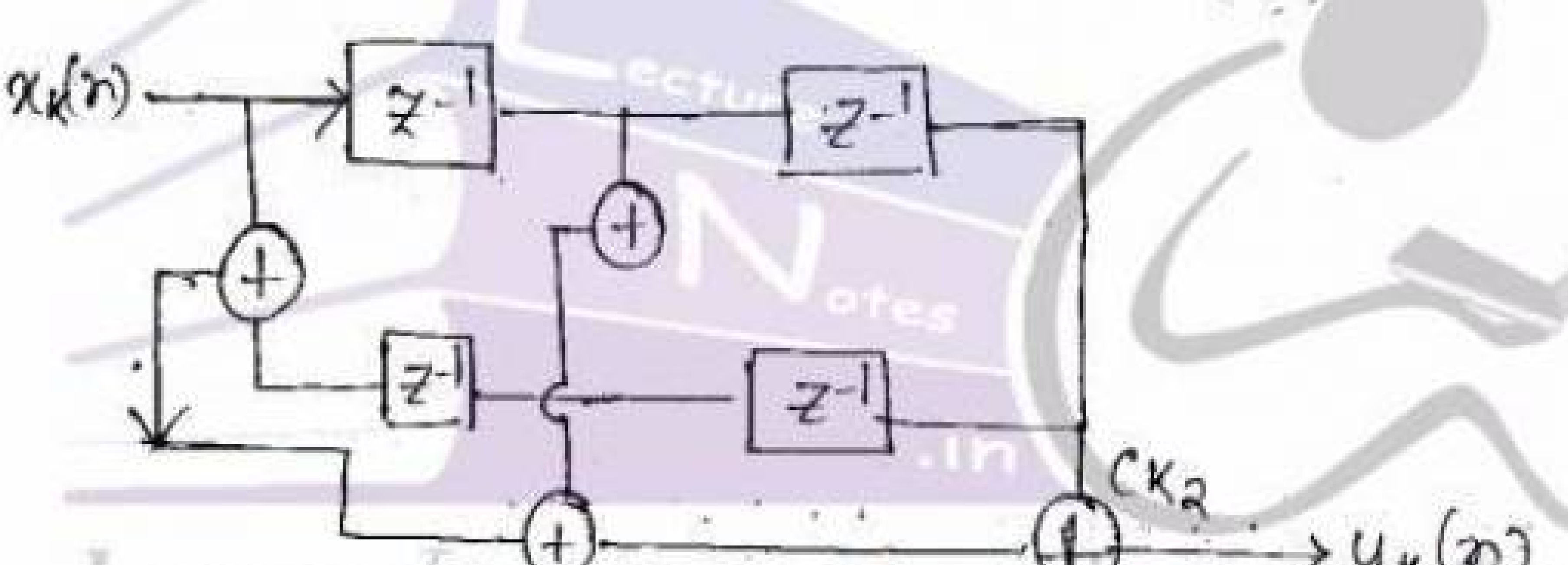


→ In case of linear-~~band~~ phase FIR system, the symmetry of FIR system implies that zeros in H(z) also have symmetry.

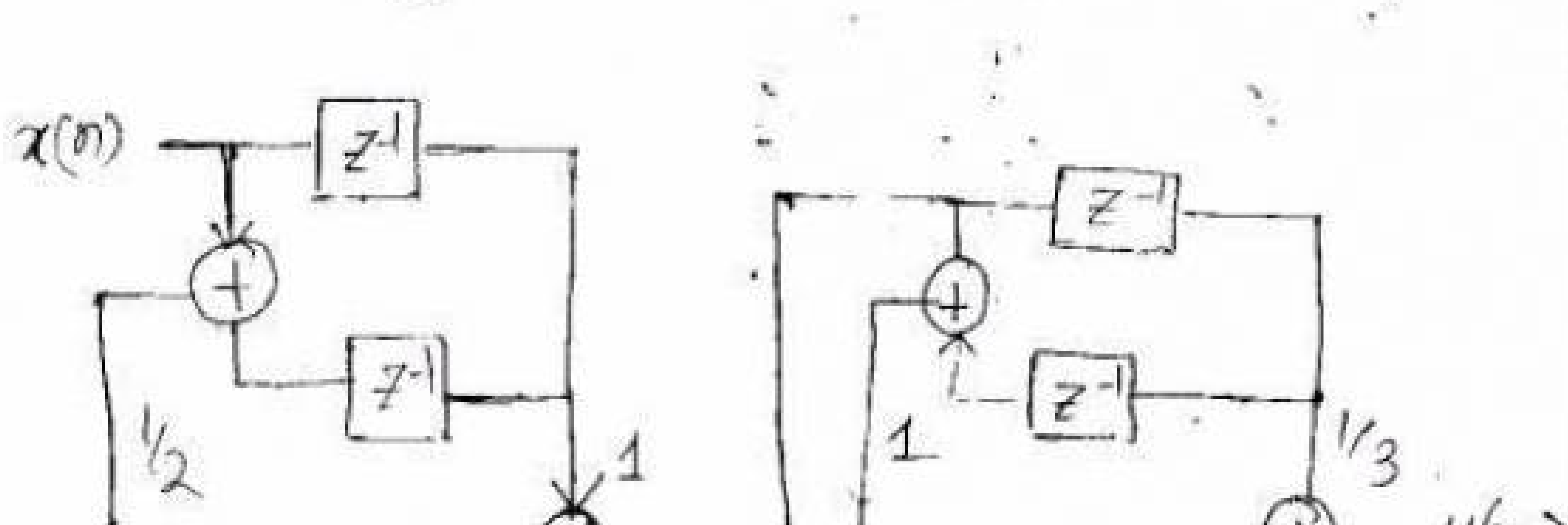
→ If z, z^* are two conjugate zeros, then $\frac{1}{z_K}$, $\frac{1}{z_K^*}$ are two conjugate poles.

$$H(z) = C_{K0}\left(1 - z_K z^{-1}\right)\left(1 - z_K^* z^{-1}\right)\left(1 - \frac{z^{-1}}{z_K}\right)\left(1 - \frac{z^{-1}}{z_K^*}\right)$$

$$= C_{K0} + C_{K1} z^{-1} + C_{K2} z^{-2} + C_{K1} z^{-3} + C_{K0} z^{-4}$$



Q) $H(z) = \left(\frac{1}{2} + z^{-1} + \frac{1}{2} z^{-2}\right)\left(1 + \frac{1}{3} z^{-1} + z^{-2}\right)$



Frequency Sampling Structure :-

Let the desired freq. response $H(\omega)$ is specif at a set equally spaced frequency samples.

$$\omega_k = \frac{2\pi}{M}(k+\alpha) \quad k = 0, 1 \ldots \frac{M-1}{2}$$

M is ~~even~~ odd

$$k = 0, 1 \ldots \frac{M}{2} - 1$$

M is even

$$H(\omega) = \sum_{n=0}^{M-1} h(n) e^{-j\omega n}$$

The values of $H(\omega)$ at frequency $\omega_k = \frac{2\pi}{M}(k+$

$$H(k+\alpha) = H\left(\frac{2\pi}{M}(k+\alpha)\right) = \sum_{n=0}^{M-1} h(n) e^{-j\frac{2\pi}{M}(k+\alpha)}$$

$$\Rightarrow h(n) = \frac{1}{M} \sum_{k=0}^{M-1} H(k+\alpha) e^{\frac{j2\pi(k+\alpha)n}{M}}$$

Now, $H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$

$$= \sum_{n=0}^{M-1} \frac{1}{M} \sum_{k=0}^{M-1} H(k+\alpha) e^{\frac{j2\pi(k+\alpha)n}{M}}$$

$$= \sum_{k=0}^{M-1} H(k+\alpha) \left[\frac{1}{M} \sum_{n=0}^{M-1} \left(e^{j2\pi(k+\alpha)/M} z^{-1}\right)^n\right]$$

$$= \sum_{k=0}^{M-1} H(k+\alpha) \left[\frac{1}{M} \frac{1 - e^{j2\pi(k+\alpha)} z^{-M}}{1 - e^{j2\pi(k+\alpha)/M} z^{-1}}\right]$$

$$\boxed{H(z) = \frac{1 - z^{-M} e^{j2\pi\alpha}}{M} \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j2\pi(k+\alpha)/M} z^{-1}}}$$

$$\sum_{n=0}^{M-1} a^{n} = \frac{1-a^{M}}{1-a}$$

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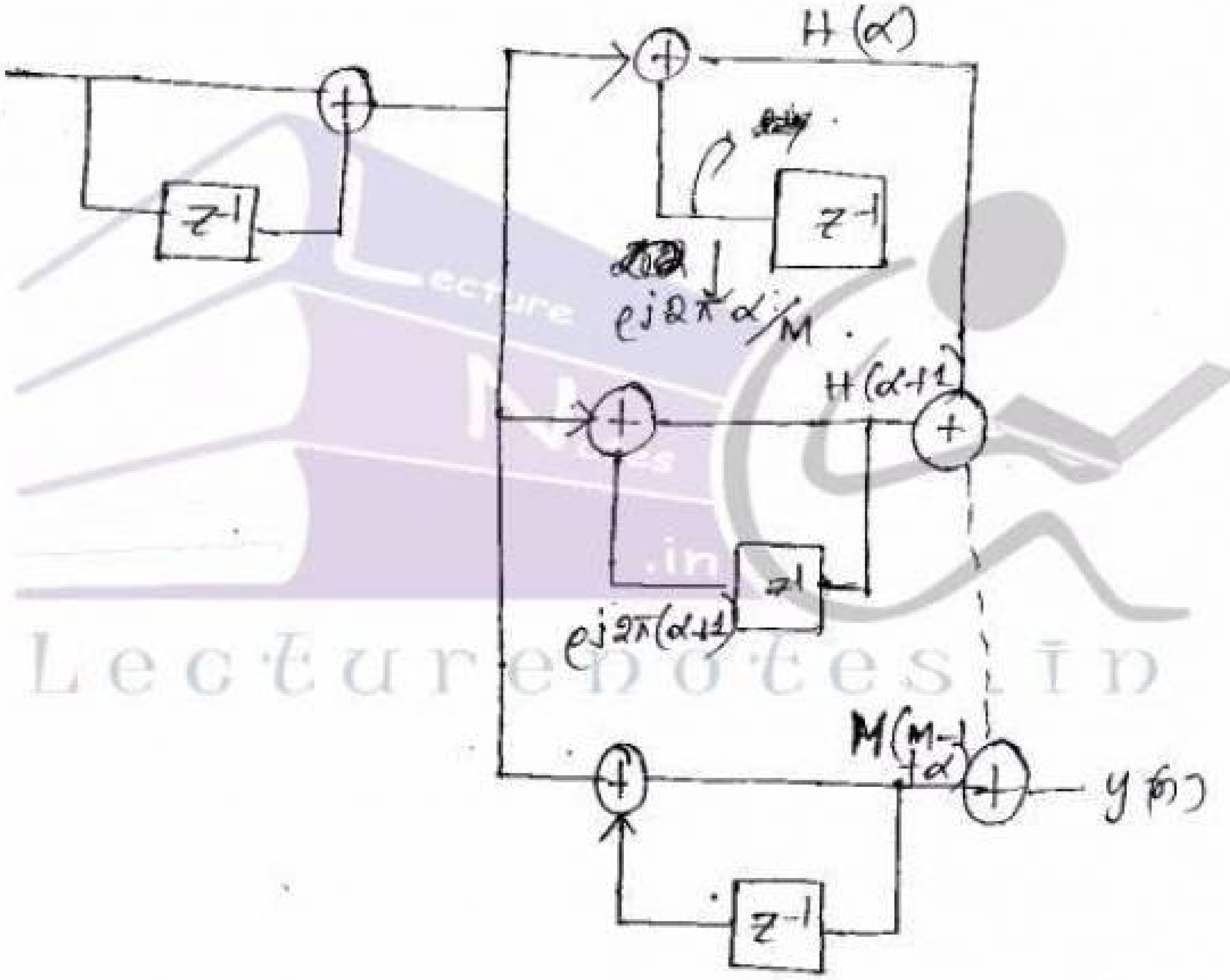
$$H(z) = \underbrace{\frac{1-z^{-M}e^{j2\pi\alpha}}{M}}_{\text{All zero filter}} \underbrace{\sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1-e^{j2\pi(k+\alpha)/M} z^{-1}}}_{\text{Single pole filter}}$$



If h(n) is symmetric

~~H(k) = H*(M-k)~~

$H(k) = H^{*}(M-K)$ for $\alpha = 0$

$H(K) = H^{*}(M-K-\frac{1}{2})$ for $\alpha = \pm\frac{1}{2}$

for $\alpha = 0$, (when M is odd)

$$H_{2}(z) = \frac{H(0)}{1-z^{-1}} + \sum_{k=1}^{M-\frac{1}{2}} \frac{A(k)+B(k)z^{-1}}{1-2\cos(2\pi k/M)\, z^{-1} + z^{-2}}$$

$$H_2(z) = \frac{H(0)}{1-z^{-1}} + \frac{H(M/2)}{1+z^{-1}} + \sum_{k=1}^{\frac{M}{2}-1} \frac{A(k) + B(k)z^{-1}}{1 - 2\cos\left(\frac{2\pi k}{M}\right)z^{-1} + z^{-2}}$$

$M \rightarrow$ even.

where $A(k) = H(k) + H(M-k)$

$$B(k) = H(k)e^{-j2\pi k/M} + H(M-k)e^{j2\pi k/M}$$

Q) $M = 32$, $\alpha = 2$ Linear ~~phas~~ phase (symmetric) FIR filters with

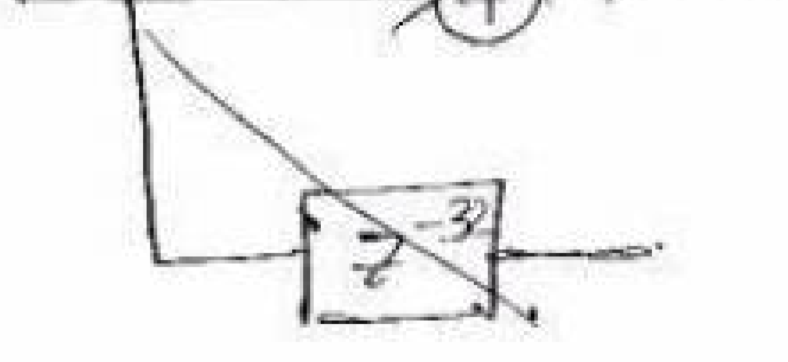
$$H\left(\frac{2\pi k}{32}\right) = \begin{cases} 1 & , k = 0, 1, 2 \\ 1/2 & , k = 3 \\ 0 & , k = 4, 5, \ldots 15 \end{cases}$$

Ans - $H_1(z) = \frac{1}{M}\left(1 - z^{-M}e^{j\pi\alpha}\right) = \frac{1}{32}\left(1 - z^{-32}\right)$

$$= \frac{H(0)}{1-z^{-1}} + \frac{H(16)}{1+z^{-1}} + \sum_{k=1}^{15} \frac{A(k) + B(k)z^{-1}}{1 - 2\cos\left(\frac{2\pi k}{32}\right)z^{-1} + z^{-2}}$$

$A(k) = H(k) + H(M-k)$

$$B(k) = H(k)e^{-j\pi k/M} + H(M-k)e^{j2\pi k/M}$$



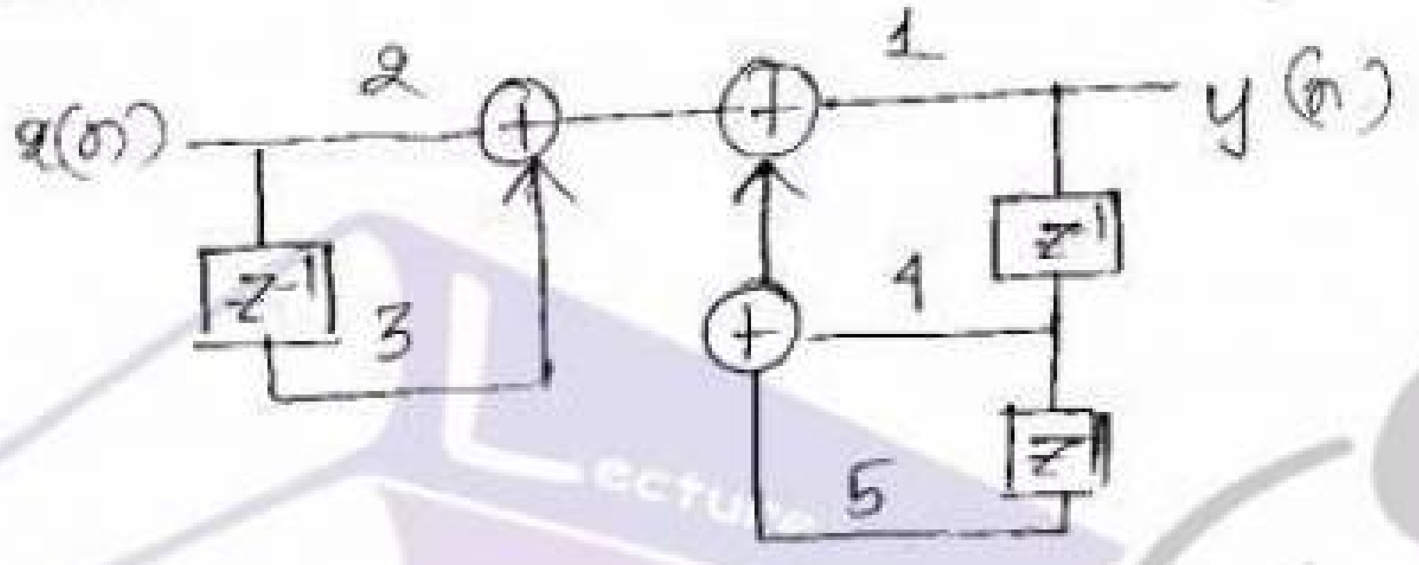
) $\cancel{y(n)}$ $y(n) = 5y(n-2) + 4y(n-1) + 2x(n) + 3x(n-1)$

Ans :- $Y(z) = 5z^{-2}Y(z) + 4z^{-1}Y(z) + 2X(z) + 3z^{-1}X$

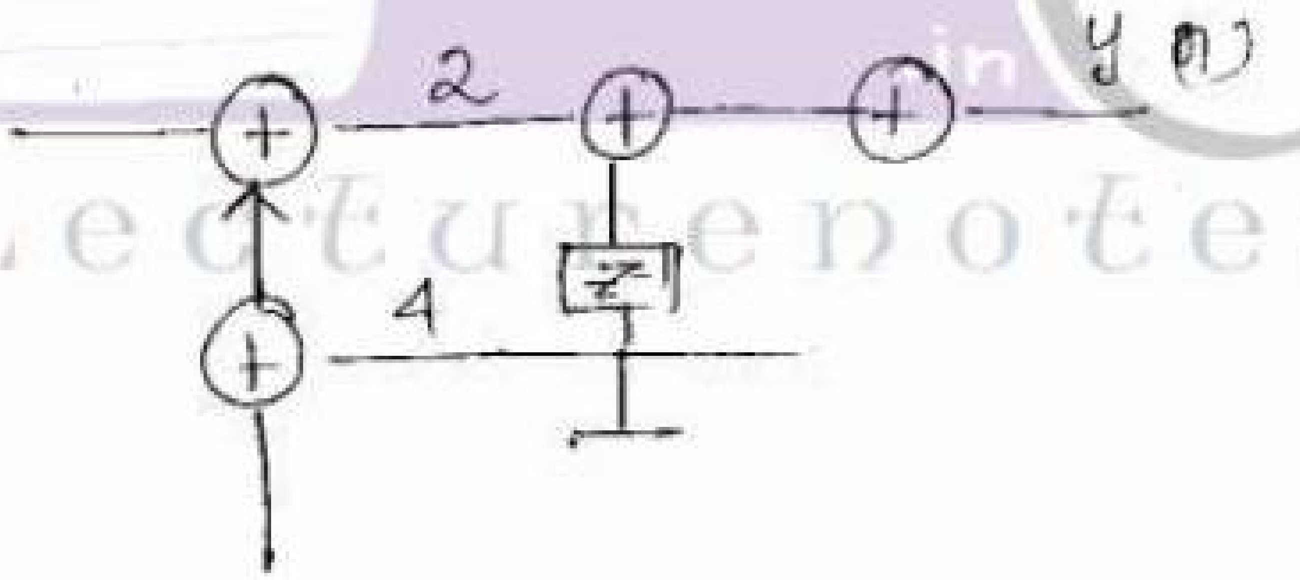
$\Rightarrow Y(z)(1 - 5z^{-2} - 4z^{-1}) = (2 + 3z^{-1})X(z)$.

$\Rightarrow \frac{Y(z)}{X(z)} = H(z) = \frac{2 + 3z^{-1}}{1 - 4z^{-1} - 5z^{-2}}$.

Direct form - 1 :-



Direct form - II



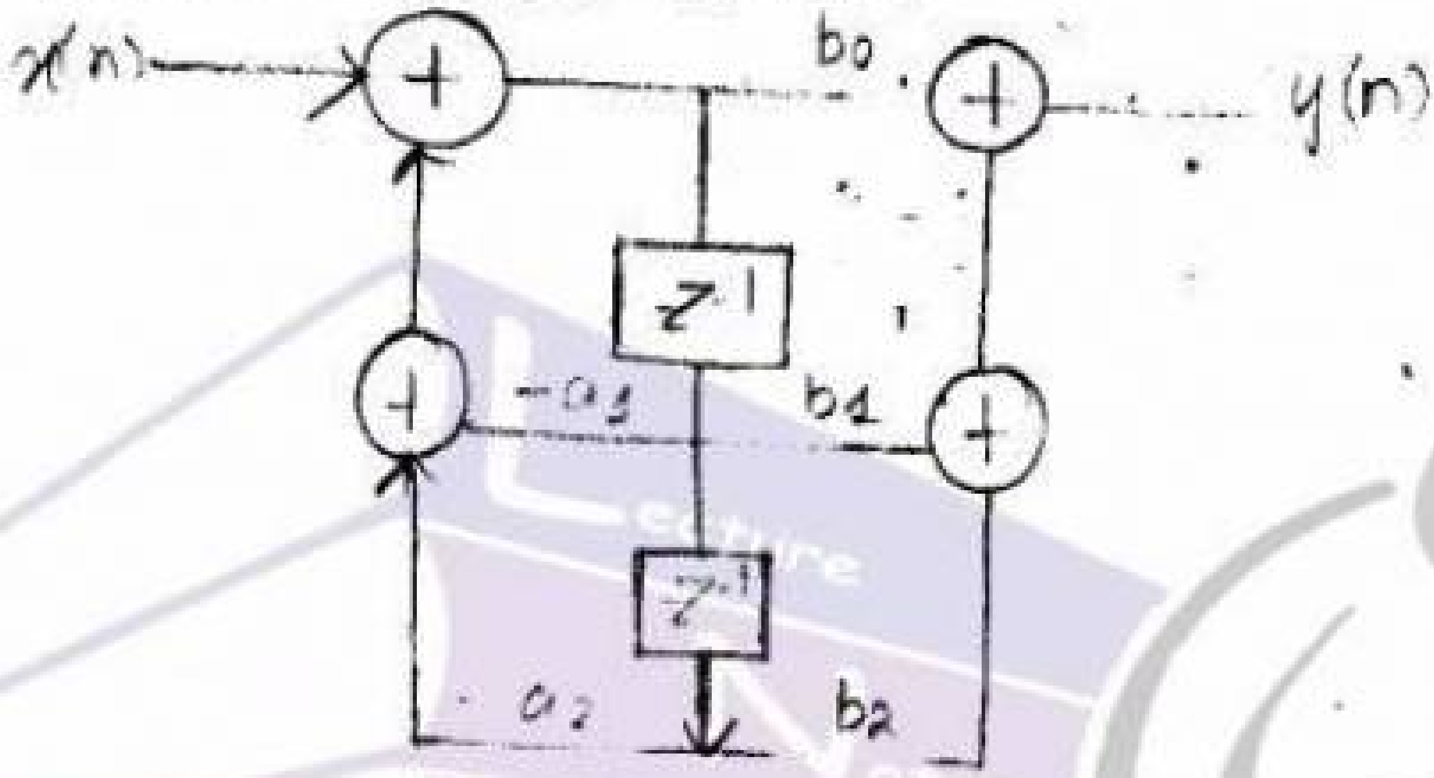
If $H(z) = \dfrac{\sum\limits_{k=0} b_k z^{-k}}{1 + \sum\limits_{k=1}^{N} a_k z^{-k}}$

then Direct-form-1 needs $(M+N+1)$ memory

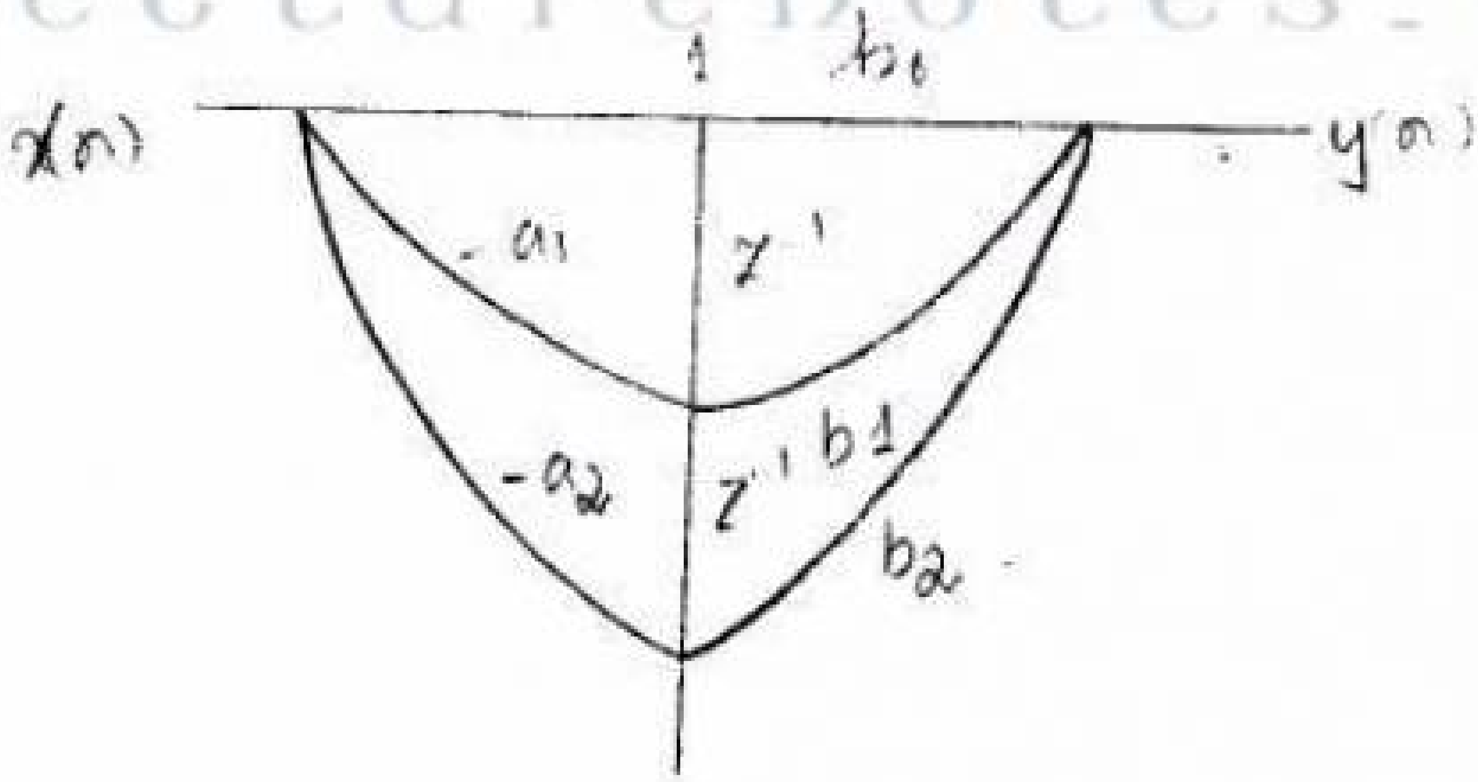
DF-II needs $\max(M,N)$ memory

DT-23.04.11

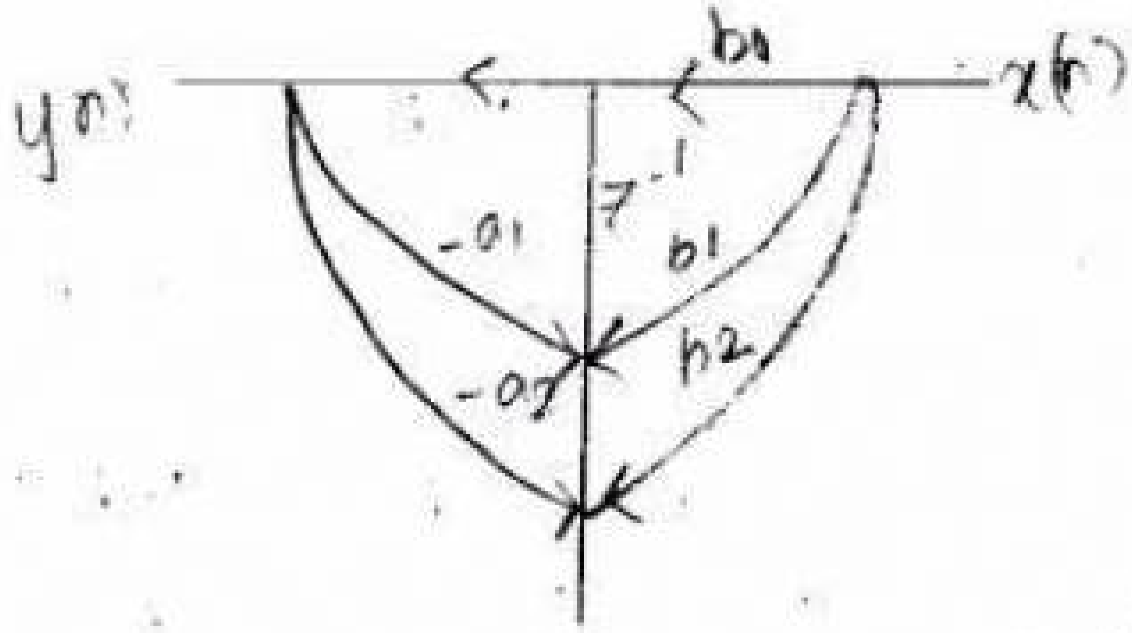
SIGNAL FLOW GRAPH :

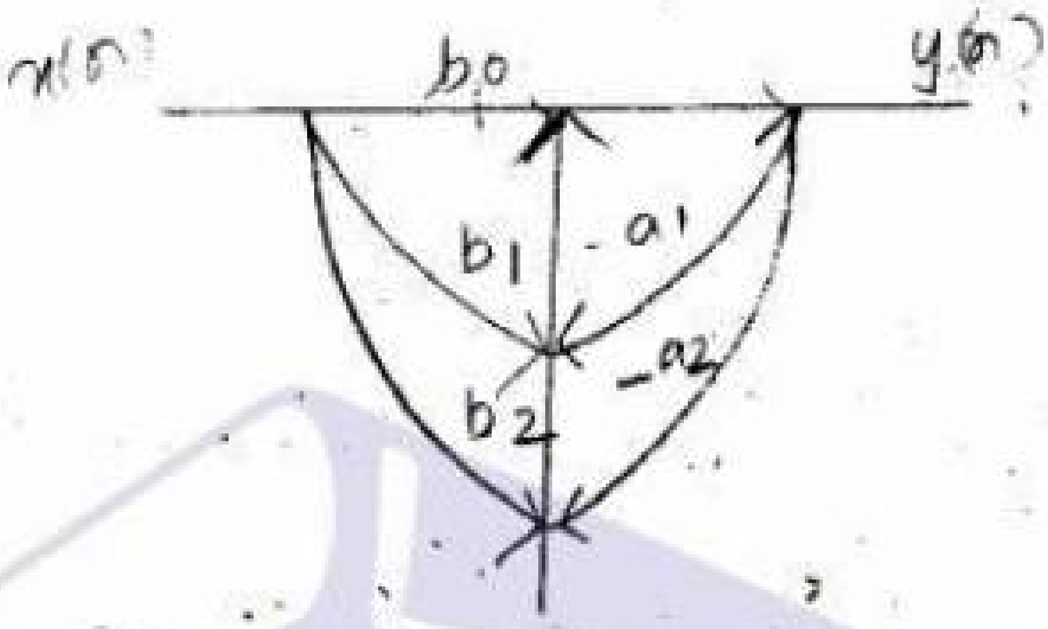


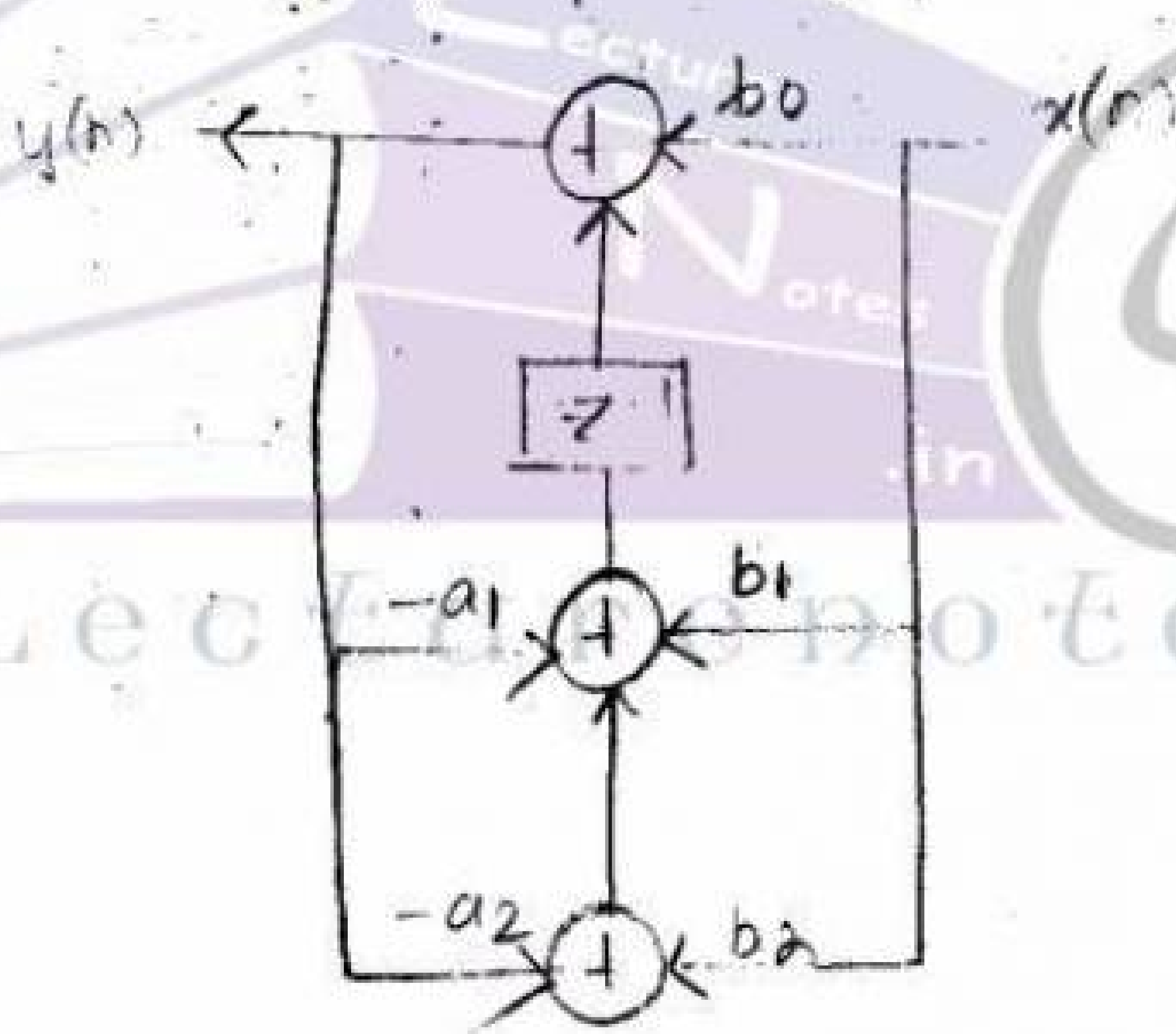
$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

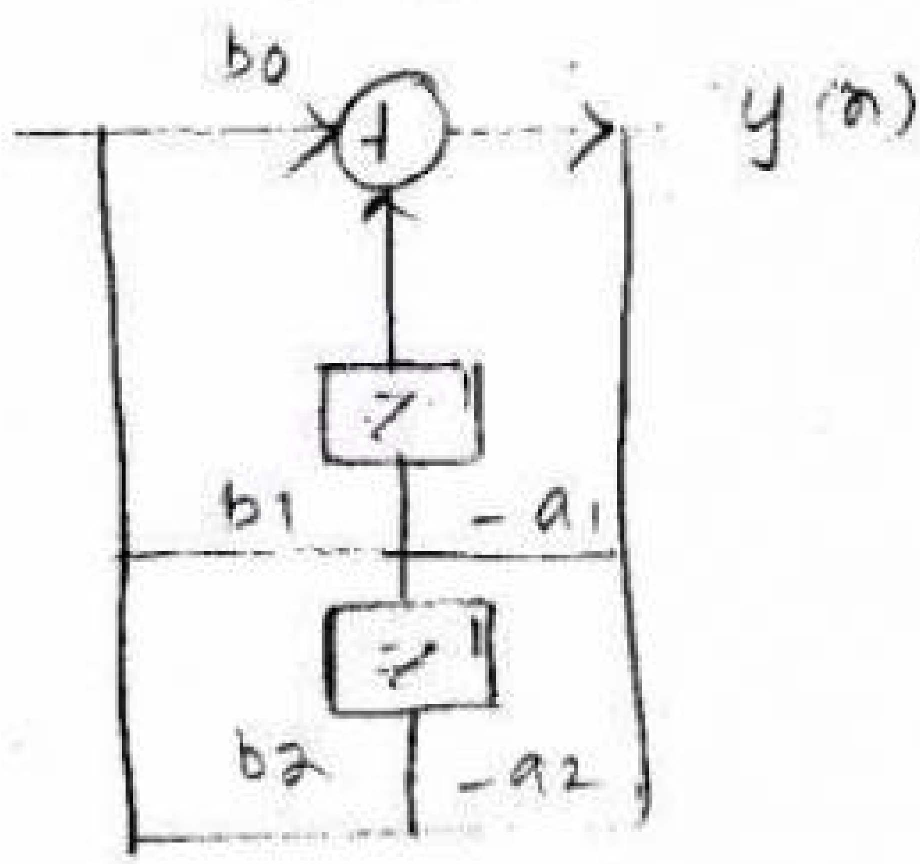


Transposed Structure :-









$$H(z) = \frac{(1-z_1 z^{-1})(1-z_2 z^{-1})(1-z_3 z^{-1})(1-z_4 z^{-1})}{(1-p_1 z^{-1})(1-p_2 z^{-1})(1-p_3 z^{-1})(1-p_4 z^{-1})}$$

$$= \frac{(1-z_1 z^{-1})(1-z_2 z^{-1})(1-z_3 z^{-1})(1-z_4 z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})(1-p_3 z^{-1})(1-p_4 z^{-1})}$$

$$= \frac{(1-z_1 z^{-1})(1-z_3 z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})} \times \frac{(1-z_2 z^{-1})(1-z_4 z^{-1})}{(1-p_3 z^{-1})(1-p_4 z^{-1})}$$

Q) Find the cascaded realisation of

$$H(z) = \frac{10\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)\left(1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right)\left(1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right)}$$

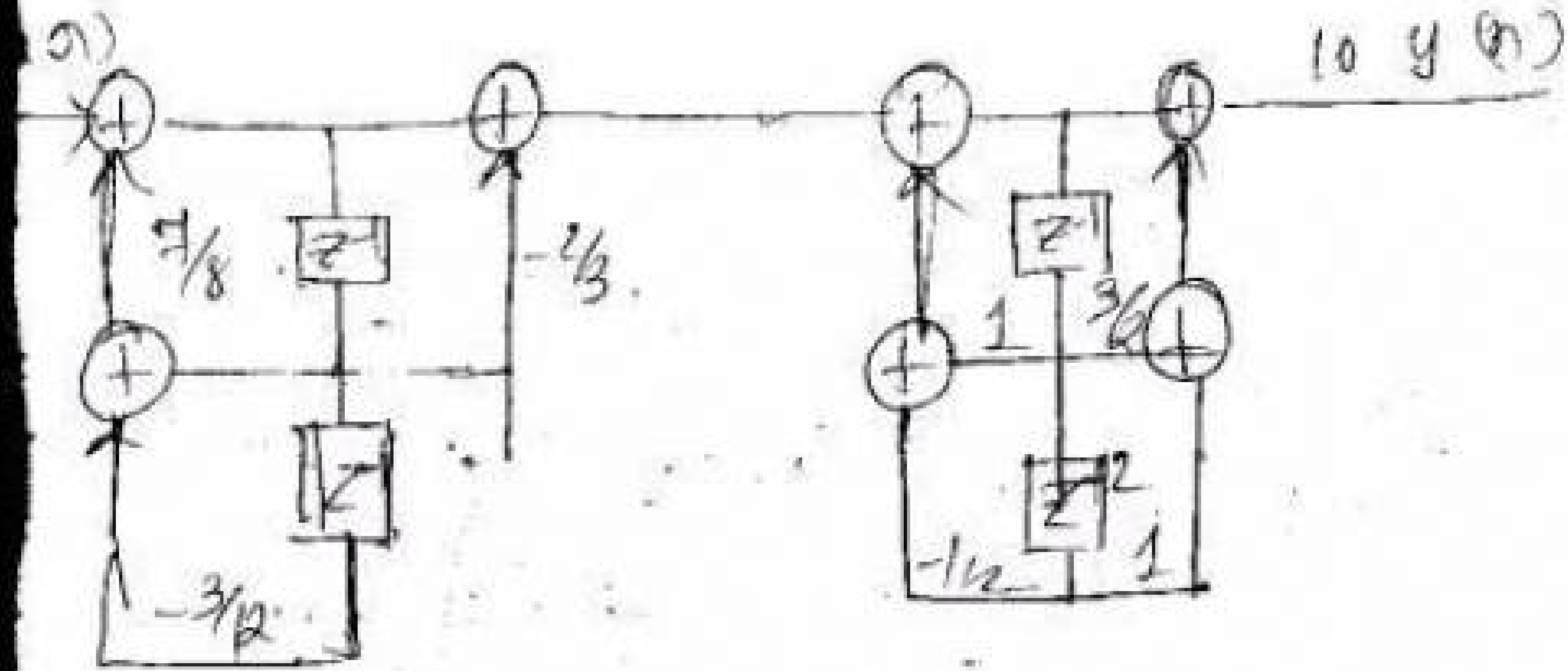
Ans

$$\frac{1-\frac{2}{3}z^{-1}}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)} \times \frac{10\left(1-\frac{1}{2}z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right)\left(1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right)}$$

$$H_1(z) = \frac{1-\frac{2}{3}z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}}$$

$$= \cancel{\frac{1+\frac{3}{2}z^{-1}+z^{-2}}{}}$$

$$= \frac{1+\frac{3}{2}z^{-1}-z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$



$$H(z) = C + \sum_{k=1}^{N} \frac{A_k}{1+P_k z^{-1}}$$

~~$\{P_k\}$ poles~~ ~~pon~~ $\{A_k\}$ = ~~Impu~~ partial fraction coefficient
$\to 1$ ~~P_k = lorem~~

$C \to$ constant. ~~$C = b_N$~~ $C = \frac{b_N}{a_N}$

~~H(z)~~ $H(z) = C + \frac{A_1}{1-P_1 z^{-1}} + \frac{A_2}{1-P_1 z^{-1}}$

~~$H(z) = C + A$~~

Parallel

~~H(z) = 10(1 - ...)~~

~~H(z) = 10(...)~~

$$H(z) = \frac{10\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1+\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)\left(1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right)\left(1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right)}$$

$$= \frac{A_1}{\left(1-\frac{3}{4}z^{-1}\right)} + \frac{A_2}{\left(1-\frac{1}{8}z^{-1}\right)} + \frac{A_3}{\left(1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right)} + \frac{A_4}{1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}}$$

$A_1 = 2.93$, $A_2 = -17.68$, $A_3 = 12.25 - j14.57$

$A_4 = 12.25 + j14.57$

$$H(z) = \frac{2.93}{1-\frac{3}{4}z^{-1}} + \frac{17.68}{1-\frac{1}{8}z^{-1}} - \frac{-14.75 - 12.90z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}}$$

$$H_2(z) = \frac{12.25 - j14.57}{1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}} + \frac{12.25 + j14.57}{1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}}$$

$$= \frac{24.50 + 26.82z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

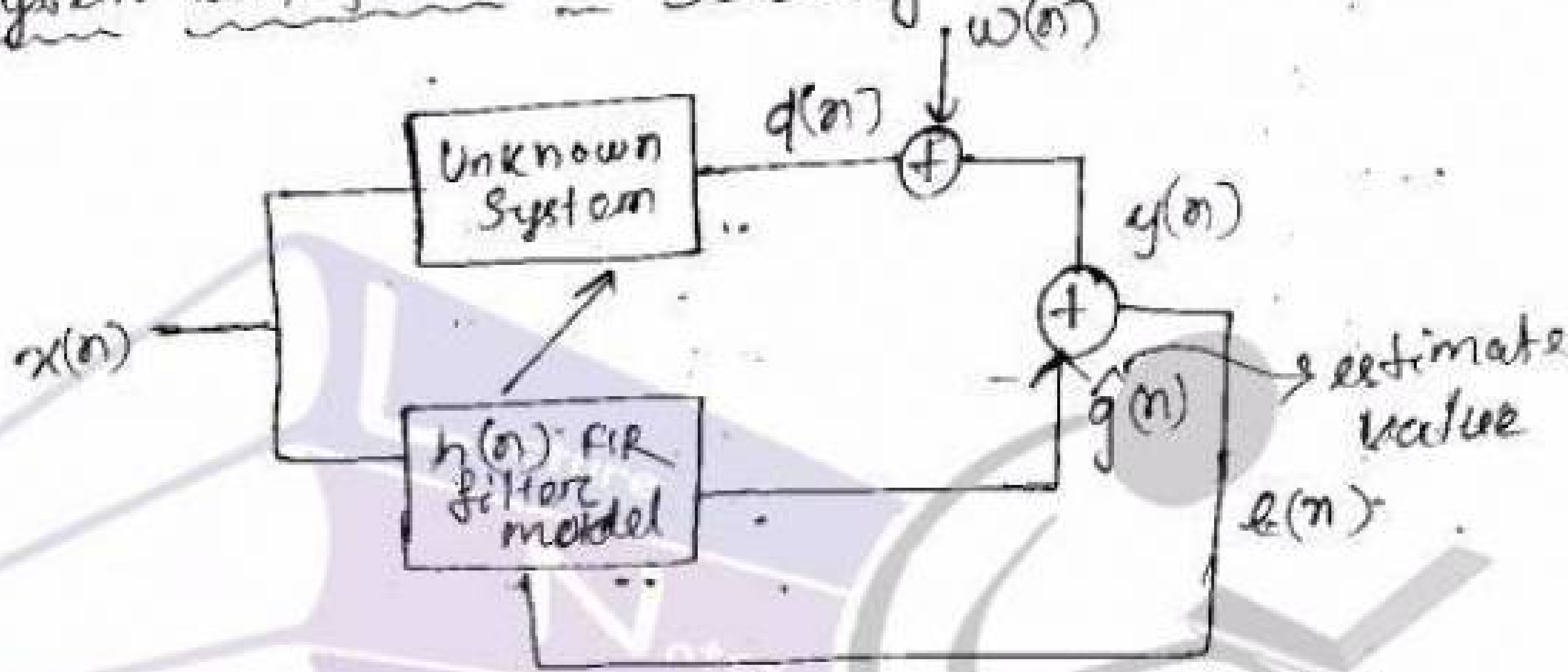
DT - 25.01.11

Adaptive filter :-

Applications :-

→ Adaptive Antenna
→ Digital Communication System (Channel equalisatio
→ ANC (Adaptive Noise Cascade)
→ System modelling

System identification & modelling :-



$$\hat{y}(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k)$$

$$e(n) = y(n) - \hat{y}(n)$$

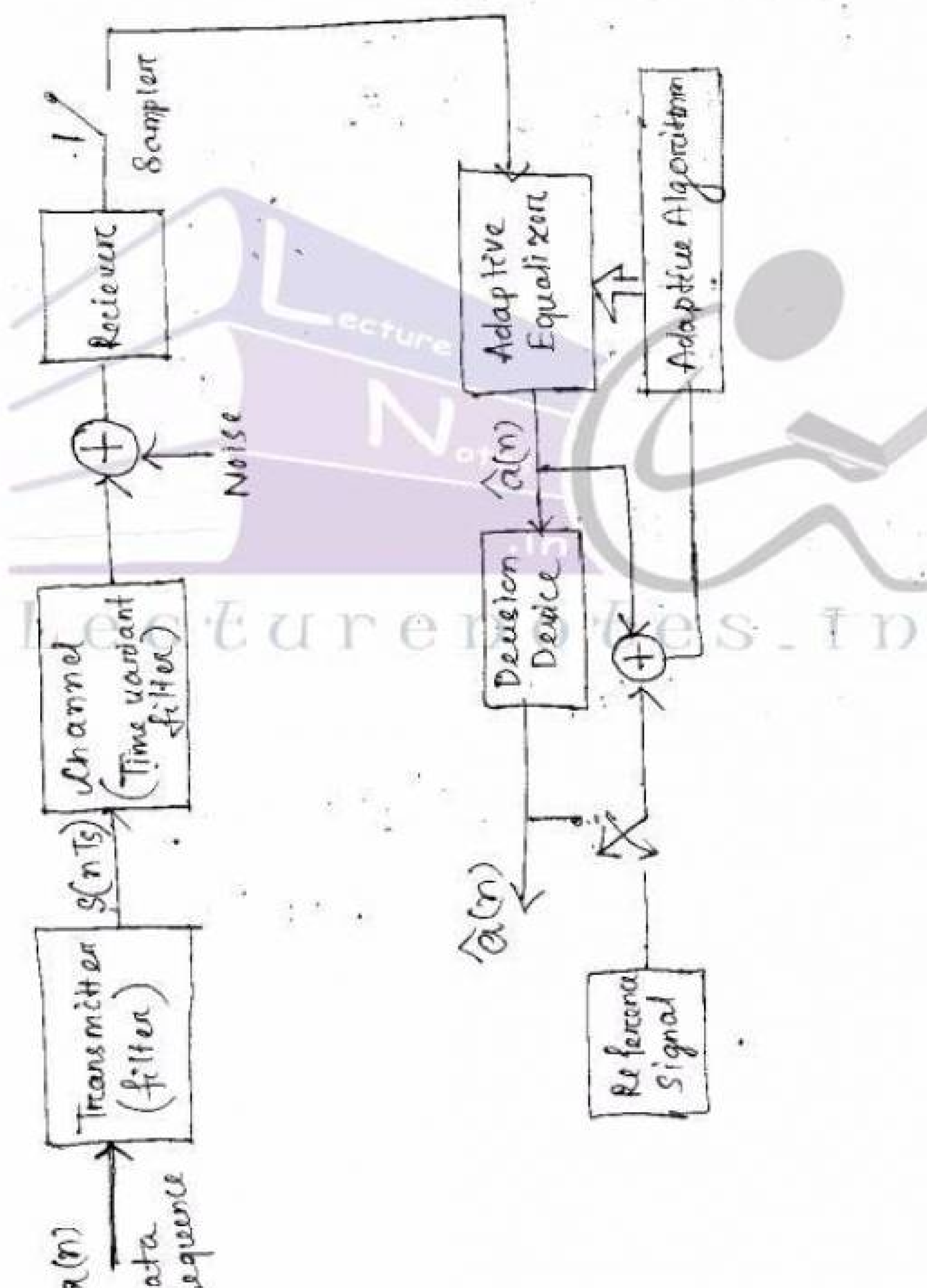
$$\mathcal{E} = \sum_{n=0}^{N} (e(n))^2$$

$$= \sum_{n=0}^{N} \left[y(n) - \sum_{k=0}^{M-1} h(k)\, x(n-k) \right]^2$$

$N+1 \leftarrow$ no. of observation

Adaptive Channel Equalisation :-

→ ~~Sybo~~ Symbol contains set of bits



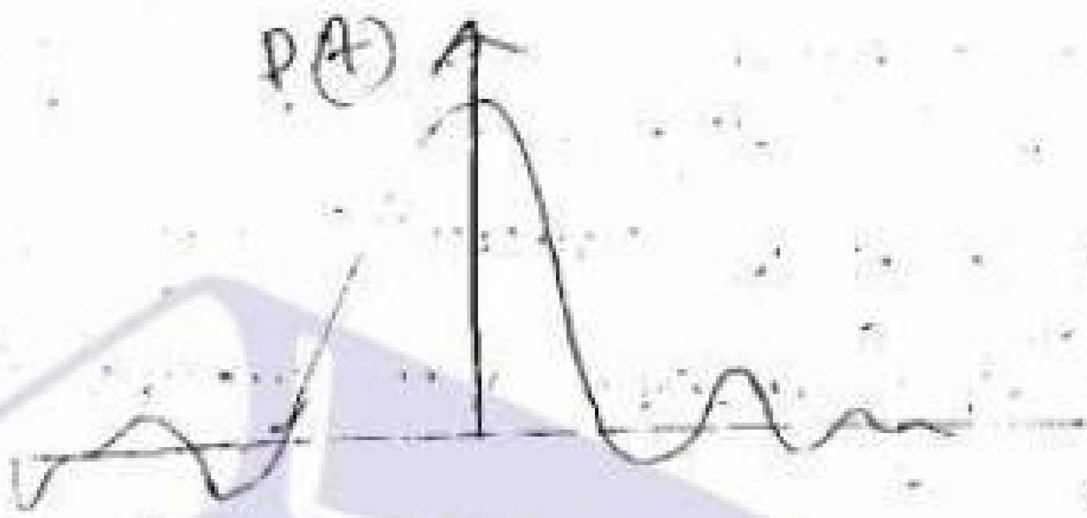
Let the input symbols are $a(n)$.
with symbol rate $\left(\frac{1}{T_s}\right)$

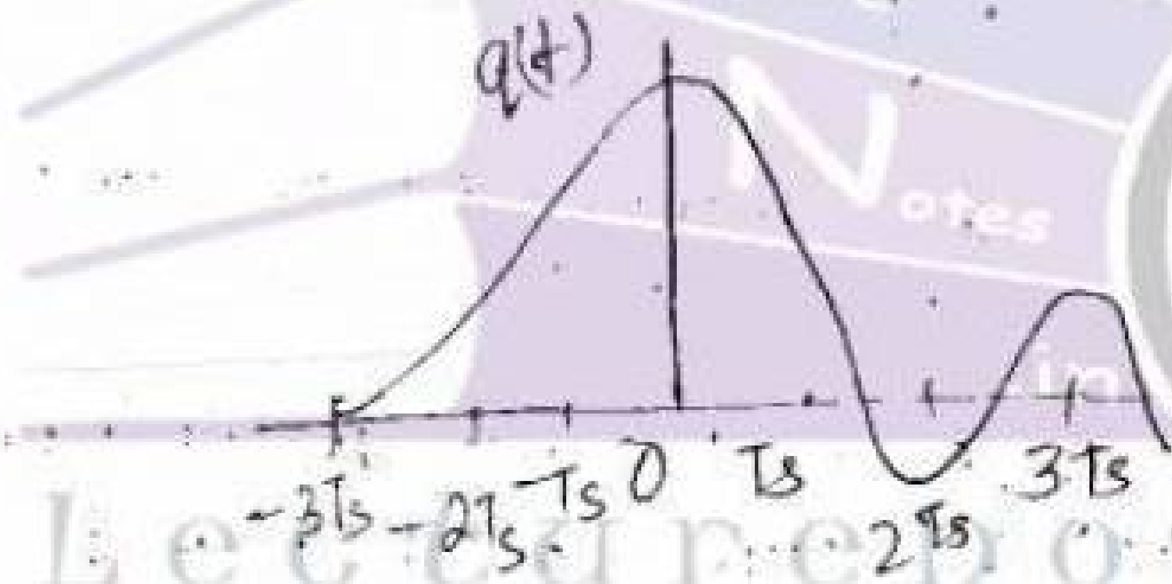
$a(n)$ multilevel symbol that can take values
$\pm 1, \pm 3, \pm 5, \ldots \pm(K+1)$

$K \rightarrow$ possible no. of symbol values

$$s(t) = \sum_{K=0}^{\infty} a(K)\, p(t - KT_s)$$

$$s(nT_s) = a(n)$$





$$x(nT_s) = \sum_{K=0}^{\infty} a(K)\, q(nT_s - KT_s) + w(nT_s)$$

$$= a(n)\, q(0) + \sum_{\substack{K=0 \\ K \neq n}}^{\infty} a(K)\, q(nT_s - KT_s)$$