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Department of Mathematics

Lecture Notes

Transform Calculus, Fourier Series & Numerical
Techniques (18MAT31)

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Module - 1

LAPLACE TRANSFORM

Definition:- Changing the function from one domain to another domain without changing its actual value is called transformation.

Mathematically, Suppose $F(t)$ be a real valid function defined on $[-\infty, +\infty]$ for any kernel or a function $K(s, t)$ such that $T[F(t)] = \int_{-\infty}^{\infty} K(s, t) \cdot F(t) dt = f(s)$

is called the Transformation of $F(t)$

where T is called the Transformation operator

s is real (or) complex

If the kernel $k(s, t)$ is convergent, we have

$$K(s, t) = \begin{cases} e^{-st}, & t > 0 \\ 0, & t < 0 \end{cases}$$

$$\text{then } T[F(t)] = \int_0^{\infty} e^{-st} F(t) dt =$$

$L[F(t)] = f(s)$ is called the Laplace transform of $F(t)$ in the positive time t , where

L is called the Laplace transform operator.

If $K(s, t) = e^{ist}$, then

$\Rightarrow T[F(t)] = \int_{-\infty}^{\infty} e^{ist} F(t) dt = F[F(t)] = f(s)$ is called the Fourier transform of $F(t)$.

where as F is the Fourier transformation operator.

Notations:- Suppose the given functions are in the capitals of alphabets $F(t), G(t), H(t), \dots$

then their Laplace transforms are $f(s), g(s), h(s), \dots$

If the functions are small letters like $f(t), g(t), h(t), \dots$ then their laplace transforms to be in the form $F(s), G(s), H(s), \dots$

Properties of Laplace transforms:-

1. Suppose $L[F(t)] = f(s)$, $L[G(t)] = g(s)$ and then for any c_1 and c_2 such that $L[c_1 F(t) + c_2 G(t)] = c_1 L[F(t)] + c_2 L[G(t)]$.
2. If $L[F(t)] = f(s)$, then

$$a) L[e^{at} F(t)] = f(s-a)$$

$$b) L[e^{-at} F(t)] = f(s+a)$$

$$3. \text{ If } L[F(t)] = f(s), \text{ then } L[t^n F(t)] = (-1)^n \frac{d^n f(s)}{ds^n}$$

$$4. \text{ If } L[F(t)] = f(s), \text{ then } L\left[\frac{F(t)}{t}\right] = \int_s^\infty f(s) ds.$$

Standard Results of Laplace transform:-

$$1) \text{ Let } F(t) = 1$$

$$\text{WKT } L[F(t)] = \int_0^\infty e^{-st} F(t) dt$$

$$L[1] = \int_0^\infty e^{-st} 1 dt$$

$$= \int_0^\infty e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^\infty$$

$$= -\frac{1}{s} [e^{-st}]_0^\infty$$

$$= -\frac{1}{s} [e^{-\infty} - e^0]$$

$$= -\frac{1}{s} [0 - 1]$$

$$\boxed{L[1] = \frac{1}{s}}$$

$$2) \text{ Let } F(t) = e^{at}$$

$$L[F(t)] = \int_0^\infty e^{-st} e^{at} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$$

$$= \frac{-1}{s-a} \left[e^{-(s-a)t} \right]_0^{\infty}$$

$$= \frac{-1}{s-a} (e^{-\infty} - e^0)$$

$$= \frac{-1}{s-a} (0 - 1)$$

$$\boxed{L[e^{at}] = \frac{1}{s-a}}$$

3) Let $F(t) = e^{-at}$

$$L[e^{-at}] = \int_0^{\infty} e^{-st} e^{-at} dt$$

$$= \int_0^{\infty} e^{-(s+a)t} dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty}$$

$$= -\frac{1}{s+a} \left[e^{-(s+a)t} \right]_0^{\infty}$$

$$= -\frac{1}{s+a} (e^{-\infty} - e^0)$$

$$= -\frac{1}{s+a} (0 - 1)$$

$$\boxed{L[e^{-at}] = \frac{1}{s+a}}$$

4) $L[t^n] = \frac{n!}{s^{n+1}}, \forall n \in \mathbb{Z}^+$

$$= \frac{n!}{s^{n+1}} \quad \forall n \in \mathbb{Q}$$

5) Let $F(t) = \cosh at$

$$F(t) = \frac{e^{at} + e^{-at}}{2}$$

$$\begin{aligned}
 L[F(t)] &= \frac{1}{2} L[e^{at} + e^{-at}] \\
 &= \frac{1}{2} \{ L(e^{at}) + L(e^{-at}) \} \\
 &= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right] \\
 &= \frac{1}{2} \left[\frac{s+a+s-a}{s^2-a^2} \right] \\
 &= \frac{1}{2} \frac{2s}{s^2-a^2}
 \end{aligned}$$

$$L[F(t)] = L[\cosh at] \Rightarrow \frac{s}{s^2 - a^2} //$$

$$||14 F(t) = \sinh at$$

$$F(t) = \frac{e^{at} - e^{-at}}{2}$$

$$\begin{aligned}
 L[F(t)] &= \frac{1}{2} L[e^{at} - e^{-at}] \\
 &= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right] \\
 &= \frac{1}{2} \left[\frac{s+a-s+a}{s^2-a^2} \right] \\
 &= \frac{1}{2} \frac{2a}{s^2-a^2}
 \end{aligned}$$

$$L[F(t)] = L[\sinh at] = \frac{a}{s^2 - a^2}$$

$$\text{WKT } e^{i\theta} = \cos \theta + i \sin \theta$$

$$\Rightarrow e^{iat} = \cos at + i \sin at$$

$$\Rightarrow L[e^{iat}] = L[\cos at] + i L[\sin at] \quad \text{--- (1)}$$

$$\therefore L[e^{iat}] = L[e^{(ai)t}]$$

$$= \frac{1}{s-ai} = \frac{s+ai}{(s-ai)(s+ai)}$$

$$= \frac{s+ai}{s^2 - (ai)^2}$$

$$L[e^{iat}] = \frac{S+ai}{S^2+a^2}$$

$$L[\cos at] + i L[\sin at] = \frac{S}{S^2+a^2} + i \frac{a}{S^2+a^2}$$

$$\therefore L[\cos at] = \frac{S}{S^2+a^2}$$

$$L[\sin at] = \frac{a}{S^2+a^2}$$

Find the laplace transform for the following functions:

① $\sin 5t \cdot \cos 2t$

② $\sin t \cdot \sin 2t \cdot \sin 3t$

③ $\cos t \cdot \cos 2t \cdot \cos 3t$

④ $(3t+4)^3 + 5t$

⑤ $t^{-5/2} + t^{5/2}$

⑥ $e^{-2t} \sinh(4t)$

① Let $F(t) = \sin 5t \cdot \cos 2t$

$$F(t) = \frac{1}{2} [2 \sin 5t \cdot \cos 2t]$$

$$= \frac{1}{2} [\sin(5t+2t) + \sin(5t-2t)]$$

$$= \frac{1}{2} [\sin(7t) + \sin(3t)]$$

$$L[F(t)] = \frac{1}{2} L[\sin 7t + \sin 3t]$$

$$\Rightarrow f(s) = \frac{1}{2} \left[\frac{7}{s^2+49} + \frac{3}{s^2+9} \right]$$

② Let $F(t) = \sin t \cdot \sin 2t \cdot \sin 3t$

$$\begin{aligned}
&= \frac{1}{2} [2 \sin 2t \sin t] \sin 3t \\
&= \frac{1}{2} [\cos(2t-t) - \cos(2t+t)] \sin 3t \\
&= \frac{1}{2} [\cos t - \cos 3t] \sin 3t \\
&= \frac{1}{2} [\sin 3t \cdot \cos t - \sin 3t \cdot \cos 3t] \\
&= \frac{1}{4} [2 \sin 3t \cdot \cos t - 2 \sin 3t \cdot \cos 3t] \\
&= \frac{1}{4} [\sin(4t) + \sin 2t - \sin 6t] \\
&= \frac{1}{4} \mathcal{L}[\sin 2t + \sin 4t - \sin 6t] \\
&= \frac{1}{4} [\mathcal{L}(\sin 2t) + \mathcal{L}(\sin 4t) - \mathcal{L}(\sin 6t)] \\
&= \frac{1}{4} \left[\frac{2}{s^2+4} + \frac{4}{s^2+16} - \frac{6}{s^2-36} \right] //
\end{aligned}$$

③ Let $F(t) = \cos t \cdot \cos 2t \cdot \cos 3t$

$$\begin{aligned}
&= \frac{1}{2} [2 \cos 2t \cos t] \cos 3t \\
&= \frac{1}{2} [\cos 3t + \cos t] \cos 3t \\
&= \frac{1}{2} [\cos^2 3t + \cos 3t \cdot \cos t] \\
&= \frac{1}{2} \left[\frac{1 + \cos 2(3t)}{2} + \frac{2 \cos 3t \cdot \cos t}{2} \right]
\end{aligned}$$

$$F(t) = \frac{1}{4} [1 + \cos 6t + \cos 4t + \cos 2t]$$

$$\mathcal{L}[F(t)] = \frac{1}{4} \mathcal{L}[1 + \cos 6t + \cos 4t + \cos 2t]$$

$$F(t) = f(s) = \frac{1}{4} \left[\frac{1}{s} + \frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4} \right] //$$

④ Let $F(t) = (3t+4)^3 + s^t$

$$= (3t)^3 + 4^3 + 3(3t)^2 + 3(3t)(4)^2 + e^{109e(5^t)}$$

$$F(t) = 27t^3 + 64 + 108t^2 + 144t + e^{109e(5^t)}$$

$$\Rightarrow L[F(t)] = 27L[t^3] + 64L[1] + 108L[t^2] + 144L[t] + L[e^{109e(5^t)}]$$

$$f(s) = 27 \frac{3!}{s^4} + \frac{64}{s} + \frac{216}{s^3} + \frac{144}{s^2} + \frac{1}{s-1095} //$$

$$\textcircled{5} \quad F(t) = e^{-2t} \sinh 4t$$

$$= e^{-2t} \left[\frac{e^{4t} - e^{-4t}}{2} \right]$$

$$F(t) = \frac{1}{2} [e^{2t} - e^{-6t}]$$

$$L[F(t)] = \frac{1}{2} L[e^{2t} - e^{-6t}]$$

$$= \frac{1}{2} \left[\frac{1}{s-2} - \frac{1}{s+6} \right] //$$

$$\textcircled{6} \quad \text{Let } F(t) = t^{-5/2} + t^{5/2}$$

$$L[F(t)] = L[t^{-5/2}] + L[t^{5/2}]$$

$$f(s) = \frac{\sqrt{-\frac{5}{2}+1}}{s^{-5/2+1}} + \frac{\sqrt{5/2+1}}{s^{5/2+1}}$$

$$= \frac{\sqrt{-\frac{3}{2}}}{s^{3/2}} + \frac{\sqrt{7/2}}{s^{7/2}}$$

$$= \frac{\left(-\frac{3}{2}\right) \sqrt{-\frac{3}{2}}}{\left(-\frac{3}{2}\right) s^{3/2}} + \frac{\left(\frac{7}{2}-1\right) \sqrt{\frac{7}{2}-1}}{s^{7/2}}$$

$$\begin{aligned}
&= \frac{\sqrt{\frac{-3}{2} + 1}}{\left(-\frac{3}{2}\right) S^{-3/2}} + \frac{\left(\frac{5}{2}\right) \sqrt{\frac{5}{2}}}{S^{7/2}} \\
&= \frac{\sqrt{\frac{-1}{2}}}{\left(-\frac{3}{2}\right) S^{-3/2}} + \frac{\left(\frac{5}{2}\right) \left(\frac{3}{2}\right) \sqrt{\frac{1}{2}}}{S^{7/2}} \\
&= \frac{\left(-\frac{1}{2}\right) \sqrt{-\frac{1}{2}}}{\left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) S^{-3/2}} + \frac{\left(\frac{5}{2}\right) \left(\frac{3}{2}\right) \left(\frac{1}{2}\right) \sqrt{\frac{1}{2}}}{S^{7/2}} \\
&= \frac{\sqrt{\frac{-1}{2} + 1}}{\frac{3}{4} S^{-3/2}} + \frac{15}{8} \frac{\sqrt{\frac{1}{2}}}{S^{7/2}} \\
&= \frac{\sqrt{\frac{1}{2}}}{\frac{3}{4} S^{-3/2}} + \frac{15}{8} \frac{\sqrt{\frac{1}{2}}}{S^{7/2}} \\
f(s) &= \frac{4}{3} \frac{\sqrt{\pi}}{S^{-3/2}} + \frac{15}{8} \frac{\sqrt{\pi}}{S^{7/2}} //
\end{aligned}$$

Find the Laplace transform of the following functions

- ① $e^{-2t} (2 \cos t - \sin t)$
- ② $e^{-t} \cos^2 3t$
- ③ $e^{3t} \sin t \cdot \sin 3t$
- ④ $e^{-4t} t^{-5/2}$
- ⑤ $e^{-3t} \sin at$
- ⑥ $e^{2t} (\sin 3t \cos 5t)$
- ⑦ $\left(\frac{4t+5}{e^{2t}}\right)^2$

① Let $F(t) = 2 \cos 5t - \sin 5t$

$$L[F(t)] = 2L[\cos 5t] - L[\sin 5t]$$

$$f(s) = 2 \left[\frac{s}{s^2 + 25} \right] - \left[\frac{5}{s^2 + 25} \right]$$

$$f(s) = \frac{2s - 5}{s^2 + 25}$$

$$\therefore \text{WKT } L[e^{at} F(t)] = f(s-a)$$

$$L[e^{-2t} F(t)] = f(s+2)$$

$$\Rightarrow L[e^{-2t} (2 \cos 5t - \sin 5t)] = \frac{2(s+2) - 5}{(s+2)^2 + 25}$$

$$= \frac{2s + 4 - 5}{s^2 + 4s + 4 + 25}$$

$$\Rightarrow L[e^{-2t} (2 \cos 5t - \sin 5t)] = \frac{2s - 1}{s^2 + 4s + 29} //$$

② Let $f(t) = \cos^2 3t$

$$F(t) = \frac{1 + \cos 6t}{2}$$

$$L[F(t)] = \frac{1}{2} L[1 + \cos 6t]$$

$$= \frac{1}{2} \{ L(1) + L(\cos 6t) \}$$

$$f(s) = \frac{1}{2} \left[\frac{1}{s} + \frac{s}{s^2 + 36} \right]$$

$$\therefore \text{WKT } L[e^{at} F(t)] = f(s-a)$$

$$L[e^{-t} F(t)] = f(s+1)$$

$$L[e^{-t} \cos^2 3t] = \frac{1}{2} \left[\frac{1}{s+1} + \frac{s+1}{(s+1)^2 + 36} \right] //$$

③ Let $F(t) = \sin 2t \cdot \sin 3t$

$$F(t) = \frac{1}{2} (2 \sin 2t \cdot \sin 3t)$$

$$F(t) = \frac{1}{2} [\cos 2t - \cos 8t]$$

$$L[F(t)] = \frac{1}{2} L[\cos 2t - \cos 8t]$$

$$f(s) = \frac{1}{2} \left[\frac{s}{s^2+4} - \frac{s}{s^2+64} \right]$$

WKT $L[e^{at} F(t)] = f(s-a)$

$$L[e^{3t} F(t)] = f(s-3)$$

$$L[e^{3t} \sin 2t \sin 3t] = \frac{1}{2} \left[\frac{(s-3)}{(s-3)^2+4} - \frac{(s-3)}{(s-3)^2+64} \right] //$$

④ Let $e^{-4t} t^{-5/2}$

$$F(t) = t^{-5/2}$$

$$L[F(t)] = L[t^{-5/2}]$$

$$f(s) = \frac{\left[\frac{-5}{2} + 1 \right]}{s^{-5/2+1}}$$

$$= \frac{\left[\frac{-3}{2} \right]}{s^{-3/2}}$$

$$= \frac{\left(\frac{-3}{2} \right) \left[\frac{-3}{2} \right]}{\left(\frac{-3}{2} \right) s^{-3/2}}$$

$$= \frac{\left[\frac{-3}{2} + 1 \right]}{\left(\frac{-3}{2} \right) s^{-3/2}}$$

$$= \frac{\left[\frac{-1}{2} \right]}{\left(\frac{-3}{2} \right) s^{-3/2}}$$

$$= \frac{\left(-\frac{1}{s}\right) \sqrt{-\frac{1}{s}}}{\left(-\frac{1}{s}\right) \left(-\frac{3}{s}\right) s^{-3/2}}$$

$$= \frac{\left(-\frac{1}{s}\right) \sqrt{-\frac{1}{s}}}{\frac{3}{4} s^{-3/2}}$$

$$= \frac{\sqrt{-\frac{1}{s}} + 1}{\frac{3}{4} s^{-3/2}}$$

$$f(s) = \frac{\sqrt{\pi}}{\frac{3}{4} s^{-3/2}}$$

$$L[e^{at} F(t)] = f(s-a)$$

$$L[e^{-4t} F(t)] = f(s+4)$$

$$L[e^{-4t} t^{3/2}] = \frac{\sqrt{\pi}}{\frac{3}{4} (s+4)^{-3/2}}$$

$$= \frac{4\sqrt{\pi}}{3(s+4)^{3/2}} //$$

⑤ Let $F(t) = \sin at$

$$L[F(t)] = L[\sin at]$$

$$f(s) = \frac{a}{s^2 + 4}$$

$$\text{WKT } L[e^{at} F(t)] = f(s-a)$$

$$L[e^{-3t} F(t)] = f(s+3)$$

$$L[e^{-3t} \sin at] = \frac{a}{(s+3)^2 + 4}$$

⑥

$$F(t) = \sin 3t \cdot \cos 5t$$

$$F(t) = \frac{1}{2} [\sin 3t \cdot \cos 5t]$$

$$= \frac{1}{2} [\sin 8t - \sin 2t]$$

$$L[F(t)] = \frac{1}{2} L[\sin 8t - \sin 2t]$$

$$= \frac{1}{2} \left[\frac{8}{s^2 + 64} - \frac{2}{s^2 + 4} \right]$$

$$\text{WKT } L[e^{at} F(t)] = f(s-a)$$

$$L[e^{-2t} F(t)] = f(s+2)$$

$$L[e^{-2t} \sin 3t \cdot \cos 5t] = \frac{1}{2} \left[\frac{8}{(s+2)^2 + 64} - \frac{2}{(s+2)^2 + 4} \right]$$

⑦

$$\frac{(4t+5)^2}{e^{4t}} = e^{-4t} (4t+5)^2$$

$$F(t) = (4t+5)^2$$

$$= 4t^2 + 5^2 + 2(4t)5$$

$$= (4t)^2 + 5^2 + 40t$$

$$L[F(t)] = 16 L[t^2] + 5^2 L[1] + 40 L[t]$$

$$f(s) = \frac{32}{s^3} + \frac{5^2}{s} + \frac{40}{s^2}$$

$$\text{WKT } L[e^{at} F(t)] = f(s-a)$$

$$L[e^{-4t} F(t)] = f(s+4)$$

$$L[e^{-4t} (4t+5)^2] = \frac{32}{(s+4)^3} + \frac{25}{(s+4)} + \frac{40}{(s+4)^2} //$$

Find the laplace transform of the following functions:

① $t \cos at$

② $t^2 \sin at$

③ $t^3 \cosh t$

④ $t^3 e^{4t} \cosh 3t$

⑤ $t e^{-2t} \sin 4t$

① Let $F(t) = \cos at$

$$L[F(t)] = L[\cos at]$$

$$f(s) = \frac{s}{s^2 + a^2}$$

$$\text{WKT } L[t^n F(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$$

$$L[t F(t)] = (-1)^1 \frac{d}{ds} f(s)$$

$$= -\frac{d}{ds} \left[\frac{s}{s^2 + a^2} \right]$$

$$= - \left[\frac{(s^2 + a^2)(1) - s(2s)}{(s^2 + a^2)^2} \right]$$

$$= - \left[\frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2} \right]$$

$$= - \left[\frac{a^2 - s^2}{(s^2 + a^2)^2} \right]$$

$$L[t \cos at] = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

②

$$\text{Let } F(t) = \sin at$$

$$L[F(t)] = L[\sin at]$$

$$f(s) = \frac{a}{s^2 + a^2}$$

$$\text{WKT } L[t^n F(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$$

$$L[t^2 F(t)] = (-1)^2 \frac{d^2}{ds^2} f(s)$$

$$L[t^2 \sin at] = \frac{d}{ds} \left[\frac{d}{ds} \left(\frac{a}{s^2 + a^2} \right) \right]$$

$$= \frac{d}{ds} \left[\frac{-2as}{(s^2 + a^2)^2} \right]$$

$$= -2a \frac{d}{ds} \left[\frac{s}{(s^2 + a^2)^2} \right]$$

$$= -2a \left[\frac{(s^2 + a^2)^2 (1) - s \cdot 2(s^2 + a^2)(2s)}{(s^2 + a^2)^4} \right]$$

$$= -2a \left[\frac{s^2 + a^2 - 4s^2}{(s^2 + a^2)^3} \right]$$

$$L[t^2 \sin at] = -2a \left[\frac{a^2 - 3s^2}{(s^2 + a^2)^3} \right] //$$

③

$$\text{Let } F(t) = \cos at$$

$$L[F(t)] = L[\cos at]$$

$$f(s) = \frac{s}{s^2 + a^2}$$

$$\text{WKT } L[t^n F(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$$

$$= (-1)^n \frac{d^n}{ds^n} f(s)$$

$$= (-1)^1 \frac{d}{ds} f(s)$$

$$= - \frac{d}{ds} \left(\frac{s}{s^2+4} \right)$$

$$= - \left[\frac{(s^2+4)(1) - 2s^2}{(s^2+4)^2} \right]$$

$$= - \left[\frac{s^2+4-2s^2}{(s^2+4)^2} \right]$$

$$= - \left[\frac{4-s^2}{(s^2+4)^2} \right]$$

$$\therefore \mathcal{L}[t \cos at] = \left[\frac{s^2-4}{(s^2+4)^2} \right] //$$

(4)

$$t \sin t$$

$$\text{Let } F(t) = \sin t$$

$$\mathcal{L}[F(t)] = \mathcal{L}(\sin t)$$

$$f(s) = \frac{1}{s^2+1}$$

WKT

$$\mathcal{L}[t^n F(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$$

$$= (-1)^1 \frac{d}{ds} f(s)$$

$$= - \frac{d}{ds} \left(\frac{1}{s^2+1} \right)$$

$$\mathcal{L}[t \sin t] = -1 \left[\frac{s^2+1-1(2s)}{(s^2+1)^2} \right]$$

$$L[t \sin t] = - \left[\frac{0 - 2s}{(s^2 + 1)^2} \right]$$

$$= \frac{2s}{(s^2 + 1)^2} //$$

⑤

$$t e^{-2t} \sin 4t$$

$$\text{Let } F(t) = t \sin 4t$$

$$L[F(t)] = - \frac{d}{ds} \left(\frac{4}{s^2 + 16} \right)$$

$$f(s) = \left[\frac{(s^2 + 16)0 - 4(2s)}{(s^2 + 16)^2} \right]$$

$$= \frac{8s}{(s^2 + 16)^2}$$

$$L[e^{-2t} F(t)] = f(s) \xrightarrow{s \rightarrow s+2}$$

$$= \frac{8(s+2)}{[(s+2)^2 + 16]^2} //$$

⑥

$$t e^{-t} \sin 4t$$

$$\text{Given } e^{at} t^n F(t) = t e^{-t} \sin 4t$$

$$F(t) = \sin 4t$$

$$L[F(t)] = L[\sin 4t]$$

$$f(s) = \frac{4}{s^2 + 16}$$

$$L[t \sin 4t] = (-1) \frac{d}{ds} \left[\frac{4}{s^2 + 16} \right]$$

$$L[t \sin 4t] = - \frac{[(s^2 + 16)(0) - 4(2s)]}{(s^2 + 16)^2}$$

$$= - \left[\frac{-8s}{(s^2 + 16)^2} \right]$$

$$L[t \sin 4t] = \frac{8s}{(s^2 + 16)^2}$$

$$L[e^{-t} \sin 4t] = f(s)$$

$\hookrightarrow s \rightarrow s+1$

$$L[e^{-t} t \sin 4t] = \frac{8(s+1)}{(s+1)^2 + 16^2} //$$

④

$$t e^{-3t} \sin 3t$$

$$\text{Given } e^{at} t^n f(t) = t e^{-3t} \sin 3t$$

$$f(t) = \sin 3t$$

$$L[f(t)] = L(\sin 3t)$$

$$f(s) = \frac{3}{s^2 + 9}$$

$$L[t \sin 3t] = (-1)^1 \frac{d}{ds} \left[\frac{3}{s^2 + 9} \right]$$

$$= - \left[\frac{(s^2 + 9)(0) - 3(2s)}{(s^2 + 9)^2} \right]$$

$$= - \left[\frac{-6s}{(s^2 + 9)^2} \right]$$

$$= \frac{6s}{(s^2 + 9)^2}$$

$$L[e^{-3t} \sin 3t] = f(s) \hookrightarrow s = s+3$$

$$L[e^{-4t} t \sin 3t] = \frac{6(s+4)}{[(s+4)^2 + 9]^2} //$$

⑧ $t e^{-4t} \sin 3t$
 Given $e^{at} t^n f(t) = t e^{-4t} \sin 3t$

$$f(t) = \sin 3t$$

$$L[f(t)] = L(\sin 3t)$$

$$f(s) = \frac{3}{s^2 + 9}$$

$$L[t \sin 3t] = -1 \frac{d}{ds} \left[\frac{3}{s^2 + 9} \right]$$

$$L[t \sin 3t] = - \left[\frac{(s^2 + 9)(0) - 3(2s)}{(s^2 + 9)^2} \right]$$

$$L[t \sin 3t] = \frac{6s}{(s^2 + 9)^2}$$

$$L[e^{-4t} \sin 3t] = f(s) \quad \hookrightarrow s = s + 4$$

$$L[e^{-4t} t \sin 3t] = \frac{6(s+4)}{[(s+4)^2 + 9]^2} //$$

⑨ $t e^{-2t} \cos 2t$

Given $e^{at} t^n f(t) = t e^{-2t} \cos 2t$

$$f(t) = \cos 2t$$

$$L[t \cos at] = - (1) \frac{d}{ds} f(s)$$

$$= - \left[\frac{s^2 + 4(1) - 4(2s)}{(s^2 + 4)^2} \right]$$

$$= - \left[\frac{s^2 + 4 - 8s}{(s^2 + 4)^2} \right]$$

$$= \frac{s^2 - 4}{(s^2 + 4)^2}$$

$$L[e^{-2t} \cos at] = f(s)_{L \rightarrow s = s+2}$$

$$L[e^{-2t} t \cos at] = \frac{(s+2)^2 - 4}{[(s+2)^2 + 4]^2} //$$

Find the Laplace transform of the following:-

① $\frac{\cos at - \cos bt}{t}$

Given $\frac{F(t)}{t} = \frac{\cos at - \cos bt}{t}$

$$F(t) = \cos at - \cos bt$$

$$L[F(t)] = L(\cos at) - L(\cos bt)$$

$$f(s) = \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}$$

$$\text{WKT } L\left[\frac{F(t)}{t}\right] = \int_s^\infty f(s) ds$$

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty \left(\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right) ds$$

$$\begin{aligned}
&= \int_s^{\infty} \frac{s}{s^2+a^2} ds - \int_s^{\infty} \frac{s}{s^2+b^2} ds \\
&= \frac{1}{2} \int_s^{\infty} \frac{2s}{s^2+a^2} ds - \frac{1}{2} \int_s^{\infty} \frac{2s}{s^2+b^2} ds \\
&= \frac{1}{2} \log(s^2+a^2) - \frac{1}{2} \log(s^2+b^2) \\
&= \frac{1}{2} [\log(s^2+a^2) - \log(s^2+b^2)]_s^{\infty} \\
&= \frac{1}{2} \log \left[\frac{s^2+a^2}{s^2+b^2} \right]_s^{\infty} \\
&= -\frac{1}{2} \log \left[\frac{s^2+a^2}{s^2+b^2} \right]
\end{aligned}$$

$$\frac{\cos at - \cos bt}{t} = \log \sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

② $2^t + \frac{\cos 2t - \cos 3t}{t} + t \sin t$

Let $F(t) = 2^t + \frac{\cos 2t - \cos 3t}{t}$

$$L[F(t)] = L[2^t] + L\left[\frac{\cos 2t - \cos 3t}{t}\right] + \textcircled{1} t \sin t$$

$$\begin{aligned}
L[2^t] &= L[e^{(\log 2)t}] \\
&= L[e^{(\log 2)t}] \\
&= \frac{1}{s - \log 2}
\end{aligned}$$

$$\begin{aligned}
L[\cos 2t - \cos 3t] &= L[\cos 2t] - L[\cos 3t] \\
&= \frac{s}{s^2+4} - \frac{s}{s^2+9}
\end{aligned}$$

$$\begin{aligned}
 L\left[\frac{\cos 2t - \cos 3t}{t}\right] &= \int_s^\infty \left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right] ds \\
 &= \frac{1}{2} \int_s^\infty \left(\frac{2s}{s^2+4} - \frac{2s}{s^2+9}\right) ds \\
 &= \frac{1}{2} \left[\log(s^2+4) - \log(s^2+9) \right]_s^\infty \\
 &= \frac{1}{2} \log \left[\frac{s^2+4}{s^2+9} \right]_s^\infty \\
 &= -\frac{1}{2} \log \left[\frac{s^2+4}{s^2+9} \right] \\
 &= \log \sqrt{\frac{s^2+9}{s^2+4}}
 \end{aligned}$$

from eqn ①

$$L[F(t)] = \frac{1}{s - \log 2} + \log \sqrt{\frac{s^2+9}{s^2+4}} + \frac{2s}{(s^2+1)^2}$$

③ $\frac{1 - \cos t}{t}$

Given $\frac{F(t)}{t} = \frac{1 - \cos t}{t}$

$$F(t) = 1 - \cos t$$

$$\begin{aligned}
 L[F(t)] &= L[1 - \cos t] \\
 &= L[1] - L[\cos t]
 \end{aligned}$$

$$f(s) = \frac{1}{s} - \frac{s}{s^2+1}$$

WKT

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty f(s) ds$$

$$\begin{aligned}
 L\left[\frac{F(t)}{t}\right] &= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds \\
 &= \int_s^\infty \frac{1}{s} ds - \int_s^\infty \frac{s}{s^2+1} ds
 \end{aligned}$$

$$\begin{aligned}
 &= \int_s^{\infty} \log s - \frac{1}{2} \int_s^{\infty} \frac{2s}{s^2+1} ds \\
 &= \left[\log s - \frac{1}{2} \log (s^2+1) \right]_s^{\infty} \\
 &= \left[\log s - \log \sqrt{s^2+1} \right]_s^{\infty} \\
 &= -\log \left[\frac{s}{\sqrt{s^2+1}} \right]_s^{\infty} \\
 &= \log \left[\frac{\sqrt{s^2+1}}{s} \right] //
 \end{aligned}$$

④

$$3^t + (4t+s)^3$$

$$\text{Let } F(t) = 3^t + (4t+s)^3$$

$$\begin{aligned}
 F(t) &= (e^{\log 3})^t + (4t)^3 + s^3 + 3(4t)^2(s) + 3(4t)s^2 \\
 &= (e^{\log 3})^t + 64t^3 + 12s + 240t^2 + 300t
 \end{aligned}$$

$$\begin{aligned}
 L[F(t)] &= L[e^{(\log 3)^t}] + 64 L[t^3] + 12s L[1] + 240 L[t^2] + \\
 &\quad 300[t]
 \end{aligned}$$

$$f(s) = \frac{1}{s - \log 3} + 64 \left(\frac{3!}{s^4} \right) + 12s \left(\frac{1}{s} \right) + 240 \left(\frac{2}{s^3} \right) + 300 \left(\frac{1}{s} \right)$$

$$f(s) = \frac{1}{s - \log 3} + \frac{384}{s^4} + \frac{12s}{s} + \frac{480}{s^3} + \frac{300}{s^2} //$$

⑤

$$\sqrt{e^{4(t+3)}} + e^{-2t} \sin 3t$$

$$= \sqrt{(e^{2(t+3)})^2} + e^{-2t} \sin 3t$$

$$= e^{2(t+3)} + e^{-2t} \sin 3t$$

$$F(t) = e^{2t+6} + e^{-2t} \sin 3t$$

$$L[F(t)] = L[e^{2t+6}] + L[e^{-2t} \sin 3t] - \text{①}$$

$$\begin{aligned} L[e^{2t+6}] &= L[e^{2t} e^6] \\ &= e^6 L[e^{2t}] \\ &= \frac{e^6}{s-2} \end{aligned}$$

$$\text{Let } L[\sin 3t] = \frac{3}{s^2+9}$$

$$\Rightarrow L[e^{-2t} \sin 3t] = \frac{3}{s^2+9} \quad s \rightarrow s+2$$

$$L[e^{-2t} \sin 3t] = \frac{3}{(s+2)^2+9}$$

from eqn ①

$$f(s) = \frac{e^6}{s-2} + \frac{3}{(s+2)^2+9} //$$

⑥ $\frac{2 \sin t \sin 2t}{t}$

$$\begin{aligned} F(t) &= 2 \sin t \sin 2t \\ &= 2 \times \frac{1}{2} [\cos(4t) - \cos(6t)] \end{aligned}$$

$$F(t) = \cos 4t - \cos 6t$$

$$L\left[\frac{F(t)}{t}\right] = \log \sqrt{\frac{s^2+36}{s^2+16}} //$$

⑦ $t^2 e^{-3t} \sin 2t$

$$\text{Let } [F(t)] = L[t^2 e^{-3t} \sin 2t]$$

$$L[\sin 2t] = \frac{2}{s^2+4}$$

$$L[t^2 \sin 2t] = (-1)^2 \frac{d^2}{ds^2} \left[\frac{2}{s^2+4} \right]$$

$$= \frac{d}{ds} \left[\frac{(s^2+4)(1) - 2(2s)}{(s^2+4)^2} \right]$$

$$= \frac{d}{ds} \left[\frac{-4s}{(s^2+4)^2} \right]$$

$$= -4 \frac{d}{ds} \left[\frac{s}{(s^2+4)^2} \right]$$

$$= -4 \left[\frac{(s^2+4)(1) - s \cdot 2(s^2+4)(2s)}{(s^2+4)^4} \right]$$

$$= -4 \left[\frac{(s^2+4) - 4s^2}{(s^2+4)^3} \right]$$

$$= -4 \left[\frac{4 - 3s^2}{(s^2+4)^3} \right]$$

$$L[e^{-3t} \sin at] = 4 \left[\frac{3s^2 - 4}{(s^2+4)^3} \right]$$

$$s \rightarrow s+3$$

$$= 4 \left[\frac{3(s+3)^2 - 4}{((s+3)^2+4)^3} \right] //$$

$$(8) \quad t^3 + 4t^2 - 3t + 5$$

$$L[t^3 + 4t^2 - 3t + 5] = L[t^3] + 4L[t^2] - 3L[t] + 5L[1]$$

$$= \frac{3!}{s^4} + 4 \cdot \frac{2!}{s^3} - 3 \cdot \frac{1!}{s^2} + 5 \cdot \frac{1}{s}$$

$$= \frac{6}{s^4} + \frac{8}{s^3} - \frac{3}{s^2} + \frac{5}{s} //$$

$$(9) \quad \int_0^{\infty} t^3 e^{-t} \sin t \, dt = 0$$

$$\text{Let } \int_0^{\infty} e^{-st} F(t) \, dt = L[F(t)] \quad \text{--- (1)}$$

$$\text{Let } F(t) = t^3 \sin t$$

$$\textcircled{1} \Rightarrow \int_0^{\infty} e^{-st} t^3 \sin t dt = L[t^3 \sin t] - \textcircled{2}$$

$$\text{WKT } L[\sin t] = \frac{1}{s^2+1}$$

$$L[t^3 \sin t] = (-1)^3 \frac{d^3}{ds^3} \left[\frac{1}{s^2+1} \right]$$

$$= - \frac{d^2}{ds^2} \left[\frac{d}{ds} \left(\frac{1}{s^2+1} \right) \right]$$

$$= - \frac{d^2}{ds^2} \left[\frac{-2s}{(s^2+1)^2} \right]$$

$$= 2 \frac{d^2}{ds^2} \left[\frac{s}{(s^2+1)^2} \right]$$

$$= 2 \frac{d}{ds} \left[\frac{d}{ds} \left(\frac{s}{(s^2+1)^2} \right) \right]$$

$$= 2 \frac{d}{ds} \left[\frac{(s^2+1)^2(1) - s \cdot 2(s^2+1)(2s)}{(s^2+1)^4} \right]$$

$$= 2 \frac{d}{ds} \left[\frac{s^2+1-4s^2}{(s^2+1)^3} \right]$$

$$= 2 \frac{d}{ds} \left[\frac{1-3s^2}{(s^2+1)^3} \right]$$

$$= 2 \left[\frac{(s^2+1)^3(-6s) - (1-3s^2)3(s^2+1)^2(2s)}{(s^2+1)^6} \right]$$

$$= 2 \left[\frac{(s^2+1)(-6s) + (-6s)(1-3s^2)}{(s^2+1)^4} \right]$$

$$= -12s \left[\frac{s^2+1-3s^2}{(s^2+1)^4} \right]$$

$$= -12s \left[\frac{-2s^2 + 2}{(s^2+1)^4} \right]$$

$$= 24s \left[\frac{s^2-1}{(s^2+1)^4} \right]$$

$$\textcircled{1} \Rightarrow \int_0^{\infty} e^{-st} t^3 \sin t dt = 24s \left[\frac{s^2-1}{(s^2+1)^4} \right] - \textcircled{2}$$

$$\text{if } s=1$$

$$\textcircled{2} \Rightarrow \int_0^{\infty} e^{-t} t^3 \sin t dt = 0$$

$$\therefore \int_0^{\infty} t^3 e^t \sin t dt = 0 //$$

10 Evaluate $\int_0^{\infty} t e^{-3t} \cos 2t dt$

$$\text{Let } \int_0^{\infty} t e^{-3t} \cos 2t dt$$

$$\int_0^{\infty} e^{-3t} F(t) dt = L[f(t)] - \textcircled{1}$$

$$\text{Let } F(t) = t \cos 2t$$

$$\textcircled{1} \Rightarrow \int_0^{\infty} t e^{-3t} \cos 2t dt = L[t(\cos 2t)] - \textcircled{2}$$

$$\text{WKT } L[\cos 2t] = \frac{s}{s^2+4}$$

$$L[t \cos 2t] = (-1) \frac{d}{ds} \left(\frac{s}{s^2+4} \right)$$

$$= - \left[\frac{1(s^2+4) - s(2s)}{(s^2+4)^2} \right]$$

$$= - \left[\frac{(s^2+4) - 2s^2}{(s^2+4)^2} \right]$$

$$= - \left[\frac{-s^2 + 4}{(s^2 + 4)^2} \right]$$

$$L[t \cos at] = \left[\frac{s^2 - 4}{(s^2 + 4)^2} \right]$$

$$\textcircled{1} \Rightarrow \int_0^{\infty} t e^{-3t} \cos at \, dt = \left[\frac{s^2 - 4}{(s^2 + 4)^2} \right]$$

$$\text{if } s = 3$$

$$\textcircled{3} \Rightarrow \int_0^{\infty} t e^{-3t} \cos at = \left[\frac{9 - 4}{(9 + 4)^2} \right]$$

$$= \int_0^{\infty} t e^{-3t} \cos at = \frac{5}{169} //$$

Laplace transform for Periodic functions:-

Suppose for any $T \geq 0$, the function $f(t)$ is said to be a periodic function for $f(t+T) = f(t)$.

The laplace transform of the periodic function $f(t)$ for the period T is defined as

$$L[f(t)] = \bar{f}(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) \, dt$$

$$\textcircled{1} \text{ If } f(t) = \begin{cases} E, & \text{if } 0 \leq t < a/2 \\ -E, & \text{if } a/2 \leq t < a \end{cases} \quad \text{where } f(t+a) = f(t),$$

then show that $L[f(t)] = \frac{E}{s} \tanh\left(\frac{as}{4}\right)$.

$$\text{Given } f(t) = \begin{cases} E, & \text{if } 0 \leq t < a/2 \\ -E, & \text{if } a/2 \leq t \leq a \end{cases}$$

$$f(t+a) = f(t) \quad T = a$$

WKT

$$\begin{aligned}
 L[f(t)] &= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt \\
 &= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st} f(t) dt + \int_{a/2}^a e^{-st} f(t) dt \right] \\
 &= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} E e^{-st} dt + \int_{a/2}^a (-E) e^{-st} dt \right] \\
 &= \frac{E}{1-e^{-as}} \left[\int_0^{a/2} e^{-st} dt - \int_{a/2}^a e^{-st} dt \right] \\
 &= \frac{E}{1-e^{-as}} \left\{ \left[\frac{e^{-st}}{-s} \right]_0^{a/2} - \left[\frac{e^{-st}}{-s} \right]_{a/2}^a \right\} \\
 &= \frac{E}{1-e^{-as}} \left\{ -\frac{1}{s} [e^{-st}]_0^{a/2} + \frac{1}{s} (e^{-st})_{a/2}^a \right\} \\
 &= \frac{E}{s(1-e^{-as})} \left\{ - (e^{-a/2} - e^0) + (e^{-as} - e^{-as/2}) \right\} \\
 &= \frac{E}{s(1-e^{-as})} \left\{ -e^{-a/2} + 1 + e^{-as} - e^{-as/2} \right\} \\
 &= \frac{E}{s(1-e^{-as})} \left\{ 1 - 2e^{-as/2} + e^{-as} \right\} \\
 &= \frac{E(1 - e^{-as/2})^2}{s[(1)^2 - (e^{-as/2})^2]} \\
 &= \frac{E(1 - e^{-as/2})^2}{s(1 - e^{-as/2})(1 + e^{-as/2})} \\
 &= \frac{E(1 - e^{-as/2})}{s(1 + e^{-as/2})}
 \end{aligned}$$

$$= \frac{E}{s} \frac{(1 - e^{-as/2}) \times e^{as/4}}{(1 + e^{-as/2}) \times e^{as/4}}$$

$$= \frac{E}{s} \frac{e^{as/4} - e^{-as/4}}{e^{as/4} + e^{-as/4}}$$

$$L[f(t)] = \frac{E}{s} \tanh\left(\frac{as}{4}\right) //$$

② If $f(t) = \begin{cases} E, & \text{if } 0 \leq t < a \\ -E, & \text{if } a \leq t < 2a \end{cases}$ where $f(t+2a) = f(t)$, then

Show that $L[f(t)] = \frac{E}{s} \tanh\left(\frac{as}{2}\right)$

$$\text{Given } f(t) = \begin{cases} E, & \text{if } 0 \leq t < a \\ -E, & \text{if } a \leq t < 2a \end{cases}$$

$$f(t+2a) = f(t) = T = 2a$$

$$\text{WKT } L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$$

$$L[f(t)] = \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t) dt$$

$$= \frac{1}{1 - e^{-2as}} \left\{ \int_0^a e^{-st} f(t) dt + \int_a^{2a} e^{-st} f(t) dt \right\}$$

$$= \frac{1}{1 - e^{-2as}} \left\{ \int_0^a e^{-st} E dt + \int_a^{2a} (-E) e^{-st} dt \right\}$$

$$= \frac{E}{1 - e^{-2as}} \left[\int_0^a e^{-st} dt + (-1) \int_a^{2a} e^{-st} dt \right]$$

$$= \frac{E}{1 - e^{-2as}} \left[\int_0^a e^{-st} dt - \int_a^{2a} e^{-st} dt \right]$$

$$= \frac{E}{1 - e^{-2as}} \left[\left[\frac{e^{-st}}{-s} \right]_0^a - \left[\frac{e^{-st}}{-s} \right]_a^{2a} \right]$$

$$\begin{aligned}
&= \frac{E}{s(1-e^{-2as})} \left\{ -(e^{-st})_0^a + (e^{-st})_a^{2a} \right\} \\
&= \frac{E}{s(1-e^{-2as})} \left\{ [-e^{-as} + e^0 + e^{-2as} - e^{-as}] \right\} \\
&= \frac{E}{s(1-e^{-2as})} [-2e^{-as} + 1 + e^{-2as}] \\
&= \frac{E}{s(1-e^{-2as})} [1 + (e^{-as})^2 - 2(1)(e^{-as})] \\
&= \frac{E(1-e^{-as})^2}{s(1-e^{-2as})} \\
&= \frac{E(1-e^{-as})^2}{s[1 - (e^{-as})^2]} \\
&= \frac{E(1-e^{-as})}{s[1+e^{-as}]} \\
&= \frac{E(1-e^{-as})e^{as/2}}{s(1+e^{-as})e^{as/2}} \\
&= \frac{E(e^{as/2} - e^{-as/2})}{s(e^{as/2} + e^{-as/2})}
\end{aligned}$$

$$L[f(t)] = \frac{E}{s} \tanh(as/2) //$$

③ If $f(t) = \begin{cases} t, & 0 \leq t < a \\ 2a-t, & a \leq t < 2a \end{cases}$ where $f(t+2a) = f(t)$ then

Show that $L[f(t)] = \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$

$$\text{Given } f(t) = \begin{cases} t, & 0 \leq t < a \\ 2a-t, & a \leq t < 2a \end{cases}$$

$$f(t+2a) = f(t) = T = 2a$$

$$\Rightarrow L[f(t)] = \frac{1}{1-e^{-2sT}} \int_0^T e^{-st} f(t) dt$$

$$\Rightarrow L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} f(t) dt + \int_a^{2a} e^{-st} f(t) dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a t e^{-st} dt + \int_a^{2a} (2a-t) e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-2as}} [I_1 + I_2]$$

$$I_1 = \int_0^a t e^{-st} dt$$

$$= t \int_0^a e^{-st} dt - \int_0^a [1 \int e^{-st} dt] dt$$

$$= -\frac{1}{s} [t e^{-st}]_0^a - \frac{1}{s^2} \int_0^a e^{-st} dt$$

$$= -\frac{1}{s} [t e^{-st}]_0^a - \frac{1}{s^2} [e^{-st}]_0^a$$

$$= -\frac{1}{s} [a e^{-as} - 0] - \frac{1}{s^2} [e^{-as} - e^0]$$

$$I_1 = -\frac{a}{s} e^{-as} - \frac{1}{s^2} e^{-as} + \frac{1}{s^2}$$

$$I_2 = \int_0^{2a} (2a-t) e^{-st} dt$$

$$= (2a-t) \int_a^{2a} e^{-st} dt - \int_a^{2a} (-1 \int e^{-st} dt) dt$$

$$= -\frac{1}{s} [(2a-t) e^{-st}]_a^{2a} - \frac{1}{s} \int_0^{2a} e^{-st} dt$$

$$= -\frac{1}{s} [(2a-t) e^{-st}]_a^{2a} + \frac{1}{s^2} [e^{-st}]_a^{2a}$$

$$= -\frac{1}{s} [0 - a e^{-as}] + \frac{1}{s^2} [e^{-2as} - e^{-as}]$$

$$\int_0^{2a} (2a-t) e^{-st} dt = \frac{a}{s} e^{-as} + \frac{1}{s^2} e^{-2as} - \frac{1}{s^2} e^{-as}$$

$$L[f(t)] = \frac{1}{1-e^{-2as}} \left[-\frac{a}{s} e^{-as} - \frac{1}{s^2} e^{-as} + \frac{1}{s^2} + \frac{a}{s} e^{-as} + \frac{1}{s^2} e^{-2as} - \frac{1}{s^2} e^{-as} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1}{s^2} - \frac{2}{s^2} e^{-as} + \frac{1}{s^2} e^{-2as} \right]$$

$$= \frac{1}{s^2(1-e^{-2as})} [1 - 2e^{-as} + (e^{-as})^2]$$

$$= \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

$$= \frac{1}{s^2} \frac{1-e^{-as}}{1+e^{-as}} \times \frac{e^{as/2}}{e^{as/2}}$$

$$= \frac{1}{s^2} \frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}$$

$$L[f(t)] = \frac{1}{s^2} \tanh\left(\frac{as}{2}\right) //$$

④ If $f(t) = \begin{cases} t, & 0 \leq t < a/2 \\ a-t, & a/2 \leq t < a \end{cases}$ show that $L[f(t)] = \frac{1}{s^2} \tanh\left(\frac{as}{4}\right)$

Given $f(t) = \begin{cases} t, & 0 \leq t < a/2 \\ a-t, & a/2 \leq t < a \end{cases}$

$$f(t+a) = f(t) = T=a$$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st} f(t) dt + \int_{a/2}^a e^{-st} f(t) dt \right]$$

$$= \frac{1}{1 - e^{-as}} \left[\int_0^{a/2} t e^{-st} dt + \int_{a/2}^a e^{-st} (a-t) dt \right]$$

$$= \frac{1}{1 - e^{-as}} [I_1 + I_2]$$

$$I_1 = \int_0^{a/2} t e^{-st} dt$$

$$= t \int_0^{a/2} e^{-st} dt - \int_0^{a/2} 1 \int e^{-st} dt$$

$$= -\frac{1}{s} [t e^{-st}]_0^{a/2} + \int_0^{a/2} \left[\frac{e^{-st}}{-s} \right]$$

$$= -\frac{1}{s} \left[\frac{a}{2} e^{-as/2} - 0 \right] - \frac{1}{s^2} [e^{-st}]_0^{a/2}$$

$$= -\frac{1}{s} \left[\frac{a}{2} e^{-as/2} \right] - \frac{1}{s^2} [e^{-as/2} - e^0]$$

$$= -\frac{a}{2s} e^{-as/2} - \frac{1}{s^2} e^{-as/2} + \frac{1}{s^2}$$

$$I_2 = \int_{a/2}^a (a-t) e^{-st} dt$$

$$= (a-t) \int_{a/2}^a e^{-st} dt - \int_{a/2}^a (-1) \int e^{-st} dt a t$$

$$= -\frac{1}{s} [(a-t) e^{-st}]_{a/2}^a - \frac{1}{s} \int_{a/2}^a e^{-st} dt$$

$$= -\frac{1}{s} [(a-t) e^{-st}]_{a/2}^a + \frac{1}{s^2} [e^{-st}]_{a/2}^a$$

$$= -\frac{1}{s} [0 - (a - a/2) e^{-as/2}] + \frac{1}{s^2} [e^{-as} - e^{-as/2}]$$

$$= -\frac{a}{2s} e^{-as/2} + \frac{1}{s^2} e^{-as} - \frac{e^{-as/2}}{s^2}$$

$$L[f(t)] = \frac{1}{1-e^{-as}} \left[\frac{-a}{2s} e^{-as/2} - \frac{1}{s^2} e^{-as/2} + \frac{1}{s^2} + \frac{a}{s} e^{-as} + \frac{1}{s^2} e^{-as} - \frac{e^{-as/2}}{s^2} \right]$$

$$= \frac{1}{1-e^{-as}} \left[\frac{-2}{s^2} e^{-as/2} + \frac{1}{s^2} + \frac{1}{s^2} e^{-as} \right]$$

$$= \frac{1}{s^2(1-e^{-as})} \left[(1)^2 - 2(1)(e^{-as/2}) + (e^{-as/2})^2 \right]$$

$$= \frac{1}{s^2(1-e^{-as})} (1 - e^{-as/2})^2$$

$$= \frac{1}{s^2} \frac{(1 - e^{-as/2})^2}{(1)^2 - (e^{-as/2})^2}$$

$$= \frac{1}{s^2} \frac{(1 - e^{-as/2})^2}{(1 - e^{-as/2})(1 + e^{-as/2})}$$

$$= \frac{1}{s^2} \frac{(1 - e^{-as/2}) \times e^{as/4}}{(1 + e^{-as/2}) e^{as/4}}$$

$$= \frac{1}{s^2} \frac{(e^{as/4} - e^{-as/4})}{(e^{as/4} + e^{-as/4})}$$

$$L[f(t)] = \frac{1}{s^2} \tanh\left(\frac{as}{4}\right)$$

⑤ If $f(t) = \begin{cases} E \sin(\omega t), & 0 \leq t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} \leq t < \frac{2\pi}{\omega} \end{cases}$ is a periodic function of the period $\frac{2\pi}{\omega}$, then st $L[f(t)] = \frac{E\omega}{(s^2 + \omega^2)(1 - e^{-\pi/\omega})}$

for ω and E constants.

$$\text{Given } f(t) = \begin{cases} E \sin(\omega t), & 0 \leq t < \pi/\omega \\ 0, & \pi/\omega \leq t < 2\pi/\omega \end{cases} \quad T = \frac{2\pi}{\omega}$$

$$\therefore L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$$

$$= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} f(t) dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} f(t) dt \right]$$

$$= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \int_0^{\pi/\omega} e^{-st} E \sin \omega t dt$$

$$= \frac{E}{1 - e^{-\frac{2\pi s}{\omega}}} \int_0^{\pi/\omega} e^{-st} \sin \omega t dt$$

$$= \frac{E}{1 - e^{-\frac{2\pi s}{\omega}}} \left\{ \frac{e^{-st}}{(s^2 + \omega^2)} \left[-s \sin \omega t - \omega \cos \omega t \right] \right\}_0^{\pi/\omega}$$

$$= \frac{E}{1 - e^{-\frac{2\pi s}{\omega}}} \left\{ \frac{e^{-\pi s/\omega}}{s^2 + \omega^2} [0 + \omega] - \left[\frac{1}{s^2 + \omega^2} (0 - \omega) \right] \right\}$$

$$= \frac{E}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\frac{\omega e^{-\pi s/\omega}}{s^2 + \omega^2} + \frac{\omega}{s^2 + \omega^2} \right]$$

$$= \frac{E \omega (1 + e^{-\pi s/\omega})}{(s^2 + \omega^2) (1 - e^{-\pi s/\omega}) (1 + e^{-\pi s/\omega})}$$

$$L[f(t)] = \frac{E \omega}{s^2 + \omega^2 (1 - e^{-\pi s/\omega})} //$$

- ⑥ The square wave function $f(t)$ with the period $2a$ is defined by $f(t) = \begin{cases} 1, & 0 \leq t < a \\ -1, & a \leq t < 2a \end{cases}$ show that

$$L[f(t)] = \frac{1}{s} \tanh\left(\frac{as}{2}\right)$$

$$L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$$

$$= \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t) dt$$

$$\begin{aligned}
&= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} f(t) dt + \int_a^{2a} e^{-st} f(t) dt \right] \\
&= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} dt - \int_a^{2a} e^{-st} dt \right] \\
&= \frac{1}{1-e^{-2as}} \left\{ \left[\frac{e^{-st}}{-s} \right]_0^a - \left[\frac{e^{-st}}{-s} \right]_a^{2a} \right\} \\
&= \frac{1}{1-e^{-2as}} \left\{ -\frac{1}{s} [e^{-st}]_0^a + \frac{1}{s} [e^{-st}]_a^{2a} \right\} \\
&= \frac{1}{s(1-e^{-2as})} \left\{ [-e^{-as} + 1 + e^{-2as} - e^{-as}] \right\} \\
&= \frac{1}{s(1-e^{-2as})} [1 - 2e^{-as} + e^{-2as}] \\
&= \frac{(1-e^{-2as})^2}{s(1-e^{-2as})^2} \\
&= \frac{(1-e^{-2as})^2}{s(1-e^{-2as})(1+e^{-2as})} \\
&= \frac{(1-e^{-2as}) \times e^{as/2}}{s(1+e^{-2as}) \times e^{as/2}} \\
&= \frac{1}{s} \frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}
\end{aligned}$$

$$L[f(t)] = \frac{1}{s} \tanh \left[\frac{as}{2} \right] //$$

⑦ The square wave function $f(t)$ with period a is defined by $f(t) = \begin{cases} +1, & 0 \leq t < a/2 \\ -1, & a/2 \leq t < a \end{cases}$ show that

$$L[f(t)] = \frac{1}{s} \tanh\left(\frac{as}{4}\right)$$

$$L[f(t)] = \frac{1}{1-e^{-as}} \int_0^T e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-as}} \left\{ \int_0^{a/2} e^{-st} dt + \int_{a/2}^a e^{-st} (-1) dt \right\}$$

$$= \frac{1}{1-e^{-as}} \left[\frac{e^{-st}}{-s} \right]_0^{a/2} + \left[\frac{e^{-st}}{-s} \right]_{a/2}^a$$

$$= \frac{1}{s(1-e^{-as})} \left[-e^{-as/2} + 1 + e^{-as} - e^{-as/2} \right]$$

$$= \frac{1}{s(1-e^{-as})} \left[1 - 2e^{-as/2} + e^{-as} \right]$$

$$= \frac{(1 - e^{-as/2})^2}{s(1 - e^{-as/2})^2}$$

$$= \frac{(1 - e^{-as/2})^2}{s(1 - e^{-as/2})(1 + e^{-as/2})}$$

$$= \frac{(1 - e^{-as/2})}{s(1 + e^{-as/2})} \times \frac{e^{as/4}}{e^{as/4}}$$

$$= \frac{e^{as/4} - e^{-as/4}}{s(e^{as/4} + e^{-as/4})}$$

$$L[f(t)] = \frac{1}{s} \tanh\left(\frac{as}{4}\right) //$$

Unit Step (or) Heaviside function:- For any $a \geq 0$ the

unit step (or) Heaviside function can be defined as

$$u(t-a) \text{ or } H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

The Laplace transform of the unit step function is

$$L[u(t-a)] = \frac{e^{-as}}{s}.$$

1. If $f(t)$ be a function, then $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$,
where $\bar{f}(s) = L[f(t)]$.

2. If $f(t) = \begin{cases} f_1(t), & \text{for } t < a \\ f_2(t), & \text{for } t \geq a, \end{cases}$ then the Heaviside form
of $f(t)$ can be written as $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$ and its Laplace transform is

$$L[f(t)] = L[f_1(t)] + L\{[f_2(t) - f_1(t)]u(t-a)\}$$

3. If $f(t) = \begin{cases} f_1(t), & \text{for } 0 \leq t < a \\ f_2(t), & \text{for } a \leq t < b \\ f_3(t), & \text{for } t \geq b \end{cases}$ then, the Heaviside

form of $f(t)$ can be written as

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$$

$$L[f(t)] = L[f_1(t)] + L\{[f_2(t) - f_1(t)]u(t-a)\} + L\{[f_3(t) - f_2(t)]u(t-b)\}$$

① Find the Laplace transform for the following Heaviside function:-

① $(t^2 - 1)u(t-1)$

② $\sin t \cdot u(t-\pi)$

③ $(2t^2 - t + 1)u(t-1)$

④ Let $f(t-1) = t^2 - 1$

$$f(t) = (t+1)^2 - 1$$

$$f(t) = t^2 + 2t + 1 - 1$$

$$f(t) = t^2 + 2t$$

$$L[f(t)] = L[t^2] + 2L[t]$$

$$\bar{f}(s) = \frac{2}{s^3} + \frac{2}{s^2}$$

$$\Rightarrow L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$$

$$L[f(t-1)u(t-1)] = e^{-s}\bar{f}(s)$$

$$L[f(t^2-1)u(t-1)] = e^{-s}\left[\frac{2}{s^3} + \frac{2}{s^2}\right] //$$

⑤ $\sin t \cdot u(t-\pi)$

Let $f(t-\pi) = \sin t$

$$f(t) = \sin(t+\pi)$$

$$f(t) = -\sin t$$

$$L[f(t)] = -L[\sin t]$$

$$\bar{f}(s) = -\frac{1}{s^2+1}$$

$$\text{NKT, } L[f(t-a)u(t-a)] = e^{-as} \bar{f}(s)$$

$$L[f(t-\pi)u(t-\pi)] = e^{-\pi s} \bar{f}(s)$$

$$L[\sin t u(t-\pi)] = \frac{e^{-\pi s}}{s^2+1} //$$

$$\textcircled{3} \quad L[2t^2 - t + 1]u(t-1)$$

$$f(t-1) = 2t^2 - t + 1$$

$$f(t) = 2(t+1)^2 - (t+1) + 1$$

$$= 2(t^2+1+2t) - t$$

$$f(t) = 2t^2 + 3t + 2$$

$$L[f(t)] = 2L[t^2] + 3L[t] + 2L[1]$$

$$= 2 \frac{2!}{s^3} + 3 \frac{1!}{s^2} + \frac{2}{s}$$

$$L[f(t)] = \frac{4}{s^3} + \frac{3}{s^2} + \frac{2}{s}$$

$$\Rightarrow L[f(t-a)u(t-a)] = e^{-as} \bar{f}(s)$$

$$L[f(t-1)u(t-1)] = e^{-s} \bar{f}(s)$$

$$L[(2t^2 - t + 1)u(t-1)] = e^{-s} \left(\frac{4}{s^3} + \frac{3}{s^2} + \frac{2}{s} \right) //$$

$$\textcircled{1} \quad \text{Express } f(t) = \begin{cases} \sin t, & \text{for } t < \pi \\ \pi - t, & \text{for } t \geq \pi \end{cases} \text{ to unit step function,}$$

hence find its laplace transform

$$\text{Given } f(t) = \begin{cases} \sin t, & \text{for } t < \pi \\ \pi - t, & \text{for } t \geq \pi \end{cases}$$

$$f(t) = \sin t + [(\pi - t) - \sin t] u(t - \pi)$$

$$L[f(t)] = L[\sin t] + L\{(\pi - t) - \sin t\} u(t - \pi) \quad \text{--- (1)}$$

$$\Rightarrow g(t - \pi) = (\pi - t) - \sin t$$

$$g(t) = \pi - (t + \pi) - \sin(t + \pi)$$

$$g(t) = \sin t - t$$

$$L[g(t)] = L[\sin t] - L[t]$$

$$\bar{g}(s) = \frac{1}{s^2 + 1} - \frac{1}{s^2}$$

$$\therefore \text{WKT } L[g(t - a)u(t - a)] = e^{-as} \bar{g}(s)$$

$$L[g(t - \pi)u(t - \pi)] = e^{-\pi s} \left(\frac{1}{s^2 + 1} - \frac{1}{s^2} \right)$$

$$\bar{f}(s) = \frac{1}{s^2 + 1} + e^{-\pi s} \left(\frac{1}{s^2 + 1} - \frac{1}{s^2} \right) //$$

② Express $f(t) = \begin{cases} \sin t, & \text{for } 0 \leq t < \pi/2 \\ \cos t, & \text{for } t \geq \pi/2 \end{cases}$ to unit Step function, hence find its Laplace transform

$$\text{Given } f(t) = \begin{cases} \sin t, & \text{for } 0 \leq t < \pi/2 \\ \cos t, & \text{for } t \geq \pi/2 \end{cases}$$

$$f(t) = \sin t + [\cos t - \sin t] u(t - \pi/2) \quad \text{--- (1)}$$

$$L[f(t)] = L[\sin t] + L\{(\cos t - \sin t) u(t - \pi/2)\} \quad \text{--- (2)}$$

$$g(t - \pi/2) = \cos t - \sin t$$

$$g(t) = \cos(t + \pi/2) - \sin(t + \pi/2)$$

$$g(t) = \sin t - \cos t$$

$$L[g(t)] = -L[\sin t] - L[\cos t]$$

$$\bar{g}(s) = \frac{-1}{s^2 + 1} - \frac{s}{s^2 + 1}$$

$$\bar{q}(s) = - \left[\frac{1}{s^2+1} + \frac{s}{s^2+1} \right]$$

NKT

$$\Rightarrow L[q(t-a)u(t-a)] = e^{-as} \bar{q}(s)$$

$$\Rightarrow L[q(t-\pi/2)u(t-\pi/2)] = e^{-\pi s/2} \left[\frac{1}{s^2+1} + \frac{s}{s^2+1} \right]$$

$$\textcircled{2} \Rightarrow \frac{1}{s^2+1} - e^{-\pi s/2} \left[\frac{1}{s^2+1} + \frac{s}{s^2+1} \right]$$

③ Express $f(t) = \begin{cases} \sin t, & 0 \leq t < \pi \\ \sin at, & \pi \leq t < 2\pi \\ \sin 3t, & t \geq 2\pi \end{cases}$ to unit step function,

hence find its Laplace transform.

$$f(t) = \sin t + (\sin at - \sin t)u(t-\pi) + (\sin 3t - \sin at)u(t-2\pi)$$

$$\Rightarrow L[f(t)] = L[\sin t] + L[(\sin at - \sin t)u(t-\pi)] + L[(\sin 3t - \sin at)u(t-2\pi)] \quad \text{--- ①}$$

$$g(t-\pi) = \sin at - \sin t$$

$$g(t) = \sin(at + \pi) - \sin(t + \pi)$$

$$g(t) = \sin at + \sin t$$

$$\Rightarrow L[g(t)] = L[\sin at] + L[\sin t]$$

$$\bar{g}(s) = \frac{a}{s^2+4} + \frac{1}{s^2+1}$$

$$L[g(t-\pi)u(t-\pi)] = e^{-\pi s} \bar{g}(s)$$

$$L[g(t-\pi)u(t-\pi)] = e^{-\pi s} \left[\frac{a}{s^2+4} + \frac{1}{s^2+1} \right]$$

$$h(t-2\pi) = \sin 3t - \sin at$$

$$h(t) = \sin(3t + 2\pi) - \sin(at + 4\pi)$$

$$h(t) = \sin 3t - \sin 2t$$

$$\rightarrow L[h(t)] = L[\sin 3t] - L[\sin 2t] = \frac{3}{s^2+9} - \frac{2}{s^2+4}$$

$$\therefore L[h(t-2\pi)u(t-2\pi)] = e^{-2\pi s} \bar{h}(s)$$

$$L[h(t-2\pi)u(t-2\pi)] = e^{-2\pi s} \left[\frac{3}{s^2+9} - \frac{2}{s^2+4} \right]$$

$$\textcircled{1} \Rightarrow \bar{f}(s) = \frac{1}{s^2+1} + e^{-\pi s} \left[\frac{2}{s^2+4} + \frac{1}{s^2+1} \right] + e^{-2\pi s} \left[\frac{3}{s^2+9} - \frac{2}{s^2+4} \right] //$$

$$\textcircled{4} \text{ Express } f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ t, & 1 \leq t < 2 \\ t^2, & t \geq 2 \end{cases} \text{ to unit step function,}$$

Hence find its Laplace transform.

$$\text{Given } f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ t, & 1 \leq t < 2 \\ t^2, & t \geq 2 \end{cases}$$

$$\Rightarrow f(t) = 1 + (t-1)u(t-1) + (t^2-t)u(t-2)$$

$$L[f(t)] = L[1] + L[(t-1)u(t-1)] + L[(t^2-t)u(t-2)] \quad \text{--- (1)}$$

$$\text{Let } q(t-1) = t-1$$

$$q(t) = t$$

$$\Rightarrow L[q(t)] = L[t]$$

$$\bar{q}(s) = \frac{1}{s^2}$$

$$L[q(t-1)u(t-1)] = e^{-s} \bar{q}(s) = \frac{e^{-s}}{s^2}$$

$$\text{Let } h(t-2) = t^2-t$$

$$h(t) = (t+2)^2 - (t+2)$$

$$h(t) = t^2 + 4t + 4 - t - 2$$

$$h(t) = t^2 + 3t + 2$$

$$\Rightarrow L[h(t)] = L[t^2] + 3L[t] + 2L[1]$$

$$\bar{h}(s) = \frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}$$

$$L[h(t-2)u(t-2)] = e^{-2s}\bar{h}(s) = e^{-2s}\left[\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right]$$

$$\textcircled{1} \Rightarrow \therefore \bar{f}(s) = \frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-2s}\left[\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right] //$$

$$\textcircled{5} \quad f(t) = \begin{cases} t^2, & 0 \leq t < 2 \\ 4t, & 2 \leq t < 4 \\ 8, & t \geq 4 \end{cases} \quad \text{to unit step function hence,}$$

find its Laplace transform

$$f(t) = t^2 + (4t - t^2)u(t-2) + (8 - 4t)u(t-4)$$

$$L[f(t)] = L[t^2] + L[(4t - t^2)u(t-2)] + L[(8 - 4t)u(t-4)]$$

$$L[4t - t^2] = 4L[t] - L[t^2]$$

$$g(t) = 4(t+2) - (t+2)^2$$

$$= 4t + 8 - (t^2 + 4t + 4)$$

$$= 4t + 8 - t^2 - 4t - 4$$

$$g(t) = 4 - t^2$$

$$L[g(t)] = L[4] - L[t^2]$$

$$\bar{g}(s) = \frac{4}{s} - \frac{2}{s^3}$$

$$L[g(t-2)u(t-2)] = e^{-2s}\bar{g}(s)$$

$$= e^{-2s}\left[\frac{2}{s^3} - \frac{4}{s}\right]$$

$$h(t-4) = 8 - 4t$$

$$h(t) = 8 - 4(t+4)$$

$$= 8 - 4t - 16$$

$$h(t) = -8 - 4t$$

$$L[h(t)] = -L[8] - 4L[t]$$

$$= -\frac{8}{s} - \frac{4}{s^2}$$

$$\bar{h}(s) = -\left[\frac{4}{s^2} + \frac{8}{s}\right]$$

$$\therefore L[h(t-4)u(t-4)] = e^{-4s}\bar{h}(s)$$

$$= e^{-4s}\left(\frac{4}{s^2} + \frac{8}{s}\right)$$

$$\textcircled{1} \Rightarrow \bar{f}(s) = \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^2} - \frac{4}{s}\right) - e^{-4s}\left(\frac{4}{s^2} + \frac{8}{s}\right) //$$

⑥ Express $f(t) = \begin{cases} \cos t, & 0 \leq t < \pi \\ \cos 2t, & \pi \leq t < 2\pi \\ \cos 3t, & t \geq 2\pi \end{cases}$ to unit step function,

hence find its laplace transform

$$\Rightarrow f(t) = \cos t + [(\cos 2t - \cos t)u(t-\pi)] + [(\cos 3t - \cos 2t)u(t-2\pi)]$$

$$L[f(t)] = L[\cos t] + L[(\cos 2t - \cos t)u(t-\pi)] + L[(\cos 3t - \cos 2t)u(t-2\pi)]$$

$$q(t-\pi) = \cos 2t - \cos t$$

$$q(t) = \cos 2(t+\pi) - \cos(t+\pi)$$

$$q(t) = \cos(2t+2\pi) - \cos(t+\pi)$$

$$q(t) = \cos 2t + \cos t$$

$$L[q(t)] = L[\cos 2t] + L[\cos t]$$

$$\bar{q}(s) = \frac{s}{s^2+4} + \frac{s}{s^2+1}$$

$$\therefore L[q(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{q}(s)$$

$$= e^{-\pi s}\left[\frac{s}{s^2+4} + \frac{s}{s^2+1}\right]$$

$$\text{Let } h(t-2\pi) = \cos 3t - \cos 2t$$

$$h(t) = \cos 3(t+2\pi) - \cos 2(t+2\pi) \\ = \cos 3t - \cos 2t$$

$$L[h(t)] = L[\cos 3t] - L[\cos 2t]$$

$$h(s) = \frac{s}{s^2+9} - \frac{s}{s^2+4}$$

$$L[h(t-2\pi)u(t-2\pi)] = e^{-2\pi s} h(s)$$

$$= e^{-2\pi s} \left[\frac{s}{s^2+9} - \frac{s}{s^2+4} \right]$$

$$\textcircled{1} \Rightarrow \bar{f}(s) = \frac{s}{s^2+1} + e^{-\pi s} \left[\frac{s}{s^2+4} + \frac{s}{s^2+1} \right] + e^{-2\pi s} \left[\frac{s}{s^2+9} - \frac{s}{s^2+4} \right] //$$

$$\textcircled{7} \quad f(t) = \begin{cases} 2, & 0 \leq t < 1 \\ \frac{t^2}{2}, & 1 \leq t < \pi/2 \\ \cos t, & t \geq \pi/2 \end{cases}$$

$$f(t) = 2 + \left[\frac{t^2}{2} - 2 \right] u(t-1) + (\cos t - \frac{t^2}{2}) u(t-\pi/2)$$

$$L[f(t)] = 2L(1) + L\left[\frac{t^2}{2} - 2\right]u(t-1) + L\left[\cos t - \frac{t^2}{2}\right]u(t-\pi/2)$$

$$\Rightarrow g(t-1) = \frac{t^2}{2} - 2$$

$$g(t) = \frac{(t+1)^2}{2} - 2$$

$$= \frac{t^2 + 2t + 1 - 4}{2}$$

$$g(t) = \frac{1}{2} [t^2 + 2t - 3]$$

$$L[g(t)] = \frac{1}{2} L[t^2] + 2L[t] - 3L(1)$$

$$\bar{g}(s) = \frac{1}{2} \left[\frac{2}{s^3} + \frac{2}{s^2} - \frac{3}{s} \right]$$

$$\begin{aligned} \mathcal{L}[q(t-1)u(t-1)] &= e^{-s} \bar{q}(s) \\ &= \frac{e^{-s}}{2} \left[\frac{2}{s^3} + \frac{2}{s^2} - \frac{3}{s} \right] \end{aligned}$$

$$\begin{aligned} h(t) &= \left[\cos(t + \pi/2) - \frac{(t + \pi/2)^2}{2} \right] \\ &= \frac{-\sin t - t^2 + t\pi + \frac{\pi^2}{4}}{2} \end{aligned}$$

$$= -\sin t - \frac{1}{2} \left[t^2 + t\pi + \frac{\pi^2}{4} \right]$$

$$\mathcal{L}[h(t)] = \mathcal{L}[-\sin t] - \frac{1}{2} \mathcal{L}[t^2] + \mathcal{L}[t]\pi + \mathcal{L}\left[\frac{\pi^2}{4}\right]$$

$$\bar{h}(s) = -\frac{1}{s^2+1} - \frac{1}{2} \left[\frac{2}{s^3} + \frac{\pi}{s^2} + \frac{\pi^2}{4} \left(\frac{1}{s} \right) \right]$$

$$\bar{h}(s) \neq$$

$$\begin{aligned} \mathcal{F}[h(t - \pi/2)u(t - \pi/2)] &= e^{-\pi s} \bar{h}(s) \\ &= e^{-\frac{\pi s}{2}} \left[\frac{1}{s^2+1} - \frac{1}{2} \left\{ \frac{2}{s^3} + \frac{\pi}{s^2} + \frac{\pi^2}{4s} \right\} \right] \end{aligned}$$

$$\bar{f}(s) = \frac{2}{s} + \frac{e^{-s}}{2} \left[\frac{2}{s^3} + \frac{2}{s^2} - \frac{3}{s} \right] - e^{-\frac{\pi s}{2}} \left[\frac{1}{s^2+1} + \frac{2}{s^3} + \frac{\pi + \pi^2}{4s} \right]$$

Inverse Laplace Transform: Suppose $f(s)$ be the Laplace transform of $F(t)$, then the inverse Laplace transform of $f(s)$ can be defined as $\mathcal{L}^{-1}[f(s)] = F(t)$, where $\mathcal{L}^{-1}$ is called the inverse Laplace Transform.

Some Important Results:

- ① $\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$
- ② $\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
- ③ $\mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$
- ④ $\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$
- ⑤ $\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a} \sin at$
- ⑥ $\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cosh at$
- ⑦ $\mathcal{L}^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{s} \sinh at$
- ⑧ $\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!} \quad \forall n=0, 1, 2, \dots$
- ⑨ $\mathcal{L}^{-1}[f(s+a)] = e^{-at} \mathcal{L}^{-1}[f(s)] = e^{-at} F(t)$
- ⑩ $\mathcal{L}^{-1}[f(s-a)] = e^{at} \mathcal{L}^{-1}[f(s)] = e^{at} F(t)$
- ⑪ $\mathcal{L}^{-1}(f^n(s)) = (-1)^n t^n F(t)$
- ⑫ $\mathcal{L}^{-1}\left[\int_s^\infty f(s) ds\right] = \frac{F(t)}{t}$

Find the laplace transform for the following:

① $\frac{1}{s(s+1)(s+2)}$

Let $f(s) = \frac{1}{s(s+1)(s+2)}$

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \quad \text{--- ①}$$

$$1 = A(s+1)(s+2) + B(s)(s+2) + C(s)(s+1) \quad \text{--- ②}$$

when $s=0$

$$\text{②} \Rightarrow 1 = A(1)(2)$$

$$1 = 2A$$

$$A = 1/2$$

when $s=-1$

$$\text{②} \Rightarrow 1 = B(-1)(-1+2)$$

$$1 = -B$$

$$B = -1$$

when $s = -2$

$$\textcircled{2} \Rightarrow 1 = c(-2)(-2+1)$$

$$1 = -2c(-1)$$

$$1 = 2c$$

$$c = \frac{1}{2}$$

$$\textcircled{1} \Rightarrow f(s) = \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{s+1} + \frac{1}{2} \cdot \frac{1}{s+2}$$

$$\mathcal{L}^{-1}[f(s)] = \frac{1}{2} \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2} \mathcal{L}^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{2} (1) - e^{-t} + \frac{1}{2} e^{-2t}$$

$$F(t) = \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} //$$

$$\textcircled{3} \quad \frac{s}{(s+2)(s+3)}$$

$$\text{Let } f(s) = \frac{s}{(s+2)(s+3)} \quad - \textcircled{1}$$

$$\frac{s}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$s = A(s+3) + B(s+2) \quad - \textcircled{2}$$

when $s = -2$

$$-2 = A$$

$$A = -2$$

when $s = -3$

$$-3 = B(-1)$$

$$B = 3$$

$$\text{eqn } \textcircled{1} \Rightarrow f(s) = \frac{-2}{(s+2)} + \frac{3}{s+3}$$

$$\mathcal{L}^{-1}[f(s)] = -2 \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + 3 \mathcal{L}^{-1}\left[\frac{1}{s+3}\right]$$

$$F(t) = -2e^{-2t} + 3e^{-3t} //$$

(3) $\frac{s+3}{s^2-4s+13}$

$$\text{Let } f(s) = \frac{s+3}{s^2-4s+13}$$

$$f(s) = \frac{s+3}{s^2-2(s)(2)+2^2-2^2+13}$$

$$= \frac{s+3}{(s-2)^2+9}$$

$$= \frac{(s-2)+2+3}{(s-2)^2+9}$$

$$= \frac{(s-2)+5}{(s-2)^2+9}$$

$$\mathcal{L}^{-1}[f(s)] = \mathcal{L}^{-1}\left[\frac{(s-2)+5}{(s-2)^2+9}\right]$$

$$= e^{2t} \mathcal{L}^{-1}\left[\frac{s+5}{s^2+9}\right]$$

$$= e^{2t} \mathcal{L}^{-1}\left[\frac{s}{s^2+9} + 5 \frac{1}{s^2+9}\right]$$

$$F(t) = e^{2t} \left[\cos 3t + \frac{5}{3} \sin 3t \right] //$$

(4) $\frac{s+2}{s^2-4s+13}$

$$\text{Let } f(s) = \frac{s+2}{s^2-2(s)(2)+(2)^2-(2)^2+13}$$

$$= \frac{s+2}{(s-2)^2-4+13}$$

$$= \frac{s+2}{(s-2)^2+9}$$

$$= \frac{(s-2)+2+2}{(s-2)^2+9}$$

$$= \frac{(s-2)+4}{(s-2)^2+9}$$

$$\mathcal{L}^{-1} f(s) = \mathcal{L}^{-1} \left[\frac{(s-2)+4}{(s-2)^2+9} \right]$$

$$= e^{2t} \mathcal{L}^{-1} \left[\frac{s+4}{s^2+9} \right]$$

$$= e^{2t} \mathcal{L}^{-1} \left[\frac{s}{s^2+9} + \frac{4}{s^2+9} \right]$$

$$\Rightarrow f(t) = e^{2t} \left[\cos 3t + \frac{4}{3} \sin 3t \right]$$

⑤ $\frac{4s+5}{(s+1)^2(s+2)}$

$$f(s) = \frac{4s+5}{(s+1)^2(s+2)}$$

$$\frac{4s+5}{(s+1)^2(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{s+2} \quad \text{--- ①}$$

$$\text{when } s = -1$$

$$\text{②} \Rightarrow -3 = c(-2+1)^2$$

$$c = -3$$

$$0 \Rightarrow 1 = B(-1+2)$$

$$B = 1$$

By comparing s^2 coefficient on both sides we

$$\text{have } A+C=0$$

$$A = -C$$

$$A = 3$$

$$f(s) = \frac{3}{s+1} + \frac{1}{(s+1)^2} - \frac{3}{s+2}$$

$$\mathcal{L}^{-1} [f(s)] = 3 \mathcal{L}^{-1} \left[\frac{1}{s+1} \right] + \mathcal{L}^{-1} \left[\frac{1}{(s+1)^2} \right] - 3 \mathcal{L}^{-1} \left[\frac{1}{s+2} \right]$$

$$= 3e^{-t} \mathcal{L}^{-1}\left[\frac{1}{s}\right] + e^{-t} \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - 3e^{-2t} \mathcal{L}^{-1}\left[\frac{1}{s}\right]$$

$$F(t) = 3e^{-t} + te^{-t} - 3e^{-2t}$$

⑥ $\frac{4s+5}{(s-1)^2(s+2)}$

$$\text{Let } f(s) = \frac{1}{(s-1)^2(s+2)}$$

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{s+2} \quad \text{--- (1)}$$

$$4s+5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$$

$$\text{When } s = 1$$

$$9 = 3B \Rightarrow B = 9/3 = 3$$

$$\text{When } s = -2$$

$$4(-2)+5 = C(-3)^2$$

$$-3 = 9C$$

$$C = -\frac{1}{3}$$

By comparing s^2 coefficient on BS

$$A + C = 0$$

$$A = -C$$

$$A = \frac{1}{3}$$

$$f(s) = \frac{1}{3(s-1)} + \frac{3}{(s-1)^2} - \frac{1}{3(s+2)}$$

$$\mathcal{L}^{-1}[f(s)] = \frac{1}{3} \mathcal{L}^{-1}\left[\frac{1}{(s-1)}\right] + 3 \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3} \mathcal{L}^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{3} e^t \mathcal{L}^{-1}\left[\frac{1}{s}\right] + 3e^t \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{3} e^{-2t} \mathcal{L}^{-1}\left[\frac{1}{s}\right]$$

$$F(t) = \frac{1}{3} e^t + 3e^t - \frac{1}{3} e^{-2t} //$$

(7)

$$\frac{3s+2}{s^2-s-2}$$

$$\begin{aligned}\text{Let } f(s) &= \frac{3s+2}{s^2-s-2} \\ &= \frac{3s+2}{s^2+s-2s-2} \\ &= \frac{3s+2}{s(s+1)-2(s+1)}\end{aligned}$$

$$f(s) = \frac{3s+2}{(s-2)(s+1)}$$

$$\frac{3s+2}{(s-2)(s+1)} = \frac{A}{(s-2)} + \frac{B}{(s+1)} \quad \text{--- (1)}$$

$$3s+2 = A(s+1) + B(s-2) \quad \text{--- (2)}$$

When $s = 2$

$$\text{(2)} \Rightarrow 8 = A(3)$$

$$A = \frac{8}{3}$$

When $s = -1$

$$\text{(2)} \Rightarrow -1 = B(-3)$$

$$B = \frac{1}{3}$$

$$\text{(1)} \Rightarrow f(s) = \frac{8}{3} \frac{1}{s-2} + \frac{1}{3} \frac{1}{s+1}$$

$$\mathcal{L}^{-1}[f(s)] = \frac{8}{3} \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{3} \mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

$$F(t) = \frac{8}{3} e^{2t} + \frac{1}{3} e^{-t}$$

(8)

$$\frac{2s^2+5s-4}{s^3+s^2-2s}$$

$$\begin{aligned}f(s) &= \frac{2s^2+5s-4}{s^3+s^2-2s} \\ &= \frac{2s^2+5s-4}{s(s^2+s-2)}\end{aligned}$$

$$= \frac{2s^2 + 5s - 4}{s(s^2 - s + 2s - 2)}$$

$$= \frac{2s^2 + 5s - 4}{s[s(s-1) + 2(s-1)]}$$

$$= \frac{2s^2 + 5s - 4}{s(s-1)(s+2)}$$

$$\frac{2s^2 + 5s - 4}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s+2)} \quad \text{--- (1)}$$

$$2s^2 + 5s - 4 = A(s-1)(s+2) + B(s)(s+2) + C(s)(s-1) \quad \text{--- (2)}$$

when $s = 1$

$$\textcircled{2} \Rightarrow 2 + 5 - 4 = B(3)$$

$$3 = B(3)$$

$$B = 1$$

when $s = -2$

$$\textcircled{2} \Rightarrow 8 - 10 - 4 = C(-2)(-2-1)$$

$$-6 = C(6)$$

$$C = -1$$

$$\textcircled{2} \Rightarrow 2 = A + B + C$$

$$2 - A = B + C$$

$$-A = 1 - 1 - 2$$

$$A = 2$$

$$\textcircled{1} \Rightarrow f(s) = 2 \cdot \frac{1}{s} + \frac{1}{(s-1)} - \frac{1}{(s+2)}$$

$$L^{-1}[f(s)] = 2 L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+2}\right]$$

$$f(t) = 2 + e^t - e^{-2t}$$

9

$$\frac{1}{(s-1)(s^2+1)}$$

$$f(s) = \frac{1}{(s-1)(s^2+1)}$$

$$\frac{1}{(s-1)(s^2+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{s^2+1} \quad \text{--- (1)}$$

$$1 = A(s^2+1) + (Bs+C)(s-1) \quad \text{--- (2)}$$

$$1 = As^2 + A + Bs^2 - Bs + Cs - C$$

$$1 = (A+B)s^2 + (C-B)s + (A-C) \quad \text{--- (3)}$$

when $s = 1$

$$\textcircled{2} \Rightarrow 1 = A(1+1)$$

$$1 = 2A$$

$$A = \frac{1}{2}$$

By comparing s^2 coefficient on both sides we have

$$A + B = 0$$

$$B = -A$$

$$B = -\frac{1}{2}$$

By comparing co-efficient of s

$$C - B = 0$$

$$C = B$$

$$C = -\frac{1}{2}$$

$$\frac{1}{(s-1)(s^2+1)} = \frac{1}{2(s-1)} + \frac{(-1/2)s}{(s^2+1)} - \frac{1}{2} \left(\frac{1}{s^2+1} \right)$$

$$= \frac{1}{2(s-1)} - \frac{s}{2(s^2+1)} - \frac{1}{2} \left(\frac{1}{s^2+1} \right)$$

$$= \frac{1}{2} \mathcal{L}^{-1} \left[\frac{1}{s-1} \right] - \frac{1}{2} \mathcal{L}^{-1} \left[\frac{s}{s^2+1} \right] - \frac{1}{2} \mathcal{L}^{-1} \left[\frac{1}{s^2+1} \right]$$

$$= \frac{1}{2} e^t - \frac{1}{2} \cos t - \frac{1}{2} \sin t //$$

Find the inverse laplace transform for the following:

① $\log \left| \frac{s+a}{s+b} \right|$

$$\text{Let } f(s) = \log \left| \frac{s+a}{s+b} \right|$$

$$f(s) = \log(s+a) - \log(s+b) \quad \text{--- ①}$$

diff wrt s

$$\text{①} \Rightarrow f'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

$$L^{-1}[f'(s)] = L^{-1}\left(\frac{1}{s+a}\right) - L^{-1}\left(\frac{1}{s+b}\right)$$

$$(-1)^1 t^1 F(t) = e^{-at} - e^{-bt}$$

$$-t F(t) = e^{-at} - e^{-bt}$$

$$F(t) = \frac{e^{-at} - e^{-bt}}{-t}$$

$$F(t) = \frac{e^{-bt} - e^{-at}}{t} //$$

$$\text{② } \log \left| \frac{s^2+1}{s(s+1)} \right|$$

$$\text{Let } f(s) = \log \left| \frac{s^2+1}{s(s+1)} \right|$$

$$f(s) = \log(s^2+1) - \log s - \log(s+1)$$

$$= \log(s^2+1) - \log s - \log(s+1) \quad \text{--- ①}$$

Diff ① wrt 's'

$$\text{①} \Rightarrow f'(s) = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$$

$$L^{-1}[f'(s)] = 2 L^{-1}\left(\frac{s}{s^2+1}\right) - L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right)$$

$$(-1)^1 t^1 F(t) = 2 \cos t - 1 - e^{-t}$$

$$F(t) = \frac{1 + e^{-t} - 2 \cos t}{t} //$$

③

$$\log \left| \frac{s(s+5)}{(s^2+2s)(s-7)} \right|$$

$$\text{Let } f(s) = \log \left| \frac{s(s+5)}{(s^2+2s)(s-7)} \right|$$

$$f(s) = \log |s(s+5)| - \log |(s^2+2s)(s-7)|$$

$$f(s) = \log s + \log(s+5) - \log(s^2+2s) - \log(s-7) \quad \text{--- ①}$$

diff ① wrt 's'.

$$\text{①} \Rightarrow f'(s) = \frac{1}{s} + \frac{1}{s+5} - \frac{2s}{s^2+2s} - \frac{1}{s-7}$$

$$L^{-1}[f'(s)] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s+5}\right] - 2L^{-1}\left[\frac{s}{s^2+2s}\right] - L^{-1}\left[\frac{1}{s-7}\right]$$

$$-t F(t) = 1 + e^{-5t} - 2 \cos st - e^{7t}$$

$$F(t) = \frac{2 \cos st + e^{7t} - e^{-5t} - 1}{t} //$$

④

$$\tan^{-1}\left(\frac{2}{s^2}\right)$$

$$\text{Let } f(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$$

Diff wrt 's'

$$f'(s) = \frac{1}{1+\left(\frac{2}{s^2}\right)^2} \cdot \frac{d}{ds} \left(\frac{2}{s^2}\right)$$

$$f'(s) = \frac{1}{1+\frac{4}{s^4}} \left(-\frac{4}{s^3}\right)$$

$$f'(s) = \frac{s^4}{s^4+4} \left(-\frac{4}{s^3}\right)$$

$$f'(s) = \frac{-4s}{s^4+4}$$

$$f'(s) = \frac{-4s}{(s^2)^2+(2)^2}$$

$$\begin{aligned}
 f'(s) &= \frac{-4s}{(s^2+2)^2 - 2(s^2)(2)} \\
 &= \frac{-4s}{(s^2+2)^2 - 4s^2} \\
 &= \frac{-4s}{(s^2+2)^2 - (2s)^2} \\
 &= \frac{-4s}{(s^2+2s+2)(s^2-2s+2)} \\
 &= \frac{-4s}{(s^2+2s+2)(s^2-2s+2)} \\
 &= \frac{(s^2-2s+2) - (s^2+2s+2)}{(s^2+2s+2)(s^2-2s+2)}
 \end{aligned}$$

$$f'(s) = \frac{1}{(s+1)^2+1} - \frac{1}{(s-1)^2+1}$$

$$\mathcal{L}^{-1}[f'(s)] = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2+1}\right]$$

$$-tF(t) = e^{-t} \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] - e^t \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$-tF(t) = e^{-t} \sin t - e^t \sin t$$

$$F(t) = \sin t \left[\frac{e^t - e^{-t}}{t} \right]$$

Laplace transform for convolution of two functions:

Suppose $f(t)$ and $g(t)$ be the two functions and let $\bar{f}(s)$ and $\bar{g}(s)$ be the laplace transforms of $f(t)$ and $g(t)$. Then the convolution of $f(t)$ and $g(t)$ can be defined as $f(t) * g(t) = \int_{u=0}^t f(u) g(t-u) du = \int_{u=0}^t f(t-u) g(u) du$

$$\therefore L[f(t) * g(t)] = L\left[\int_{u=0}^t f(u) g(t-u) du\right] = L\left[\int_{u=0}^t f(t-u) g(u) du\right] = \bar{f}(s) \cdot \bar{g}(s)$$

$$\therefore L^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t) = \int_{u=0}^t f(u) g(t-u) du = \int_{u=0}^t f(t-u) g(u) du$$

Find the following by using laplace transform with the convolution theorem:

① $\frac{1}{s(s+1)}$

$$\text{Let } \bar{f}(s) \cdot \bar{g}(s) = \frac{1}{s(s+1)} = \frac{1}{s} \cdot \frac{1}{s+1}$$

$$\bar{f}(s) = \frac{1}{s}, \quad \bar{g}(s) = \frac{1}{s+1}$$

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s}\right] = f(t)$$

$$L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{1}{s+1}\right] = e^{-t} = g(t)$$

$$\begin{aligned} \text{WKT } L^{-1}[\bar{f}(s) \cdot \bar{g}(s)] &= f(t) * g(t) = \int_{u=0}^t f(t-u) g(u) du = \int_{u=0}^t 1 \cdot e^{-u} du \\ &= \int_{u=0}^t \frac{e^{-u}}{-1} du = -[e^{-u}]_0^t = -[e^{-t} - e^0] = -[e^{-t} - 1] = 1 - e^{-t} // \end{aligned}$$

②

$$\frac{1}{s(s^2+1)}$$

$$\text{Let } \bar{f}(s), \bar{g}(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} \cdot \frac{1}{s^2+1}$$

$$\bar{f}(s) = \frac{1}{s} \quad \bar{g}(s) = \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}[\bar{f}(s)] = \mathcal{L}^{-1}\left(\frac{1}{s}\right) = f(t)$$

$$\mathcal{L}^{-1}[\bar{g}(s)] = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t = g(t)$$

WKT

$$\mathcal{L}[\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t)$$

$$= \int_{u=0}^t f(t-u)g(u)du$$

$$= \int_{u=0}^t 1 \cdot \sin u du$$

$$= \int_{u=0}^t \sin u du$$

$$= -[\cos u]_0^t$$

$$= -[\cos t - \cos 0]$$

$$= -[\cos t - 1]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - \cos t //$$

③

$$\frac{1}{(s-1)(s^2+4)}$$

$$\text{Let } \bar{f}(s), \bar{g}(s) = \frac{1}{s-1} \cdot \frac{1}{s^2+4}$$

$$\bar{f}(s) = \frac{1}{s-1} \quad \bar{g}(s) = \frac{1}{s^2+4}$$

$$\mathcal{L}^{-1}[\bar{f}(s)] = \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t = f(t)$$

$$\mathcal{L}^{-1}[\bar{q}(s)] = \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2} \sin 2t = q(t)$$

WKT $\mathcal{L}^{-1}[\bar{f}(s) \cdot \bar{q}(s)] = f(t) * q(t)$

$$= \int_{u=0}^t f(t-u) q(u) du$$

$$= \int_{u=0}^t e^{t-u} \frac{1}{2} \sin 2u du$$

$$= \frac{1}{2} \int_{u=0}^t e^t e^{-u} \sin 2u du$$

$$= \frac{1}{2} e^t \int_{u=0}^t e^{-u} \sin 2u du$$

$$= \frac{1}{2} e^t \left\{ \frac{e^{-u}}{(-1)^2 + 2^2} [-\sin 2u - 2 \cos 2u] \right\}_0^t$$

$$= \frac{-e^t}{2} \left\{ \frac{e^{-u}}{5} [\sin 2u + 2 \cos 2u] \right\}_0^t$$

$$= \frac{-e^t}{2} \left\{ \frac{e^{-t}}{5} [\sin 2t + 2 \cos 2t] - \left\{ \frac{1}{5} (0 + 2) \right\} \right\}$$

$$= \frac{-e^t}{2} \left[\frac{e^{-t}}{5} (\sin 2t + 2 \cos 2t) - \frac{2}{5} \right] //$$

(4) $\frac{1}{s(s^2+a^2)}$

Let $\bar{f}(s) \cdot \bar{q}(s) = \frac{1}{s(s^2+a^2)} = \frac{1}{s} \cdot \frac{1}{s^2+a^2}$

$$\bar{f}(s) = \frac{1}{s}, \quad \bar{q}(s) = \frac{1}{s^2+a^2}$$

$$\mathcal{L}^{-1}[\bar{f}(s)] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1 = f(t)$$

$$\mathcal{L}^{-1}[\bar{q}(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a} \sin at = q(t)$$

$$\mathcal{L}^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t)$$

$$= \int_{u=0}^t f(t-u) g(u) du$$

$$= \int_{u=0}^t \frac{1}{a} \sin au du$$

$$= \frac{1}{a} \int_{u=0}^t \sin au du$$

$$= \frac{1}{a} \left[-\frac{\cos au}{a} \right]_0^t$$

$$= -\frac{1}{a^2} [\cos au]_0^t$$

$$= -\frac{1}{a^2} [\cos at - 1]$$

$$= \frac{1 - \cos at}{a^2} //$$

$$\frac{s}{(s^2 + a^2)^2}$$

$$\text{Let } \bar{f}(s) \cdot \bar{g}(s) = \frac{s}{(s^2 + a^2)^2} = \frac{1}{s^2 + a^2} \cdot \frac{s}{s^2 + a^2}$$

$$\bar{f}(s) = \frac{1}{s^2 + a^2} \Rightarrow \mathcal{L}^{-1}[\bar{f}(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2 + a^2}\right] = \frac{1}{a} \sin at = f(t)$$

$$\bar{g}(s) = \frac{s}{s^2 + a^2} \Rightarrow \mathcal{L}^{-1}[\bar{g}(s)] = \mathcal{L}^{-1}\left[\frac{s}{s^2 + a^2}\right] = \cos at = g(t)$$

$$\mathcal{L}^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t) = \int_{u=0}^t f(t-u) g(u) du$$

$$= \frac{1}{a} \int_{u=0}^t \sin(at - au) (\cos au) du$$

$$= \frac{1}{2a} \int_{u=0}^t 2 \sin(at - au) \cos au du$$

$$= \frac{1}{2a} \int_{u=0}^t \sin(at-au+au) + \sin(at-au-au) du$$

$$= \frac{1}{2a} \int_{u=0}^t \sin at + \sin(at-2au) du$$

$$= \frac{1}{2a} \left\{ \sin at \int_{u=0}^t 1 du + \int_{u=0}^t \sin(at-2au) du \right\}$$

$$= \frac{1}{2a} \left[u \sin at + \frac{\cos(at-2au)}{-2a} \right]_{u=0}^t$$

$$= \frac{1}{2a} \left[u \sin at + \frac{1}{2a} \cos(at-2au) \right]_{u=0}^t$$

$$= \frac{1}{2a} \left\{ t \sin at + \frac{1}{2} \cos at \right\} - \left[0 + \frac{1}{2a} \cos at \right]$$

$$= \frac{1}{2a} \left\{ t \sin at + \frac{1}{2a} [\cos at - \cos at] \right\}$$

$$= \frac{1}{2a} t \sin at //$$

⑥

$$\frac{s^2}{(s^2+a^2)^2}$$

$$\mathcal{L}^{-1} \{ \bar{f}(s) \cdot \bar{g}(s) \} = \frac{s}{s^2+a^2} \cdot \frac{s}{s^2+a^2}$$

$$\bar{f}(s) = \frac{s}{s^2+a^2} \Rightarrow \mathcal{L}^{-1} [\bar{f}(s)] = \mathcal{L}^{-1} \left[\frac{s}{s^2+a^2} \right] = \cos at = f(t)$$

$$\bar{g}(s) = \frac{s}{s^2+a^2} \Rightarrow \mathcal{L}^{-1} [\bar{g}(s)] = \mathcal{L}^{-1} \left[\frac{s}{s^2+a^2} \right] = \cos at = g(t)$$

$$\mathcal{L}^{-1} [\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t) = \int_{u=0}^t f(t-u) g(u) du$$

$$= \int_{u=0}^t \cos(at-au) \cos au du$$

$$\begin{aligned}
&= \frac{1}{2} \int_{u=0}^t 2 \cos(at-au) \cos au \, du \\
&= \frac{1}{2} \int_{u=0}^t [\cos(at-au+au) + \cos(at-au-au)] \, du \\
&= \frac{1}{2} \int_{u=0}^t [\cos at + \cos(at-2au)] \, du \\
&= \frac{1}{2} \left\{ \cos at \int_{u=0}^t 1 \, du + \int_{u=0}^t \cos(at-2au) \, du \right\} \\
&= \frac{1}{2} \left[u \cos at + \frac{\sin(at-2au)}{-2a} \right]_{u=0}^t \\
&= \frac{1}{2} \left[u \cos at - \frac{1}{2a} \sin(at-2au) \right]_{u=0}^t \\
&= \frac{1}{2} \left\{ t \cos at - \frac{1}{2a} \sin(at-2at) - \frac{1}{2a} \sin(at) \right\} \\
&= \frac{1}{2} \left\{ t \cos at + \frac{1}{2a} \sin at \right\} //
\end{aligned}$$

(7)

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

$$\text{Let } \bar{f}(s) \bar{q}(s) = \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{s}{s^2+a^2} \cdot \frac{s}{s^2+b^2}$$

$$\bar{f}(s) = \frac{s}{s^2+a^2} = \mathcal{L}^{-1}[\bar{f}(s)] = \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at = f(t)$$

$$\bar{q}(s) = \frac{s}{s^2+b^2} = \mathcal{L}^{-1}[\bar{q}(s)] = \mathcal{L}^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos bt = q(t)$$

$$\mathcal{L}^{-1}[\bar{f}(s) \cdot \bar{q}(s)] = f(t) * q(t) = \int_{u=0}^t f(t-u) q(u) \, du$$

$$\begin{aligned}
&= \int_{u=0}^t \cos(at-au) \cos b u \, du \\
&= \frac{1}{2} \int_{u=0}^t 2 \cos(at-au) \cos b u \, du \\
&= \frac{1}{2} \int_{u=0}^t [\cos(at-au+bu) + \cos(at-au-bu)] \, du \\
&= \frac{1}{2} \int_{u=0}^t \{ \cos(at+(b-a)u) + \cos(at-(b+a)u) \} \, du \\
&= \frac{1}{2} \left\{ \sin \frac{(at+(b-a)u)}{b-a} + \cos \frac{(at-(b+a)u)}{b+a} \right\} \bigg|_{u=0}^t \\
&= \frac{1}{2} \left\{ \left[\frac{\sin bt}{b-a} + \frac{\sin bt}{b+a} \right] - \left[\frac{\sin at}{b-a} - \frac{\sin at}{b+a} \right] \right\} \\
&= \frac{1}{2} \left\{ \left[\frac{1}{b-a} + \frac{1}{b+a} \right] \sin bt - \left[\frac{1}{b-a} - \frac{1}{b+a} \right] \sin at \right\} \\
&= \frac{1}{2} \left\{ \frac{2b}{b^2-a^2} \sin bt - \frac{2a}{b^2-a^2} \sin at \right\} \\
&= \frac{1}{b^2-a^2} [b \sin bt - a \sin at] //
\end{aligned}$$

(8)

$$\frac{S}{(S-1)(S^2+4)}$$

$$\text{Let } \bar{f}(s) \cdot \bar{g}(s) = \frac{S}{(S-1)(S^2+4)} = \frac{1}{(S-1)} \cdot \frac{S}{S^2+4}$$

$$\therefore \bar{f}(s) = \frac{1}{s-1} \quad \bar{g}(s) = \frac{S}{s^2+4}$$

$$\mathcal{L}^{-1}[\bar{f}(s)] = \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] = e^t = f(t)$$

$$\mathcal{L}^{-1}[\bar{g}(s)] = \mathcal{L}^{-1}\left[\frac{S}{s^2+4}\right] = \cos 2t = g(t)$$

WKT $L^{-1} [\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t)$

$$= \int_{u=0}^t f(t-u) g(u) du$$

$$= \int_{u=0}^t e^{t-u} \cos u du$$

$$= \int_{u=0}^t e^t e^{-u} \cos u du$$

$$= e^t \int_{u=0}^t e^{-u} \cos u du$$

$$= e^t \left\{ \frac{e^{-u}}{(-1)^2 + (1)^2} [-\cos u + \sin u] \right\}_0^t$$

$$= e^t \left\{ \frac{e^{-u}}{2} [2 \sin u - \cos u] \right\}_0^t$$

$$= e^t \left\{ \frac{e^{-t}}{2} [2 \sin t - \cos t] - \frac{1}{2} (0-1) \right\}$$

$$= e^t \left\{ \frac{e^{-t}}{2} [2 \sin t - \cos t] + \frac{1}{2} \right\} //$$

⑨ $\frac{1}{(s-1)(s^2+1)}$

Let $\bar{f}(s) \cdot \bar{g}(s) = \frac{1}{(s-1)(s^2+1)} = \frac{1}{s-1} \cdot \frac{1}{s^2+1}$

$\bar{f}(s) = \frac{1}{s-1} \Rightarrow L^{-1} [\bar{f}(s)] = L^{-1} \left[\frac{1}{s-1} \right] = e^t = f(t)$

$\bar{g}(s) = \frac{1}{s^2+1} \Rightarrow L^{-1} [\bar{g}(s)] = L^{-1} \left[\frac{1}{s^2+1} \right] = \sin t = g(t)$

$L^{-1} [\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t) = \int_{u=0}^t f(t-u) g(u) du$

$$= \int_{u=0}^t e^{(t-u)} \sin u \, du$$

$$= \int_{u=0}^t e^t \cdot e^{-u} \sin u \, du$$

$$= e^t \int_{u=0}^t e^{-u} \sin u \, du$$

$$= e^t \left\{ \frac{e^{-u}}{(-1)^2 + 1^2} [-\sin u - \cos u] \right\}_0^t$$

$$= e^t \left\{ \frac{e^{-t}}{2} [-\sin t - \cos t] - \frac{1}{2} (-1) \right\}$$

$$= e^t \left\{ \frac{e^{-t}}{2} [-\sin t - \cos t] + \frac{1}{2} \right\} //$$

(10) $\frac{1}{(s^2+a^2)(s^2+b^2)}$

$$\bar{f}(s) \cdot \bar{g}(s) = \frac{1}{s^2+a^2} \cdot \frac{1}{s^2+b^2}$$

$$\bar{f}(s) = \frac{1}{s^2+a^2} \Rightarrow \mathcal{L}^{-1}[\bar{f}(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \sin at$$

$$\bar{g}(s) = \frac{1}{s^2+b^2} \Rightarrow \mathcal{L}^{-1}[\bar{g}(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2+b^2}\right] = \sin bt$$

$$\mathcal{L}^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t) = \int_{u=0}^t f(t-u) g(u) \, du$$

$$= \int_{u=0}^t \sin(at-u) \sin bu \, du$$

$$= \frac{1}{2ab} \int_{u=0}^t 2 \sin(at-u) \sin bu \, du$$

$$\begin{aligned}
&= \frac{1}{2ab} \int_{u=0}^t [\cos(at-au-bu) - \cos(at-au+bu)] du \\
&= \frac{1}{2ab} \left[\frac{\sin at - u(a+b)}{-(a+b)} - \frac{\sin at - u(a-b)}{-(a-b)} \right]_0^t \\
&= \frac{1}{2ab} \left[\frac{\sin at - u(a-b)}{a-b} - \frac{\sin at - u(a+b)}{a+b} \right]_0^t \\
&= \frac{1}{2ab} \left[\frac{\sin at - at + bt}{a-b} - \frac{\sin at - at - bt}{a+b} \right] - \left[\frac{\sin at}{a-b} - \frac{\sin at}{a+b} \right] \\
&= \frac{1}{2ab} \left\{ \left[\frac{\sin bt}{a-b} - \frac{\sin(-bt)}{a+b} \right] - \left[\frac{\sin at}{a-b} - \frac{\sin at}{a+b} \right] \right\} \\
&= \frac{1}{2ab} \left\{ \left[\frac{\sin bt}{a-b} + \frac{\sin bt}{a+b} \right] - \left[\frac{\sin at}{a-b} - \frac{\sin at}{a+b} \right] \right\} \\
&= \frac{1}{2ab} \left\{ \sin bt \left[\frac{1}{a-b} + \frac{1}{a+b} \right] - \sin at \left[\frac{1}{a-b} - \frac{1}{a+b} \right] \right\} \\
&= \frac{1}{2ab} \left\{ \sin bt \left[\frac{2a}{a^2-b^2} \right] - \sin at \left[\frac{2b}{a^2-b^2} \right] \right\} \\
&= \frac{1}{2ab} \left\{ \frac{\sin bt}{b} - \frac{\sin at}{a} \right\} //
\end{aligned}$$

(ii) S
 $(s^2+a^2)(s^2+b^2)$

$$\bar{f}(s) \cdot \bar{g}(s) = \frac{1}{s^2+a^2} \cdot \frac{s}{s^2+b^2}$$

$$\mathcal{L}^{-1} [\bar{f}(s)] = \mathcal{L}^{-1} \left[\frac{1}{s^2+a^2} \right] = \frac{1}{a} \sin at = f(t)$$

$$\mathcal{L}^{-1} [\bar{g}(s)] = \mathcal{L}^{-1} \left[\frac{s}{s^2+b^2} \right] = \cos bt = g(t)$$

WKT

$$\mathcal{L}^{-1} [\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t)$$

$$= \int_{u=0}^t f(t-u)g(u) du$$

$$= \int_{u=0}^t \frac{1}{a} \sin(at-au) \cos bu du$$

$$= \frac{1}{2a} \int_{u=0}^t 2 \sin(at-au) \cos bu du$$

$$= \frac{1}{2a} \int_{u=0}^t [\sin(at-au+bu) + \sin(at-u(a+b))] du$$

$$= \frac{1}{2a} \left[\frac{-\cos at + u(a-b)}{-(a-b)} - \frac{\cos(at-u(a+b))}{-(a+b)} \right]_0^t$$

$$= \frac{1}{2a} \left\{ \left[\frac{\cos at - u(a+b)}{a+b} - \frac{\cos at - u(a-b)}{-a-b} \right]_0^t - \left[\frac{\cos at}{a+b} + \frac{\cos at}{a-b} \right] \right\}$$

$$= \frac{1}{2a} \left\{ \left[\frac{\cos(-bt)}{a+b} + \frac{\cos bt}{a-b} \right] - \left[\frac{\cos at}{a+b} + \frac{\cos at}{a-b} \right] \right\}$$

$$= \frac{1}{2a} \left\{ \cos bt \left[\frac{1}{a+b} + \frac{1}{a-b} \right] - \cos at \left[\frac{1}{a+b} + \frac{1}{a-b} \right] \right\}$$

$$= \frac{1}{2a} \left\{ \cos bt \left[\frac{a-b+a+b}{a^2-b^2} \right] - \cos at \left[\frac{a-b+a+b}{a^2-b^2} \right] \right\}$$

$$= \frac{1}{2a} \left\{ \cos bt \frac{2a}{a^2-b^2} - \cos at \frac{2a}{a^2-b^2} \right\}$$

$$= \frac{2a}{2a(a^2-b^2)} [\cos bt - \cos at]$$

$$= \frac{1}{a^2-b^2} [\cos bt - \cos at] //$$

(12)

$$\frac{1}{(s^2+a^2)^2}$$

$$\text{Let } \bar{f}(s) \bar{g}(s) = \frac{1}{(s^2+a^2)^2} = \frac{1}{s^2+a^2} \cdot \frac{1}{s^2+a^2}$$

$$\mathcal{L}^{-1}[\bar{f}(s)] = \frac{1}{a} \sin at = f(t)$$

$$\mathcal{L}^{-1}[\bar{g}(s)] = \frac{1}{a} \sin at = g(t)$$

$$\mathcal{L}^{-1}[\bar{f}(s) \bar{g}(s)] = f(t) * g(t)$$

$$= \int_0^t \frac{1}{a} \sin au \cdot \frac{1}{a} \sin a(t-u) du$$

$$= \frac{1}{a^2} \int_0^t [\sin au - \sin a(t-u)] du$$

$$= \frac{1}{2a^2} \int_0^t (\sin au - \sin(at-au)) du$$

$$= \frac{1}{2a^2} \int_0^t 2 \sin(au) - \sin(at-au) du$$

$$= \frac{1}{2a^2} \int_0^t [\cos(au-at+au) - \cos(au+at-au)] du$$

$$= \frac{1}{2a^2} \int_0^t [\cos(2au-at) - \cos(at)] du$$

$$= \frac{1}{2a^2} \left[\frac{\sin(2au-at)}{-a} - \cos(at)(u) \right]_0^t$$

$$= \frac{1}{2a^2} \left[\frac{1}{a} (\sin 2au - at) + \cos at (u) \right]_0^t$$

$$= \frac{1}{2a^2} \left[\frac{1}{a} (\sin(at - at)) + \cos at(t) \right]$$

$$= \frac{1}{2a^2} t \cos at //$$

Applications of Laplace transform:

Working rule:-

Step 1:- Express the given differential eq. with the notations $y(t), y'(t), y''(t), \dots$

Step 2:- Apply Laplace transform on both sides and substitute $L[y(t)] = \bar{y}(s)$

$$L[y'(t)] = s\bar{y}(s) - y(0)$$

$$L[y''(t)] = s^2\bar{y}(s) - sy(0) - y'(0)$$

$$L[y'''(t)] = s^3\bar{y}(s) - s^2y(0) - sy'(0) - y''(0)$$

for the initial conditions $y(0), y'(0), y''(0)$.

Step 3:- Write the function of the variable 's'.

Step 4:- Take the inverse Laplace transform on the both side and hence find $y(t)$.

① using Laplace transform $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}$ where $y(0) = 0, y'(0) = 0$.

$$\text{Given } \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t} \quad y(0) = 0 \quad y'(0) = 0$$

$$\Rightarrow y''(t) + 4y'(t) + 4y(t) = e^{-t}$$

$$L[y''(t)] + 4L[y'(t)] + 4L[y(t)] = L[e^{-t}]$$

$$[s^2\bar{y}(s) - sy(0) - y'(0)] + 4[s\bar{y}(s) - y(0)] + 4\bar{y}(s) = \frac{1}{s+1}$$

$$s^2 \bar{y}(s) + 4s\bar{y}(s) + 4\bar{y}(s) = \frac{1}{s+1}$$

$$(s^2 + 4s + 4)\bar{y}(s) = \frac{1}{s+1}$$

$$(s+2)^2 \bar{y}(s) = \frac{1}{s+1}$$

$$\bar{y}(s) = \frac{1}{(s+1)(s+2)^2} \quad \text{--- (1)}$$

$$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1) \quad \text{--- (2)}$$

When $s = -1$

$$\begin{aligned} \text{(2)} \Rightarrow 1 &= A(1)^2 \\ A &= 1 \end{aligned}$$

When $s = -2$

$$\begin{aligned} \text{(2)} \Rightarrow 1 &= C(-2+1) \\ C &= -1 \end{aligned}$$

$$A + B = 0$$

$$B = -A$$

$$B = -1$$

$$\text{(1)} \Rightarrow \bar{y}(s) = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$\mathcal{L}^{-1}[\bar{y}(s)] = \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$y(t) = e^{-t} - e^{-2t} - te^{-2t}$$

② Solve the DE $\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-t}$ with the initial conditions $y(0) = 0$, $y'(0) = 1$.

$$\text{Given } \frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-t}$$

$$\Rightarrow y''(t) + 3y'(t) + 2y(t) = e^{-t}$$

$$\mathcal{L}[y''(t)] + 3\mathcal{L}[y'(t)] + 2\mathcal{L}[y(t)] = \mathcal{L}[e^{-t}]$$

$$[s^2 \bar{y}(s) - s y(0) - y'(0)] + 3 [s \bar{y}(s) - y(0)] + 2 \bar{y}(s) = \frac{1}{s+1}$$

$$s^2 \bar{y}(s) - 1 + 3 s \bar{y}(s) + 2 \bar{y}(s) = \frac{1}{s+1}$$

$$(s^2 + 3s + 2) \bar{y}(s) = \frac{1}{s+1} + 1$$

$$(s+1)(s+2) \bar{y}(s) = \frac{1}{s+1} + 1$$

$$= \frac{s+2}{s+1}$$

$$\bar{y}(s) = \frac{s+2}{(s+1)^2(s+2)}$$

$$\bar{y}(s) = \frac{1}{(s+1)^2}$$

$$\mathcal{L}^{-1}[\bar{y}(s)] = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$= e^{-t} \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]$$

$$y(t) = t e^{-t}$$

③ using laplace transform solve $\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} - 3y = \sin t$

given $y(0) = 0$ $\frac{dy}{dt} = 0$ where $t = 0$

$$\text{Given } \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} - 3y = \sin t \quad y(0) = 0 \quad y'(0) = 1$$

$$y''(t) + 2y'(t) - 3y(t) = \sin t$$

$$\mathcal{L}[y''(t)] + 2\mathcal{L}[y'(t)] - 3\mathcal{L}[y(t)] = \mathcal{L}[\sin t]$$

$$[s^2 \bar{y}(s) - s y(0) - y'(0)] + 2 [s \bar{y}(s) - y(0)] - 3 \bar{y}(s) = \frac{1}{s^2 + 1}$$

$$(s^2 + 2s - 3) \bar{y}(s) = \frac{1}{s^2 + 1}$$

$$(s-1)(s+3)\bar{y}(s) = \frac{1}{s^2+1}$$

$$\bar{y}(s) = \frac{1}{(s-1)(s+3)(s^2+1)}$$

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$$

$$1 = A(s+3)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3) \quad \text{--- (2)}$$

$$1 = A(s^3+s+3s^2+3) + B(s^3+s-s^2-1) + (Cs+D)(s^2+2s-3)$$

$$1 = A(s^3+3s^2+s+3) + B(s^3-s^2+s-1) + Cs^3+2Cs^2-3Cs+Ds^2+2Ds-3D$$

$$1 = (A+B+C)s^3 + (3A-B+2C+D)s^2 + (A+B-3C+2D)s + (3A-B-3D) \quad \text{--- (3)}$$

when $s=1$

$$\text{(2)} \Rightarrow 1 = 8A$$

$$A = \frac{1}{8}$$

when $s=-3$

$$1 = -40B$$

$$B = -\frac{1}{40}$$

$$A+B+C=0$$

$$C = -A-B$$

$$C = -\frac{1}{10}$$

$$3A-B-3D=1$$

$$3D-3A-B=1$$

$$3D = \frac{3}{8} + \frac{1}{40} - 1$$

$$D = -\frac{3}{5}$$

$$\bar{y}(s) = \frac{1}{8} \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{40} \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10} \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{5} \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{1}{8} e^t - \frac{1}{40} e^{-3t} - \frac{1}{10} \cos t - \frac{1}{5} \sin t //$$

④ Solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-t}$ given $y(0) = y'(0) = 1$

Given $y''(t) + 4y'(t) + 3y(t) = e^{-t}$

$L[y''(t)] + 4L[y'(t)] + 3L[y(t)] = L[e^{-t}]$

$[s^2\bar{y}(s) - sy(0) - y'(0)] + 4[s\bar{y}(s) - y(0)] + 3\bar{y}(s) = \frac{1}{s+1}$

$s^2\bar{y}(s) - s - 1 + 4s\bar{y}(s) - 4 + 3\bar{y}(s) = \frac{1}{s+1}$

$(s^2 + 4s + 3)\bar{y}(s) - (s + 5) = \frac{1}{s+1}$

$(s^2 + 4s + 3)\bar{y}(s) = \frac{1}{s+1} + (s + 5)$

$(s+1)(s+3)\bar{y}(s) = \frac{1 + (s+1)(s+5)}{s+1}$

$= \frac{s^2 + 6s + 6}{s+1}$

$\bar{y}(s) = \frac{s^2 + 6s + 6}{(s+1)^2(s+3)} \quad \text{--- ①}$

$\frac{s^2 + 6s + 6}{(s+1)^2(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+3)}$

$s^2 + 6s + 6 = A(s+1)(s+3) + B(s+3) + C(s+1)^2 \quad \text{--- ②}$

when $s = -1$

② $\Rightarrow 1 = 2B$

$B = \frac{1}{2}$

when $s = -3$

$-3 = 4C$

$C = -\frac{3}{4}$

when

$A + C = 1$

$A = 1 - C$

$A = \frac{7}{4}$

① $\Rightarrow \bar{y}(s) = \frac{7}{4} \cdot \frac{1}{s+1} + \frac{1}{2} \cdot \frac{1}{(s+1)^2} - \frac{3}{4} \cdot \frac{1}{s+3}$

$$\Rightarrow \mathcal{L}^{-1}[\bar{y}(s)] = \frac{7}{4} \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{8} \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] - \frac{3}{4} \mathcal{L}^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{7}{4} e^{-t} + \frac{1}{8} t e^{-t} - \frac{3}{4} e^{-3t}$$

⑤ By using Laplace transform $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^{2t}$
with $x(0) = 0, \frac{dx}{dt}(0) = -1$

Given $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x(t) = e^{2t}, x(0) = 0, x'(0) = -1$

$$x''(t) - 2x'(t) + x(t) = e^{2t}$$

$$\mathcal{L}[x''(t)] - 2\mathcal{L}[x'(t)] + \mathcal{L}[x(t)] = \mathcal{L}[e^{2t}]$$

$$[s^2 \bar{x}(s) - s x(0) - x'(0)] - 2[s \bar{x}(s) - x(0)] + \bar{x}(s) = \frac{1}{s-2}$$

$$\Rightarrow s^2 \bar{x}(s) - 0 + 1 - 2s \bar{x}(s) + 0 + \bar{x}(s) = \frac{1}{s-2}$$

$$(s^2 - 2s + 1) \bar{x}(s) = \frac{1}{s-2} - 1$$

$$(s-1)^2 \bar{x}(s) = \frac{1-s+2}{(s-2)}$$

$$(s-1)^2 \bar{x}(s) = \frac{3-s}{(s-2)}$$

$$\bar{x}(s) = \frac{3-s}{(s-1)^2(s-2)} \quad \text{--- (1)}$$

$$\frac{3-s}{(s-1)^2(s-2)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s-2)}$$

$$3-s = A(s-1) + B(s-1)^2 + C(s-1)^2$$

when $s = 1$

$$\textcircled{2} \Rightarrow 2 = -B$$

$$B = -2 \quad \text{when } s = 2 \quad \text{and } A + C = 0$$

$$\begin{aligned} \textcircled{2} \Rightarrow 1 &= C & A &= -C \\ C &= 1 & A &= -1 \end{aligned}$$

$$\textcircled{1} \Rightarrow \bar{x}(s) = \frac{-1}{s-1} - \frac{2}{(s-1)^2} + \frac{1}{s-2}$$

$$\mathcal{L}^{-1}[\bar{x}(s)] = \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$x(t) = e^{2t} - e^t - 2te^t$$

⑥ Solve the equation $y'' - 3y' + 2y = e^{3t}$, $y(0) = 1$, $y'(0) = 0$ using Laplace technique.

$$\text{Given } \frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$$

$$y''(t) - 3y'(t) + 2y(t) = e^{3t}$$

$$\mathcal{L}[y''(t)] - 3\mathcal{L}[y'(t)] + 2\mathcal{L}[y(t)] = \mathcal{L}e^{3t}$$

$$[s^2\bar{y}(s) - sy(0) - y'(0)] - 3[s\bar{y}(s) - y(0)] + 2\bar{y}(s) = \frac{1}{s-3}$$

$$s^2\bar{y}(s) - s - 3s\bar{y}(s) + 3 + 2\bar{y}(s) = \frac{1}{s-3}$$

$$(s^2 - 3s + 2)\bar{y}(s) - s + 3 = \frac{1}{s-3}$$

$$(s^2 - 3s + 2)\bar{y}(s) = \frac{1}{(s-3)} + (s-3)$$

$$(s-1)(s-2)\bar{y}(s) = \frac{1 - s^2 + 9 - 3 \times 2s}{(s-3)(s-2)(s-3)}$$

$$\bar{y}(s) = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} \quad \text{--- ①}$$

$$\frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

$$s^2 - 6s + 10 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2) \quad \text{--- (2)}$$

when $s = 1$

$$\textcircled{2} \Rightarrow 1 - 6 + 10 = A(-1)(-2)$$

$$5 = 2A$$

$$A = \frac{5}{2}$$

when $s = 3$

$$\textcircled{2} \Rightarrow 9 - 18 + 10 = C(3-1)(3-2)$$

$$1 = C(2)$$

$$C = \frac{1}{2}$$

$$\textcircled{1} \Rightarrow \bar{y}(s) = \frac{5}{2} \frac{1}{s+1} + \left(-\frac{1}{2}\right) \frac{1}{s-2} + \frac{1}{2} \left(\frac{1}{s-3}\right)$$

$$\mathcal{L}^{-1}[\bar{y}(s)] = \frac{5}{2} \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \left(-\frac{1}{2}\right) \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{2} \mathcal{L}^{-1}\left[\frac{1}{s-3}\right]$$

$$y(t) = \frac{5}{2} e^{-t} - \frac{1}{2} e^{2t} + \frac{1}{2} e^{3t}$$

$\textcircled{7}$ Solve the equation $y'' + 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$ using Laplace transform technique.

$$\text{Given } \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = 0$$

$$\Rightarrow y''(t) + 3y'(t) + 2y(t) = 0$$

$$\mathcal{L}[y''(t)] + 3\mathcal{L}[y'(t)] + 2\mathcal{L}[y(t)] = 0$$

$$[s^2 \bar{y}(s) - sy(0) - y'(0)] + 3[s\bar{y}(s) - y(0)] + 2\bar{y}(s) = 0$$

$$s^2 \bar{y}(s) - sy(0) - y'(0) + 3s\bar{y}(s) - 3 + 2\bar{y}(s) = 0$$

$$s^2 \bar{y}(s) - s + 3s\bar{y}(s) - 3 + 2\bar{y}(s) = 0$$

$$(s^2 + 3s + 2) \bar{y}(s) - (s+3) = 0$$

$$(s+1)(s+2) \bar{y}(s) = (s+3)$$

$$\bar{y}(s) = \frac{(s+3)}{(s+1)(s+2)} \quad \text{--- (1)}$$

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$s+3 = A(s+2) + B(s+1) \quad \text{--- (2)}$$

$$\text{when } s = -2$$

$$\textcircled{2} \Rightarrow 1 = -B$$

$$B = -1$$

$$\text{when } s = -1$$

$$\textcircled{2} \Rightarrow 2 = A$$

$$A = 2$$

$$\textcircled{1} \Rightarrow \bar{y}(s) = \frac{2}{(s+1)} + \frac{(-1)}{(s+2)}$$

$$\mathcal{L}^{-1}[\bar{y}(s)] = 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - 1\mathcal{L}^{-1}\left[\frac{1}{s+2}\right]$$

$$y(t) = 2e^{-t} - e^{-2t} //$$

⑧ Solve the equation $x'' - 2x' + x = e^{2t}$ $x(0) = 1$ $\frac{dx}{dt}(0) = -1$ using Laplace transform technique.

$$\text{Given } \frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^{2t} \quad x(0) = 1 \quad x'(0) = -1$$

$$x''(t) - 2x'(t) + x(t) = e^{2t}$$

$$\mathcal{L}[x''(t)] - 2\mathcal{L}[x'(t)] + \mathcal{L}[x(t)] = \mathcal{L}[e^{2t}]$$

$$[s^2 \bar{x}(s) - sx(0) - x'(0)] - 2[s\bar{x}(s) - x(0)] + \bar{x}(s) = \frac{1}{s-2}$$

$$s^2 \bar{x}(s) - s + 1 - 2s\bar{x}(s) + 2 + \bar{x}(s) = \frac{1}{(s-2)}$$

$$(s^2 - 2s + 1) \bar{x}(s) - s + 3 = \frac{1}{(s-2)}$$

$$(s-1)^2 \bar{x}(s) = \frac{1 + (s-3)(s-2)}{(s-2)}$$

$$(s-1)^2 (\bar{x}(s)) = \frac{1 + s^2 - 2s - 3s + 6}{(s-2)}$$

$$(s-1)^2 (\bar{x}(s)) = \frac{s^2 - 5s + 7}{(s-2)}$$

$$\bar{x}(s) = \frac{s^2 - 5s + 7}{(s-2)(s-1)^2} \quad \text{--- (1)}$$

$$\frac{s^2 - 5s + 7}{(s-2)(s-1)^2} = \frac{A}{(s-2)} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2}$$

$$s^2 - 5s + 7 = A(s-1)^2 + B(s-1)(s-2) + C(s-2) \quad \text{--- (2)}$$

when $s=2$

$$4 - 10 + 7 = A(2-1)^2$$

$$1 = A$$

$$A = 1$$

$$1 = A + B$$

$$B = 1 - A$$

$$= 1 - 1$$

$$B = 0$$

when $s=1$

$$-5 + 7 = C(1-2)$$

$$3 = -C$$

$$C = -3$$

$$\bar{x}(s) = \frac{1}{(s-2)} - \frac{3}{(s-1)^2}$$

$$\mathcal{L}^{-1}[\bar{x}(s)] = \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] - 3 \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$y(t) = e^{2t} - 3e^t t //$$

⑦ Solve the equation $\frac{d^2 y}{dt^2} - \frac{dy}{dt} = 0$, $y(0) = 3$, $y'(0) = 3$

$$\text{Given } \frac{d^2 y}{dt^2} - \frac{dy}{dt} = 0$$

$$y''(t) - y'(t) = 0$$

$$L[y''(t)] - L[y'(t)] = 0$$

$$[s^2 \bar{y}(s) - s y(0) - y'(0)] - [s \bar{y}(s) - y(0)] = 0$$

$$[s^2 \bar{y}(s) - 3s - 3] - [s \bar{y}(s) - 3] = 0$$

$$s^2 \bar{y}(s) - 3s - 3 - s \bar{y}(s) + 3 = 0$$

$$s^2 \bar{y}(s) - s \bar{y}(s) = 3s$$

$$(s^2 - s) \bar{y}(s) = 3s$$

$$s(s-1) \bar{y}(s) = 3s$$

$$\bar{y}(s) = \frac{3}{(s-1)}$$

$$L^{-1}[\bar{y}(s)] = 3 L^{-1}\left(\frac{1}{s-1}\right)$$

$$y(t) = 3e^t //$$