Module - 4

Numerical Solution For First Order And First Degree Differential Equation

Taylor's series method :-

Step ①:- Write the given differential equation has

$\frac{dy}{dx} = y' = f(x,y)$ to the initial condition $y(x_0) = y_0$.

Step ②:- Find $y'(x_0), y''(x_0), y'''(x_0), \ldots$

Step ③:- Write the Taylor series expansion has

$$y(x) = y(x_0) + \frac{(x-x_0)^1}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \ldots$$

and simplify.

① Employ the taylor series method to find y at $x = 0.1$ correct to 4 - decimal places, given $\frac{dy}{dx} = 2y + 3e^x, y_{(0)} = 0$.

Soln:- Given $\frac{dy}{dx} = y' = 2y + 3e^x, \; y_{(0)} = 0$

$\Rightarrow x_0 = 0, \; y_0 = 0$

$y'(x_0) = 2y_0 + 3e^{x_0} = (0) + 3e^0 = 3$

$y''(x) = 2y' + 3e^x$

$\Rightarrow y''(x_0) = 2y'(x_0) + 3e^{x_0}$

$= 2(3) + 3e^0$

$= 9$

$y'''(x) = 2y'' + 3e^{x}$

$\Rightarrow y'''(x_0) = 2y''(x_0) + 3e^{x_0}$

$= 2(9) + 3$

$= 21$

$\Rightarrow y^{iv}(x) = 2y''' + 3e^{x}$

$y^{iv}(x_0) = 2y'''(x_0) + 3e^{x_0}$

$= 2(21) + 3$

$= 45$

$\therefore$ WKT

$$\Rightarrow y(x) = y(x_0) + \frac{(x-x_0)^1}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots$$

$$\Rightarrow y(x) = 0 + \frac{x}{1!}(3) + \frac{x^2}{2!}(9) + \frac{x^3}{3!}(21) + \frac{x^4}{4!}(45) + \cdots$$

$$\Rightarrow y(x) = 3 \cdot x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{45}{4!}x^4 + \cdots$$

$$\Rightarrow y(0.1) = 3(0.1) + \frac{9}{2}(0.1)^2 + \frac{7}{2}(0.1)^3 + \frac{45}{4!}(0.1)^4 + \cdots$$

$$\Rightarrow y(0.1) \cong 0.3487$$

② If $y' + y + 2x = 0$, $y(0) = -1$, then find $y(0.1)$ using Taylor's series method.

Solu :- Given $y' + y + 2x = 0$

$\Rightarrow y' = -(y + 2x)$, $y(0) = -1$

$\Rightarrow x_0 = 0$, $y_0 = -1$

$y'(x_0) = -(y_0 + 2x_0) = -(-1+0) = 1$

$y''(x) = -(y' + 2)$

$$\Rightarrow y''(x_0) = -(y'_{(x_0)} + 2)$$
$$= -(1+2)$$
$$= -3$$
$$\Rightarrow y'''(x) = -(y'')$$
$$\Rightarrow y'''(x_0) = -(y''_{(x_0)})$$
$$= -3$$
$$\Rightarrow y^{iv}(x) = -(y''')$$
$$\Rightarrow y^{iv}(x_0) = -(y'''_{(x_0)})$$
$$= 3$$

WKT

$$\Rightarrow y(x) = y(x_0) + \frac{(x-x_0)^1}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots$$
$$\Rightarrow y(x) = -1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-3) + \frac{x^2}{3!}(3) + \frac{x^4}{4!}(-3) + \cdots$$
$$\Rightarrow y(x) = -1 + x - \frac{3}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{8}x^4 + \cdots$$
$$\Rightarrow y_{(0.1)} = -1 + (0.1) - \frac{3}{2}(0.1)^2 + \frac{1}{2}(0.1)^3 - \frac{1}{8}(0.1)^4$$
$$\Rightarrow y_{(0.1)} \simeq -0.1945$$

③ If $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$, then find $y(0.1)$ using Taylor's Series method.

Soln :- Given $y' = x^2y - 1$ ⟶ ①

$$\Rightarrow y' = x^2y - 1, \quad y_0 = 1 \quad x_0 = 0$$
$$\Rightarrow y'(x_0) = x_0^2 y_0 - 1 = 0(1) - 1 = -1$$
$$\Rightarrow y''(x) = 2xy + x^2y' - 0$$
$$= 2xy + x^2y'$$

$$y''(x_0) = 2x_0y_0 + x_0^2 y_0'$$
$$= 2(0)(1) + 0(-1)$$
$$= 0$$

$$\Rightarrow y'''(x) = 2(1.y + xy') + (2xy' + x^2y'')$$
$$= 2y + 2xy' + 2xy' + x^2y''$$
$$= 2y + 4xy' + x^2y''$$

$$y'''(x_0) = 2y_0 + 4x_0y_0' + x_0^2y_0''$$
$$= 2(1)$$
$$= 2$$

$\therefore$ WKT

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \cdots\cdots$$

$$\Rightarrow y(x) = 1 + \frac{x}{1!}(-1) + \frac{x^3}{3!}(2) + \cdots\cdots$$

$$\Rightarrow y(x) = 1 - \frac{x}{1!} + \frac{x^3}{3} + \cdots\cdots$$

$$\therefore y(0.1) = 1 - \frac{0.1}{1!} + \frac{(0.1)^3}{3} + \cdots$$

$$\Rightarrow y(0.1) = 1 - 0.1 + 0.0003$$

$$\Rightarrow y(0.1) \simeq 0.9003$$

④ Employ the Taylor series method to find y' at $x = 0.1$ correct to 4 decimal places, given $\frac{dy}{dx} = 3x + y^2$, $y(0) = 1$

<u>Soln</u> :- Given $\frac{dy}{dx} = y' = 3x + y^2$ ~~~ ①

$$x_0 = 0 \qquad y_0 = 1$$

$$\Rightarrow y'(x_0) = 3x_0 + y_0^2$$
$$= 3(0) + 1^2$$
$$= 1$$

$$\Rightarrow y''(x) = 3 + 2yy'$$

$$\Rightarrow y''(x_0) = 3 + 2y_0y_0'$$

$$= 3 + 2(1)(1)$$

$$= 5$$

$$\Rightarrow y'''(x) = 0 + 2[yy'' + (y')^2]$$

$$= 2[yy'' + (y')^2]$$

$$\Rightarrow y'''(x_0) = 2[y_0y_0'' + (y_0')^2]$$

$$= 2[(1)(5) + 1^2]$$

$$= 12$$

$\therefore$ WKT

$$y(x) = y(x_0) + \frac{(x-x_0)^1}{1!} y(x_0) + \frac{(x-x_0)^2}{2!} y^2(x_0) + \cdots$$

$$y(x) = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(5) + \frac{x^3}{3!}(12) + \cdots$$

$$= 1 + x + \frac{5}{2}x^2 + 2x^3 + \cdots$$

$$y(0.1) = 1 + (0.1) + \frac{5}{2}(0.1)^2 + 2(0.1)^3 + \cdots$$

$$= 1 + 0.1 + 0.025 + 0.002$$

$$y(0.1) \simeq 1.127$$

(5) Solve $\frac{dy}{dx} = e^x - y$, $y(0) = 1$ using taylor series method considering upto the 4^{th} degree terms and find $y(0.1)$.

Solu :- Given $\frac{dy}{dx} = y' = e^x - y$, $y(0) = 1$

$$x_0 = 0 \quad y_0 = 1$$

$$\Rightarrow y'(x_0) = e^{x_0} - y_0$$

$$= e^0 - 1$$

$$= 1 - 1$$

$$= 0$$

$$\Rightarrow y''(x_0) = e^{x_0} - y_0'$$
$$= 1 - 0$$
$$= 1$$

$$\Rightarrow y'''(x_0) = e^{x} - y''$$

$$y'''(x_0) = e^{x_0} - y_0''$$
$$= 1 - 1$$
$$= 0$$

$$\Rightarrow y^{iv}(x) = e^{x} - y'''$$

$$y^{iv}(x_0) = e^{x_0} - y_0'''$$
$$= 1 - 0$$
$$= 1$$

$\therefore$ WKT

$$y(x) = y(x_0) + \frac{(x - x_0)^1}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \cdots$$

$$\Rightarrow y(x) = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \cdots$$

$$\Rightarrow y(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{24}$$

$$\Rightarrow y(0.1) = 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^4}{24} + \cdots$$

$$\Rightarrow y(0.1) \simeq \underline{\underline{1.0050}}$$

⑥ Solve $\frac{dy}{dx} = x^3 + y$ $y(1) = 1$ using Taylor series method considering upto the 4^{th} degree terms and find $y(1.1)$

<u>Soln</u>: Given $\frac{dy}{dx} = y' = x^3 + y$

$$y(1) = 1 \Rightarrow x_0 = 1,\ y_0 = 1$$

$$\Rightarrow y'(x_0) = x_0^3 + y_0$$
$$= 1 + 1$$
$$= 2$$

$$\Rightarrow y''(x) = 3x^2 + y'$$

$$\Rightarrow y''(x_0) = 3x_0^2 + y_0'$$

$$= 3(1) + 2$$

$$= 5$$

$$\Rightarrow y'''(x) = 6x + y''$$

$$y'''(x_0) = 6x_0 + y_0''$$

$$= 6(1) + 5$$

$$= 11$$

$$\Rightarrow y^{iv}(x) = 6 + y'''$$

$$\Rightarrow y^{iv}(x_0) = 6 + y_0'''$$

$$= 6 + 11$$

$$= 17$$

∴ WKT

$$\Rightarrow y(x) = y(x_0) + \frac{(x-x_0)^1}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots$$

$$\Rightarrow y(x) = 2 + \frac{(x-x_0)^1}{1!}(2) + \frac{(x-1)^2}{2!}(5) + \frac{(x-1)^3}{3!}(11) + \frac{(x-1)^4}{4!}(17)$$

$$\Rightarrow y(1.1) = 2 + 2x - 2 + \frac{(1.1-1)^2}{2}(5) + \frac{(1.1-1)^3}{6}(11) + \frac{(1.1-1)^4}{4!}(17)$$

$$= 2 + 2.2 - 2 + \frac{(0.1)^2}{2}(5) + \frac{(0.1)^3}{6}(11) + \frac{(0.1)^4}{24}(17)$$

$$= 2.2 + 0.025 + 0.0018 + 0.0012$$

$$\Rightarrow \underline{\underline{y(1.1) \simeq 2.228}}$$

7. Solve $\frac{dy}{dx} = e^x - y$ to the initial condition $y(0) = 2$ and hence find by using Taylor series expansion $y(1.1)$ to the 4 decimal places

Soln :- $y' = e^x - y$, $y(0) = 2$, $x_0 = 0$, $y_0 = 2$

$$\Rightarrow y'(x_0) = e^{x_0} - y_0$$

$$= e^{0} - 2$$

$$= 1 - 2$$

$$= -1$$

$$\Rightarrow y''_{(x)} = e^{x} - y'$$

$$\Rightarrow y''_{(x_0)} = e^{x_0} - y'_{(x_0)}$$

$$= e^{0} - (-1)$$

$$= 1 + 1$$

$$= 2$$

$$\Rightarrow y'''(x) = e^{x} - y''$$

$$\Rightarrow y'''_{(x_0)} = e^{x_0} - y''_{(x_0)}$$

$$= e^{0} - 2$$

$$= 1 - 2$$

$$= -1$$

$$\Rightarrow y^{iv}(x) = e^{x} - y'''$$

$$y^{iv}_{(x_0)} = e^{x_0} - y'''_{(x_0)}$$

$$= e^{0} - (-1)$$

$$= 1 + 1$$

$$= 2$$

WKT

$$y(x) = y(x_0) + \frac{(x - x_0)^1}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \cdots$$

$$\Rightarrow y(x) = 2 + \frac{(x - 0)(-1)}{1!} + \frac{(x-0)^2}{2!}(2) + \frac{(x-0)^3}{3!}(-1) + \frac{(x-x_0)^4}{4!}(2) + \cdots$$

$$\Rightarrow y(x) = 2 - x + x^2 - \frac{x^3}{6} + \frac{x^4}{12} + \cdots$$

$$\Rightarrow y(1.1) = 2 - 1.1 + (1.1)^2 - \frac{(1.1)^3}{6} + \frac{(1.1)^4}{12} + \cdots$$

$$= 2 - 1.1 + 1.21 - 0.2218 + 0.1220$$

$$\Rightarrow y(1.1) \simeq 2.0102$$

(8) Solve $\frac{dy}{dx} = x+y$ initial condition $y(0)=0$ by using taylor series method and find $y(0.1)$ up to the 4 decimal places

Solu:- Given DE $= \frac{dy}{dx} = y' = x+y$, $y(0)=0$

$x_0 = 0$, $y_0 = 0$, $y'_{(x)} = x+y$

$\Rightarrow y'_{(x_0)} = x_0 + y_0$
$= 0+0$
$= 0$

$\Rightarrow y''_{(x_0)} = 1+y'$
$y''_{(x_0)} = 1+y'_{(x_0)}$
$= 1+0$
$= 1$

$\Rightarrow y'''_{(x)} = 0+y''$
$y'''_{(x_0)} = y''_{(x_0)}$
$= 1$

$\Rightarrow y^{IV}_{(x)} = y^{IV}_{(x)}$
$y^{IV}_{(x)} = y^{IV}_{(x_0)}$
$= y'''_{(x_0)}$
$= 1$

WKT

$$\Rightarrow y(x) = y(x_0) + \frac{(x-x_0)^1}{1!} y'_{(x_0)} + \frac{(x-x_0)^2}{2!} y''_{(x_0)} + \cdots$$

$$y(x) = 0 + \frac{(x-0)}{1}(0) + \frac{(x-0)^2}{2!}(1) + \frac{(x-0)^3}{3!}(1) + \frac{(x-0)^4}{4!}(1)$$

$$y(x) = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$$

$$y(0.1) = \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} + \frac{(0.1)^4}{6}$$

$$y(0.1) = 0.005 + 0.00016 + 0$$

$$y(0.1) \simeq \underline{\underline{0.00516}}$$

(9) Solve $y' = x^2 y^2 + 1$, $y(0) = 1$ and find y at $x = 0.1$ using Taylor series expansion.

<u>Soln</u>:- $y' = x^2 . y^2 + 1$ —— (1)

$y(0) = 1$, $x_0 = 0$, $y_0 = 1$

$$\Rightarrow y'_{(x_0)} = x_0^2\, y_0^2 + 1 = (0)^2 . (1)^2 + 1 = 1$$

$$\Rightarrow y'' = [(x^2 2y)\, y' + y^2 . 2x]$$

$$\Rightarrow y''_{(x_0)} = 0^2 . 2(2) + (1)^2\, 2(2)$$

$$= 0$$

$$\Rightarrow y''' = [x^2 . 2y)\, y'' + y'(2x . 2y') + y^2(2) + 2x(2y . y')$$

$$= x^2 . 2yy'' + 4x(y')^2 + 2y^2 + 4xyy'$$

$$y'''_{(x_0)} = 0 + 0 + 2 + 0$$

$$= 2$$

WKT

$$y(x) = y(x_0) + \frac{(x - x_0)^1}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \dots$$

$$y(x) = 1 + x + \frac{x^2}{2}(0) + \frac{x^3}{3}(2)$$

$$\Rightarrow y(x) = 1 + x + \frac{x^3}{3}$$

$$\Rightarrow y(0.1) = 1 + 0.1 + \frac{(0.1)^3}{3}$$

$$= 1 + 0.1 + 0.00033$$

$$\Rightarrow y(0.1) = 1.1003$$

(10) using taylor series method, find the 4^{th} order approximate solution at $x = 0.4$ of the problem

$$\frac{dy}{dx} = x^2y + 1 \text{ with } y(0) = 0$$

Sol^n:- $\frac{dy}{dx} = y(x) = x^2y + 1 = f(x, y)$

$$\& \; y(0) = 0, \; x_0 = 0, \; y_0 = 0$$

$$\Rightarrow y'(x) = x^2 y + 1$$

$$y'(x_0) = x_0^2 y_0 + 1$$

$$= 0 + 1$$

$$= 1$$

$$\Rightarrow y''(x) = 2xy + x^2 y' + 0$$

$$\Rightarrow y''(x_0) = 2x_0 y_0 + x_0^2 y_0'$$

$$= 2(0)(0) + (0)^2 (0)^2$$

$$= 0 + 0$$

$$= 0$$

$$\Rightarrow y'''(x) = 2[y + xy'] + 2xy' + x^2 y''$$

$$\Rightarrow y'''_{(x_0)} = 2y + 4xy' + x^2 y''$$

$$y'''_{(x_0)} = 2y_0 + 4x_0 y'_0 + x_0^2 y_0''$$

$$= 2(0) + 4(0)(0) + (0)(0)$$

$$= 0 + 0 + 0$$

$$= 0$$

$$\Rightarrow y^{iv}_{(x)} = 2y' + 4(y' + xy'') + 2xy'' + x^2 y'''$$

$$y^{iv}_{(x_0)} = 6y' + 6xy'' + x^2 y'''_0$$

$$= 6(1) + 0 + 0$$

$$= 6$$

WKT

$$\Rightarrow y(x) = y(x_0) + \frac{(x-x_0)^1}{1!} y'_{(x_0)} + \frac{(x-x_0)^2}{2!} y''_{(x_0)} + \cdots$$

$$= 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(6)$$

$$\Rightarrow y(0.4) = 0.4 + \frac{(0.4)^2}{4}$$

$$= 0.4 + 0.0064$$

$$y_{(0.4)} \cong 0.4064$$

Modified Euler's Method :-

Step ① :- consider $\frac{dy}{dx} = f(x,y)$ to the initial condition $y(x_0) = y_0$ having the step size 'h'.

Step ② :- To get $y(x_1) = y_1$

I-1: $y(x_1) = y_1^{(1)} = y_0 + hf(x_0, y_0)$

I-2 : $y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

I-3: $y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$

I-4: $y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$

Step ③ :-

To find $y(x_2) = y_2$

consider $y(x_1) = y_1$ has the initial condition

I-1: $y(x_2) = y_2^{(1)} = y_1 + hf[x_1, y_1]$

I-2 : $y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$

I-3 : $y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$

① Use modified Euler's method to compute $y_{(0.2)}$. Given $\frac{dy}{dx} - xy^2 = 0$, under the initial condition $y_{(0)} = 2$ Perform three iterations at each Step, taking $h = 0.1$

Soln :- Given $\frac{dy}{dx} - xy^2 = 0$

$$\Rightarrow \frac{dy}{dx} = xy^2 = f(x,y) \sim ①$$

and $y_{(0)} = 2$, $x_0 = 0$, $y_0 = 2$

To find $y(x_1) = y_1$

$$\Rightarrow y(0.1) = ?$$

$$\Rightarrow y(x_0+h) = y(x_1) = y_1^{(1)} = y_0 + hf(x_0, y_0)$$

$$\Rightarrow y_1^{(1)} = 2 + (0.1) f(0,2)$$

$$\Rightarrow y_1^{(1)} = 2 + 0 = 2$$

$$\Rightarrow y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 2 + \frac{0.1}{2}\left[f(0,2) + f(0.1, 2)\right]$$

$$= 2 + 0.05[0 + 0.4]$$

$$y_1^{(2)} = 2.02$$

$$\Rightarrow y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 2 + \frac{0.1}{2}\left[f(0,2) + f(0.1, 2.02)\right]$$

$$= 2 + 0.05[0 + 0.4080]$$

$$y_1^{(3)} = 2.0204$$

$$\therefore y_{(0.1)} = 2.0204$$

$$\Rightarrow x_1 = 0.1, \quad y_1 = 2.0204$$

To find $y(x_2) = y_2$

$\Rightarrow y(0.2) = ?$

$\Rightarrow y(x_1+h) = y(x_2) = y_2^{(1)} = y_1 + h f(x_1, y_1)$

$= 2.0204 + (0.1) f(0.1, 2.0204)$

$= 2.0204 + (0.1)(0.4082)$

$y_2^{(1)} = 2.0612$

$\Rightarrow y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$

$= 2.0204 + \frac{0.1}{2}\left[f(0.1, 2.0204) + f(0.2, 2.0612)\right]$

$= 2.0204 + 0.05\left[0.4082 + 0.8497\right]$

$y_2^{(2)} = 2.0833$

$\Rightarrow y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$

$= 2.0204 + \frac{0.1}{2}\left[f(0.1, 2.0204) + f(0.2, 2.0833)\right]$

$= 2.0204 + 0.05\left[0.4082 + 0.8680\right]$

$\Rightarrow y_2^{(3)} = 2.0842$

$\therefore y(0.2) \simeq 2.0842$

② Use modified Euler's method to compute $y(0.1)$ Given $\frac{dy}{dx} = -xy^2$ under the initial condition $y(0) = 2$ perform three iterations at each step, taking $h = 0.05$

Solu:- Given $\frac{dy}{dx} = -xy^2 = f(x,y)$ ⟶ ①

and $y(0)=2$, $x_0=0$, $y_0=2$

To find $y(x_1) = y_1^{(1)} = y_0 + hf(x_0, y_0)$

$= 2 + (0.05) f(0,2)$

$= 2 - 0$

$y_1^{(1)} = 2$

$\Rightarrow y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(1)})]$

$= 2 + \frac{0.05}{2}[f(0,2) + f(0.05,2)]$

$= 2 + 0.025[0 - 0.2]$

$\Rightarrow y_1^{(2)} = 1.995$

$\Rightarrow y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(2)})]$

$= 2 + \frac{0.05}{2}[f(0,2) + f(0.05, 1.995)]$

$= 2 + 0.025[0 - 0.1990]$

$y_1^{(3)} = 1.9950$

$\Rightarrow x_1 = 0.05$, $y_1 = 1.9950$

To find $y(x_2) = y_2 \Rightarrow y(0.1) = ?$

$\Rightarrow y(x_1+h) = y(x_2) = y_2^{(1)} = y_1 + hf(x_1, y_1)$

$= 1.9950 + (0.05) f(0.05, 1.9950)$

$y_2^{(1)} = 1.9850$

$\Rightarrow y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1,y_1) + f(x_2, y_2^{(1)})]$

$= 1.9950 + 0.025[f(0.05, 1.9950) + f(0.1, 1.9850)]$

$= 1.9950 + 0.025[-0.19900 - 0.3940]$

$\Rightarrow y_2^{(2)} = 1.9802$

$\Rightarrow y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$

$= 1.9950 + 0.025[f(0.05, 1.9950) + f(0.1, 1.9802)]$

$= 1.9950 + 0.025[-0.19900 - 0.3921]$

$\Rightarrow y_2^{(3)} = 1.9802$

$y(0.1) \simeq 1.9802$

③ Use modified Euler's method to compute $y(0.1)$ Given $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$. Perform three iterations by taking $h = 0.1$.

Soln:- Given $\frac{dy}{dx} = 3x + \frac{y}{2} = f(x, y)$

$y(0) = 1 \Rightarrow x_0 = 0,\ y_0 = 1,\ h = 0.1$

$\therefore y(x_0 + h) = y(x_1) = y_1^{(1)} = y_0 + hf(x_0, y_0)$

$y_1^{(1)} = 1 + (0.1)(0.5)$

$= 1 + (0.1)(0.5)$

$= 1.05$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$= 1 + \frac{0.1}{2}[f(0, 1) + f(0.1, 1.05)]$

$= 1.0662$

$$\Rightarrow y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$
$$= 1 + \frac{0.1}{2}\left[f(0,1) + f(0.1, 1.0662)\right]$$
$$= 1 + 0.05\left[0.5 + 0.8331\right]$$
$$= 1.0665$$
$$\therefore y(0.1) \cong 1.0665$$

(4) Using modified Euler's method to find $y(0.1)$ Given $\frac{dy}{dx} = x^2 - y = f(x, y)$, $y(0) = 1$, $x_0 = 0$, $y_0 = 1$ Perform three iterations by taking $h = 0.05$

<u>Soln</u>:- Given $\frac{dy}{dx} = x^2 - y = f(x, y)$

$y(0) = 1 \Rightarrow x_0 = 0,\ y_0 = 1,\ h = 0.05$

To find $y(x_1) = y_1$

$$\Rightarrow y(x_0 + h) = y(x_1) = y_1^{(1)} = y_0 + hf(x_0, y_0)$$
$$= 1 + (0.05)f(0,1)$$
$$= 1 - 0.05$$
$$= 0.95$$

$$\Rightarrow y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 1 + \frac{0.05}{2}\left[f(0,1) + f(0.05, 0.95)\right]$$
$$= 1 + \frac{0.05}{2}\left[-1 - 0.9475\right]$$
$$\Rightarrow y_1^{(2)} = 0.9513$$

$$\Rightarrow y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 1 + \frac{0.05}{2}[f(0,1) + f(0.05, 0.9513)]$$

$$= 1 + \frac{0.05}{2}[-1 - 0.9488]$$

$$\Rightarrow y_1^{(3)} = 0.9513$$

$$\therefore y(0.05) = 0.9513$$

$$x_1 = 0.05 \quad , y_1 = 0.9513$$

To find $y(x_2) = y_2$

$$y(x_1 + h) = y_2^{(1)} = y_1 + hf(x_1, y_1)$$

$$= 0.9513 + (0.05) f(0.05, 0.9513)$$

$$= 0.9513 + (0.05)(-0.9488)$$

$$= 0.90386$$

$$\Rightarrow y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 0.9513 + \frac{0.05}{2}[f(0.05, 0.9513) + f(0.1, 0.90386)]$$

$$= 0.9513 + \frac{0.05}{2}[-0.9488 - 0.8938]$$

$$= 0.9052$$

$$\Rightarrow y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

$$= 0.9513 + \frac{0.05}{2}[f(0.05, 0.9513) + f(0.1, 0.9052)]$$

$$= 0.90513 + \frac{0.05}{2}[-0.9488 - 0.8932]$$

$$= 0.9052$$

$$\therefore y(0.1) \simeq 0.9052$$

⑤ Solve the DE $\frac{dy}{dx} = x\sqrt{y}$ under the initial condition $y(1)=1$ by using modified Euler's method find y at $x = 1.4$. Perform three iterations at each step, taking $h = 0.2$.

Soln :- Given $\frac{dy}{dx} = x\sqrt{y} = f(x,y)$

$y(1) = 1 \Rightarrow x_0 = 1,\ y_0 = 1,\ h = 0.2$

To find $y(x_1) = y_1$

$$y(x_0+h) = y(x_1) = y_0 + hf(x_0, y_0)$$
$$= 1 + (0.2)\, f(1,1)$$
$$= 1 + 0.2 \times 1$$
$$= 1.2$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 1 + \frac{0.2}{2}\left[f(1,1) + f(1.2, 1.2)\right]$$
$$= 1 + 0.1\left[1 + 1.3145\right]$$
$$= 1.2314$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$$
$$= 1 + \frac{0.2}{2}\left[f(1,1) + f(1.2, 1.2314)\right]$$
$$= 1 + (0.1)\left[1 + 1.3316\right]$$
$$= 1.2331$$

$\therefore\ y(1.2) \simeq 1.2331$

$\Rightarrow x_1 = 1.2,\ y_1 = 1.2331$

To find $y(x_2) = y_2$

$$y(x_1+h) = y_2^{(1)} = y_1 + hf(x_1, y_1)$$
$$= 1.2331 + (0.2) f(1.2, 1.2331)$$
$$= 1.2331 + (0.2)(1.3325)$$
$$= 1.4996$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$
$$= 1.2331 + \frac{0.2}{2}\left[f(1.2, 1.2331) + f(1.4, 1.4996)\right]$$
$$= 1.2331 + (0.\cancel{2})[(1.3325) + (1.7144)]$$
$$= 1.5377$$

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$
$$= 1.2331 + \frac{0.2}{2}\left[f(1.2, 1.2331) + f(1.4, 1.5377)\right]$$
$$= 1.2331 + 0.1\,[1.3325 + 1.7360]$$
$$= 1.5399$$

$$\therefore y(1.4) \simeq 1.5399$$

⑥ Solve $y' = x^2 - y$ and find $y(0) = 1$ by using modified Euler's method by taking $h = 0.1$

Soln :- Given $y' = x^2 - y$

$$y' = f(x, y) = x^2 - y \quad \text{--- (1)}$$

$y(0) = 1$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

To find $y(x_1) = y_1$

$$\Rightarrow y(x_0+h) = y(x_1) = y_1^{(1)} = y_0 + hf[x_0, y_0]$$

$$= 1 + 0.1 f[0,1]$$

$$= 1 + (0.1)(-1)$$

$$= 1 - 0.1$$

$$= 0.9$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + \frac{0.1}{2}\left[f(0,1) + f(0.1, 0.9)\right]$$

$$= 1 + \frac{0.1}{2}\left[-1 - 0.89\right]$$

$$= 1 - 0.0945$$

$$= 0.9055$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1 + \frac{0.1}{2}\left[f(0,1) + f(0.1, 0.9055)\right]$$

$$= 1 + \frac{0.1}{2}\left[-1 - 0.8955\right]$$

$$= 1 + 0.05\left[-1 - 0.8955\right]$$

$$y_1^{(3)} = 0.9052$$

$$y_{(0.1)} \simeq 0.9052$$

7. Given $\frac{dy}{dx} = x + \sin y$, $y(0) = 1$, compute $y(0.4)$ with $h = 0.2$ by using Euler's modified method.

Soln:- $\frac{dy}{dx} = x + \sin y$

$y(0) = 1$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$

To find $y(x_1) = y_1$

$y(x_0 + h) = y(x_1) = y_1^{(1)} = y_0 + hf(x_0, y_0)$

$= 1 + 0.2f(0, 1)$

$= 1 + 0.2(0.8414)$

$= 1 + 0.16828$

$= 1.16828$

$y_2^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$= 1 + \frac{0.2}{2}\left[f(0, 1) + f(0.2, 1.16828)\right]$

$= 1 + 0.1\left[f(0.8414) + f(1.12004)\right]$

$= 1 + 0.1\, f(0.8414 + 1.94)$

$= 1.197$

$y_3^{(1)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$

$= 1 + \frac{0.2}{2}\left(f(0.8414) + f(0.2, 1.197)\right]$

$= 1 + 0.1\left[(f\ 0.8414 + 1.128)\right]$

$= 1 + 0.1\, f(1.9694)$

$y_3^{(1)} = 1.1972$

$x_1 = 0.2 \qquad y_1 = 1.1972$

To find $y(x_2) = y_2$

$y(x_1+h) = y_2^{(1)} = y_1 + hf(x_1, y_1)$

$= 1.1972 + 0.2(0.2, 1.1972)$

$= 1.1972 + 0.2(1.1310)$

$= 1.4234$

$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2 y_2^{(1)})\right]$

$= 1.1972 + \frac{0.2}{2}\left[f(0.2) + f(0.2(1.4234)\right]$

$= 1.1972 + 0.1 f(0.2 + 0.28468)$

$= 1.1972 + 0.1(0.48468)$

$= 1.1972 + 0.048468$

$= 1.4492$

$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1] + f(x_2, y_2^{(2)})\right]$

$= 1.1972 + \frac{0.2}{2}\left[f(0.2 + f(0.2, 1.4492)\right]$

$= 1.1972 + 0.1 f(0.2 + f(1.19261)$

$= 1.1972 + 0.1(0.2 + 1.19261)$

$= 1.1972 + 0.1(1.39261)$

$= 1.1972 + 0.139261$

$= 1.4492$

$y(0.4) \cong 1.4492$

(8) Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ $y(0)=1$, compute $y(0.2)$ with $h=0.1$ by using Euler's modified method

Soln: $\frac{dy}{dx} = \frac{y-x}{y+x}$

$y(0)=1$, $x_0=0$, $y_0=1$, $h=0.1$

$y(x_1) = y_1$

$$y(x_0+h) = y(x_1) = y_1^{(1)} = y_0 + hf(x_0, y_0)$$
$$= 1 + 0.1f(0,1)$$
$$= 1 + 0.1(1)$$
$$= 1.1$$

$$y_2^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 1 + \frac{0.1}{2}\left[f(0,1) + f(0.1, 1.1)\right]$$
$$= 1 + 0.05\left[f(1 + 0.833)\right.$$
$$= 1 + 0.05(1.833)$$
$$= 1.09165$$

$$y_3^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$
$$= 1 + \frac{0.1}{2}\left[f(0,1) + f(0.1, 1.09165)\right]$$
$$= 1 + 0.05\left[f(1 + 0.8321)\right]$$
$$= 1 + 0.05(1.8321)$$
$$= 1.0916$$

$x_1 = 0.1$, $y_1 = 1.0916$

$$y_2^{(1)} = y_1 + h f(x_1, y_1)$$
$$= 1.0916 + 0.1 f(0.1, 1.0916)$$
$$= 1.0916 + 0.1 f(0.83215)$$
$$= 1.1748$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$
$$= 1.0916 + \frac{0.1}{2}\left[f(0.8321) + f(0.2, 1.1748)\right]$$
$$= 1.0916 + 0.05(0.8321 + f(0.7090))$$
$$= 1.0916 + 0.05(1.5411)$$
$$= 1.0916 + 0.0770$$
$$= 1.1686$$

$$y_3^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$
$$= 1.0916 + \frac{0.1}{2}\left[f(0.8321) + 1.1686\right]$$
$$= 1.0916 + 0.05(0.7077)$$

$y(0.2) = 1.168$

$y(0.2) \cong 1.168$

Runge - Kutta Method of 4^{th} Order

Step ①: Write the given differential equation

$$\frac{dy}{dx} = f(x,y) \text{ to the initial condition}$$

$y(x_0) = y_0$ having the step size 'h'.

Step ②:- Find $y(x_0+h) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

where $K_1 = hf(x_0, y_0)$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$K_4 = hf(x_0 + h, y_0 + K_3)$

① Use 4^{th}-order Runge-Kutta method to solve $(x+y)\frac{dy}{dx} = 1$ $y(0.4) = 1$, to find $y(0.5)$ taking $h = 0.1$.

Solu:- Given $(x+y)\frac{dy}{dx} = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(x+y)} = f(x,y)$$

and $y(0.4) = 1$, $x_0 = 0.4$, $y_0 = 1$, $h = 0.1$

$\Rightarrow K_1 = hf(x_0, y_0)$

$= (0.1) f(0.4, 1)$

$= (0.1)(0.7147)$

$K_1 = 0.07142$

$$\Rightarrow K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$= (0.1) f(0.45, 1.0357)$$

$$K_2 = 0.0673$$

$$\Rightarrow K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$= (0.1) f(0.45, 1.0336)$$

$$K_3 = 0.0674$$

$$\Rightarrow K_4 = hf(x_0 + h, y_0 + K_3)$$

$$= (0.1) f(0.5, 1.0674)$$

$$K_4 = 0.06379$$

$$\therefore y(x_1) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$y(0.5) = 1 + \frac{1}{6}[0.07142 + 2(0.0673) + 2(0.0674) + 0.06379]$$

$$y(0.5) = 1 + \frac{1}{6}[0.40461]$$

$$\Rightarrow y(0.5) \simeq 1.0674$$

(2) Use R-K method of 4^{th} order to solve $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$ to find $y(0.2)$ take $h = 0.2$

Soln:- Given $\frac{dy}{dx} = 3x + \frac{y}{2} = f(x, y)$

$$\Rightarrow K_1 = hf(x_0, y_0)$$

$$= 0.2 f(0, 1)$$

$$K_1 = 0.1$$

$$\Rightarrow K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$= (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1}{2}\right)$$

$$= (0.2) f\ (0.1, 1.05)$$

$$K_2 = 0.165$$

$$\Rightarrow K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$= (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.165}{2}\right)$$

$$= (0.2) f\ (0.1, 1.0825)$$

$$\Rightarrow K_3 = 0.16825$$

$$\Rightarrow K_4 = hf\ (x_0 + h, y_0 + K_3)$$

$$= (0.2) f (0 + 0.2, 1 + 0.16825)$$

$$= (0.2)\ f\ (0.2, 1.16825)$$

$$K_4 = 0.23682$$

$$\Rightarrow y_{(0.2)} = 1 + \frac{1}{6}\left[0.1 + 2\times 0.165 + 2\times 0.16825 + 0.23682\right]$$

$$= 1 + \frac{1}{6}\ (0.1 + 0.33 + 0.3365 + 0.23682]$$

$$= 1 + (0.16722)$$

$$y_{(0.2)} \simeq 1.16722$$

③ Use R-K method to find $y_{(0.2)}$, given $\frac{dy}{dx} = \sqrt{x+y}$, taking $h = 0.2$ initial condition $y_{(0)} = 1$

<u>Soln</u>:- Given $\frac{dy}{dx} = \sqrt{x+y} = f(x, y)$

$$y_{(0)} = 1 \Rightarrow x_0 = 0,\ y_0 = 1,\ h = 0.2$$

$\Rightarrow K_1 = hf(x_0, y_0)$

$= 0.2 f(0, 1)$

$= 0.2$

$\Rightarrow K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$

$= (0.2) f(0.1, 1.1)$

$= 0.2190$

$\Rightarrow K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$= (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2190}{2}\right)$

$= (0.2) f(0.1, 1.1095)$

$= 0.2199$

$\Rightarrow K_4 = hf(x_0 + h, y_0 + K_3)$

$= (0.2) f(0 + 0.2, 1 + 0.2199)$

$= (0.2) f(0.2, 1.2199)$

$= 0.2383$

$\Rightarrow y(x_0 + h) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 1 + \frac{1}{6}[0.2 + 2 \times 0.2190 + 2 \times 0.2199$

$+ 0.2383]$

$y(0.2) = 1.21935$

(4) Using R-K method of 4^{th} order find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$ $y(0)=1$ taking $h=0.2$

Soln :- Given $\frac{dy}{dx} = \frac{y-x}{y+x} = f(x,y)$

$x_0 = 0, \quad y_0 = 1$

$\Rightarrow K_1 = hf(x_0, y_0)$

$= 0.2 f(0,1)$

$K_1 = 0.2$

$\Rightarrow K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$

$= (0.2) f(0.1, 1.1)$

$K_2 = 0.1667$

$\Rightarrow K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$= (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1667}{2}\right)$

$= 0.2 f(0.1, 1.0833)$

$K_3 = 0.1662$

$\Rightarrow K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.2 f(0.2, 1 + 0.1662)$

$= 0.1414$

$\Rightarrow y(x_0 + h) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 1 + \frac{1}{6}[0.2 + 0.3334 + 0.3324 + 0.1414]$

$= 1.1679$ //

⑤ Use fourth order R-K method to compute $y(1.1)$ given that $\frac{dy}{dx} = xy^{1/3}$, $y(1) = 1$

Sol^n :- Given $f(x,y) = xy^{1/3}$, $x_0 = 1$, $y_0 = 1$

Given, $f(x,y) = xy^{1/3}$

$\Rightarrow K_1 = hf(x_0, y_0)$

$= (0.1) f((1)(1)^{1/3})$

$= 0.1$

$\Rightarrow K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= (0.1) f(1.05, 1.05)$

$= 0.1067$

$\Rightarrow K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$= (0.1) f(1.05, 1.05335)$

$= 0.1068$

$\Rightarrow K_4 = hf(x_0 + h, y_0 + K_3)$

$= (0.1) f(1.1, 1.1068)$

$= 0.1138$

$\therefore y(x_0 + h) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 1 + \frac{1}{6}[0.1 + 2 \times 0.1067 + 2 \times 0.1068 + 0.1138]$

$y(1.1) = 1.1068$

⑥ Using R-K method of 4th order find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$ $y(0) = 1$ taking $h = 0.2$

Soln: Given $\frac{dy}{dx} = \frac{y-x}{y+x} = f(x,y)$

$y(0)=1, \quad x_0=0, \quad y_0=1, \quad h=0.2$

$\Rightarrow K_1 = hf(x_0, y_0)$

$= 0.2 f(0,1)$

$K_1 = 0.2$

$\Rightarrow K_2 = hf\left[x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right]$

$= (0.2) f\left[0+\frac{0.2}{2}, 1+\frac{0.2}{2}\right]$

$= (0.2) f(0.1, 1.1)$

$K_2 = 0.1667$

$\Rightarrow K_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right)$

$= (0.2) f\left(0+\frac{0.2}{2}, 1+\frac{0.1667}{2}\right)$

$= 0.2 f(0.1, 1.0833)$

$= 0.1662$

$\Rightarrow K_4 = hf(x_0+h, y_0+K_3)$

$= 0.2 f(0.2, 1+0.1662)$

$= 0.1414$

$y(x_0+h) = y_0 + \frac{1}{6}\left[K_1+2K_2+2K_3+K_4\right]$

$= 1+\frac{1}{6}\left[0.2+0.3334+0.3324+0.1414\right]$

$y(0.2) \cong 1.1679$ //

⑦ using R-K method of 4th order to find y at $x = 1.1$, given $\frac{dy}{dx} + y = 2x$ $y(1) = 3$ [take $h = 0.1$]

Soln :- $\frac{dy}{dx} + y = 2x$

$\frac{dy}{dx} = 2x - y \Rightarrow f(x, y)$

$y(1) = 3$, $x_0 = 1$, $y_0 = 3$, $h = 1$

$\Rightarrow k_1 = h f(x_0, y_0)$

$= 0.1 f(1, 3)$

$= 0.1(-1)$

$= -0.1$

$\Rightarrow k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.1 f\left(1 + \frac{0.1}{2}, 3 + \frac{(-0.1)}{2}\right)$

$= 0.1 f((1 + 0.05), 1.45)$

$= 0.1 f(1.05, 2.95)$

$= 0.1 f(-0.85)$

$= -0.085$

$\Rightarrow k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= (0.1) f\left(1 + \frac{0.1}{2}, 3 - \frac{0.085}{2}\right)$

$= (0.1) f(1.05, 2.9575)$

$= (0.1)(-0.8575)$

$= -0.08575$

$\Rightarrow k_4 = h f(x_0 + h, y_0 + k_3)$

$= 0.1 f(1.1, 2.91425)$

$= (0.1)(-0.71425)$

$= -0.071425$

$$y(x_1) = y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$= 3 + \frac{1}{6}[(-0.1) + 2(-0.085) + 2(-0.08575) + (-0.071425)$$

$$= 3 + \frac{1}{6}[-0.1 - 0.17 - 0.1715 - 0.071425)$$

$$= 3 + \frac{1}{6}[-0.512925]$$

$$= 3 - 0.0854875$$

$$y(1.1) \simeq 2.9145125$$

Milne's, Adams - Bashforth Predictor and Corrector method :-

Step ① :- Consider the given differential equation as $\frac{dy}{dx} = f(x, y)$, to the initial conditions, $y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2$, $y(x_3) = y_3$.

Step ② :- Find $f_0 = f(x_0, y_0)$, $f_1 = f(x_1, y_1)$, $f_2 = (x_2, y_2)$, $f_3 = f(x_3, y_3)$ and find $y(x_4) = y_4$ using

1. Milne's method :-

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$f_4^{(P)} = f(x_4, y_4^{(P)})$$

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4^{(P)}\right]$$

2. Adams method :-

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$$

$$f_4^{(P)} = f(x_4, y_4^{(P)})$$

$$y_4^{(c)} = y_3 + \frac{h}{24}\left[9f_4^{(P)} + 19f_3 - 5f_2 + f_1\right]$$

① Given $\frac{dy}{dx} = x^2(1+y)$, $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$. Evaluate $y(1.4)$ by

(1) Milne's Predictor - corrector method.

(2) Adams - Bashforth Predictor - corrector method

Soln :- Given $\frac{dy}{dx} = x^2(1+y) = f(x,y)$

$y(1) = 1 \Rightarrow x_0 = 1, y_0 = 1$

$y(1.1) = 1.233 \Rightarrow x_1 = 1.1, y_1 = 1.233$

$y(1.2) = 1.548 \Rightarrow x_2 = 1.2, y_2 = 1.548$

$y(1.3) = 1.979 \Rightarrow x_3 = 1.3\ y_3 = 1.979$ & $h = 0.1$

$f_0 = f(x_0, y_0) = f(1,1) = 2$

$f_1 = f(x_1, y_1) = f(1.1, 1.233) = 2.7019$

$f_2 = f(x_2, y_2) = f(1.2, 1.548) = 3.6691$

$f_3 = f(x_3, y_3) = f(1.3, 1.979) = 5.0345$

1) Milne's method :-

$$\Rightarrow y_4^{(P)} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2(2.7019) - 3.6691 + 2(5.0345)\right]$$

$$y_4^{(P)} = 2.5738$$

$$\therefore f_4^{(P)} = f(1.4, 2.5738) = 7.0046$$

$$\therefore y_4^{(C)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4^{(P)}\right]$$

$$= 1.548 + \frac{0.1}{3}\left[3.6691 + 4(5.0345) + 7.0046\right]$$

$$y_4^{(C)} = 2.5750$$

$$\therefore y(x_4) = y(1.4)$$

$$= 2.5750$$

2) Adams method :-

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$$

$$= 1.979 + \frac{0.1}{24}\left[276.8975 - 216.4769 + 99.9703 - 18\right]$$

$$y_4^{(P)} = 2.5722$$

$$\therefore f_4^{(P)} = f(1.4, 2.5722) = 7.0015$$

$$\therefore y_4^{(C)} = y_3 + \frac{h}{24}\left[9f_4^{(P)} + 19f_3 - 5f_2 + f_1\right]$$

$$= 1.979 + \frac{0.1}{24}\left[63.0135 + 95.6555 - 18.3455 + 2.7019\right]$$

$$y_4^{(C)} = 2.5750$$

② Given $\frac{dy}{dx} = \frac{1}{x+y}$ $y(0) = 2$, $y(0.2) = 2.0933$, $y(0.4) = 2.1755$ $y(0.6) = 2.2493$ Compute y at $x = 0.8$ by using

(i) Milne's - Predictor corrector method

(ii) Adams - Bashforth Predictor corrector method.

Soln:- Given, $\frac{dy}{dx} = \frac{1}{x+y} = f(x, y)$

$y(0) = 2 \Rightarrow$ $x_0 = 0$, $y_0 = 2$

$y(0.2) = 2.0933$ $x_1 = 0.2$ $y_1 = 2.1755$

$y(0.4) = 2.1755$ $x_2 = 0.4$ $y_2 = 2.1755$

$y(0.6) = 2.2493$ $x_3 = 0.6$ $y_3 = 2.2493$ and $h = 0.2$

$f_0(x_0, y_0) = f(0, 2) = 0.5$

$f_1(x_1, y_1) = f(0.2, 2.0933) = 0.4360$

$f_2(x_2, y_2) = f(0.4, 2.1755) = 0.3882$

$f_3(x_3, y_3) = f(0.6, 2.2493) = 0.3509$

(i) Milne's method :-

$\Rightarrow y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$

$= 2 + \frac{4(0.2)}{3}[2 \times 0.4360 - 0.3882 + 2 \times 0.3509]$

$= 2.3162$

$\therefore f_4^{(P)} = f(0.8, 2.3162) = 0.32091$

$\therefore y_4^{(P)} = 2.3162$

$\therefore y_4^{(c)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4^{(P)}\right]$

$= 2.1755 + \frac{0.2}{3}\left[0.3882 + 4\times 0.3509 + 0.32091\right]$

$y_4^{(c)} = 2.3163$

$y(x_4) = y(0.8)$

$= 2.3163$

(ii) Adams method :-

$y_4^{(P)} = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$

$= 2.2493 + \frac{0.2}{24}\left[55\times 0.3509 - 59\times 0.3882 + 37\times 0.4360 - 9\times 0.5\right]$

$y_4^{(P)} = 2.3162$

$\therefore f_4^{(P)} = f(0.8, 2.3162) = 0.32090$

$\Rightarrow y_4^{(c)} = y_3 + \frac{h}{24}\left[9f_4^{(P)} + 19f_3 - 5f_2 + f_1\right]$

$= 2.2493 + \frac{0.2}{24}\left[9\times 0.32090 + 19\times 0.3509 - 5\times 0.3882 + 0.4360\right]$

$y_4^{(c)} = 2.3164$

$\therefore y(x_4) = y(0.8)$

$= 2.3164$

③ Apply Milne's and Adam's method to compute $y(0.4)$ given $\frac{dy}{dx} = x + y^2$ with condition

x	y
0.0	1.0000
0.1	1.1000
0.2	1.2310
0.3	1.4020

Soln :- Given $\frac{dy}{dx} = x+y^2 = f(x,y)$

$y(0.0) = 1.0000 \Rightarrow x_0 = 0.0 \quad , \quad y_0 = 1.0000$

$y(0.1) = 1.1000 \Rightarrow x_1 = 0.1 \quad , \quad y_1 = 1.1000$

$y(0.2) = 1.2310 \Rightarrow x_2 = 0.2 \quad , \quad y_2 = 1.2310$

$y(0.3) = 1.4020 \Rightarrow x_3 = 0.3 \quad , \quad y_3 = 1.4020$

$f_0 = f(x_0, y_0) = f(0.0, 1.0000) = 1$

$f_1 = f(x_1, y_1) = f(0.1, 1.1000) = 1.31$

$f_2 = f(x_2, y_2) = f(0.2, 1.2310) = 1.7154$

$f_3 = f(x_3, y_3) = f(0.3, 1.4020) = 2.2656$

(i) Milne's method :-

$$\Rightarrow y_4^{(P)} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 1.0000 + \frac{4\times 0.1}{3}\left[2\times 1.31 - 1.7154 + 2\times 2.2656\right]$$

$$= 1.72477$$

$$\therefore f_4^{(P)} = f(0.4, 1.72477) = 3.37483$$

$$\Rightarrow y_4^{(C)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4^{(P)}\right]$$

$$= 1.2310 + \frac{0.1}{3}\left[1.7154 + 4\times 2.2656 + 3.37483\right]$$

$$= 1.7027$$

(ii) Adams method :-

$$\Rightarrow y_4^{(P)} = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$$

$$= 1.4020 + \frac{0.1}{24}\left[55\times 2.2656 - 59\times 1.7154 + 37\times 1.31 - 9\times 1\right]$$

$$\Rightarrow y_4^{(P)} = 1.66395$$

$\therefore f_4^{(P)} = f(x_4, y_4^{(P)}) = f(0.4, 1.66395) = 3.1687$

$y_4^{(c)} = y_3 + \frac{h}{24}[9f_4^{(P)} + 19f_3 - 5f_2 + f_1]$

$= 1.4020 + \frac{0.1}{24}[9 \times 3.1687 + 19 \times 2.2656 - 5 \times 1.7154$

$+ 1.31]$

$\Rightarrow y_4^{(c)} = 1.6699$

(4) Given $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$, $y_{(1)} = 1$; $y_{(1.1)} = 0.9960$, $y_{(1.2)} = 0.9860$

$y_{(1.3)} = 0.9720$ find $y_{(1.4)}$ using Adams Bashforth Predictor corrector method.

Soln :- Given $\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x}$

$\frac{dy}{dx} = \frac{1-xy}{x^2} = f(x,y)$

$f_0 = f(x_0, y_0)$

$y_{(1)} = 1 \quad x_0 = 1, \quad y_0 = 1$

$y_{(1.1)} = 0.9960 \Rightarrow x_1 = 1.1, \quad y_1 = 0.9960$

$y_{(1.2)} = 0.9860 \Rightarrow x_2 = 1.2, \quad y_2 = 0.9860$

$y_{(1.3)} = 0.9720 \Rightarrow x_3 = 1.3, \quad y_3 = 0.9720 \quad \& \quad h = 0.1$

$f_0 = f(x_0, y_0) = f(1, 1) = 0$

$f_1 = f(x_1, y_1) = f(1.1, 0.9960) = -0.0790$

$f_2 = f(x_2, y_2) = f(1.2, 0.9860) = -0.1272$

$f_3 = f(x_3, y_3) = f(1.3, 0.9720) = -0.15597$

(i) Milne's method :-

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2(-0.0790) + 0.1272 + 2(-0.15597)\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[-0.34274\right]$$

$$= 0.95535$$

$$f_4^{(P)} = f(x_4, y_P^{(4)}) = 1.33602$$

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4^{(P)}\right]$$

$$= 0.9860 + \frac{0.1}{3}\left[-0.1272 + 4(-0.15597) + 1.24059\right]$$

$$= 0.95552$$

(ii) Adams method :-

$$\Rightarrow y_4^{(P)} = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$$

$$= 0.9720 + \frac{h}{24}\left[55\times(-0.15597) + 59\times 0.1272 - 37\times 0.0790\right]$$

$$y_4^{(P)} = 0.95535$$

$$\therefore f_4^{(P)} = f(x_4, y_4^{(P)}) = -0.17219$$

$$\Rightarrow y_4^{(c)} = y_3 + \frac{h}{24}\left(9f_4^{(P)} + 19f_3 - 5f_2 + f_1\right]$$

$$= 0.9720 + \frac{0.1}{24}\left[9\times(-0.17219) + 19\times(-0.15597) - 5\times(-0.1272) + (-0.0790)\right]$$

$$y_4^{(c)} = 0.95552 \;//$$

(5) Apply milne's Predictor corrector method to compute $y(2.0)$, given $\frac{dy}{dx} = \frac{1}{2}(x+y)$ and

x	y
0.0	2.0000
0.5	2.6360
1.0	3.5950
1.5	4.9680

Soln :- Given $\frac{dy}{dx} = \frac{x+y}{2} = f(x,y)$

$y(0.0) = 2.0000$, $x_0 = 0.0$, $y_0 = 2.0000$

$y(0.5) = 2.6360$, $x_1 = 0.5$, $y_1 = 2.6360$

$y(1.0) = 3.5950$, $x_2 = 1.0$, $y_2 = 3.5950$

$y(1.5) = 4.9680$, $x_3 = 1.5$, $y_3 = 4.9680$

$h = 0.5$

$f_0 = f(x_0, y_0) = f(0.0, 2.0000) = 1$

$f_1 = f(x_1, y_1) = f(0.5, 2.6360) = 1.568$

$f_2 = f(x_2, y_2) = f(1.0, 3.5950) = 2.2975$

$f_3 = f(x_3, y_3) = f(1.5, 4.9680) = 3.234$

(1) Milne's method :-

$$y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 2.0000 + \frac{4 \times 0.5}{3}[2 \times 1.568 - 2.2975 + 2 \times 3.234]$$

$$= 6.871$$

$$f_4^{(P)} = f(2.0, 6.871) = 4.4355$$

$$\Rightarrow y_4^{(c)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4^{(P)}\right]$$

$$= 3.5950 + \frac{0.5}{3}\left[2.2975 + 4\times 3.234 + 4.4355\right]$$

$$= 6.8732$$

$$y(x_4) = y(2.0) = 6.8732$$

(ii) Adams method :-

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$$

$$= 4.9680 + \frac{0.5}{24}\left[55(3.234) - 59(2.2975) + 37(1.568) - 9(1)\right]$$

$$= 6.8707$$

$$f_4^{(P)} = 4.4355$$

$$y_4^{(c)} = y_3 + \frac{h}{24}\left[f_4^{(P)} + 19f_3 - 3f_2 + f_1\right]$$

$$= 6.8732$$ //

6. Apply milne's predictor corrector method to compute $y(0) = 1$, $y(0.1) = 0.9117$, $y(0.2) = 0.8494$, $y(0.3) = 0.8061$

Soln :- $\frac{dy}{dx} = x - y^2$

$y(0) = 1$	$x_0 = 0$	$y_0 = 1$
$y(0.1) = 0.9117$	$x_1 = 0.1$	$y_1 = 0.9117$
$y(0.2) = 0.8494$	$x_2 = 0.2$	$y_2 = 0.8494$
$y(0.3) = 0.8061$	$x_3 = 0.3$	$y_3 = 0.8061$

$f_0 = (x_0 - 2y_0^2) = -1$

$f_1 = (x_1 - y_1^2) = 0.7312$

$f_2 = (x_2 - y_2^2) = -0.52148$

$f_3 = (x_3 - y_3^2) = -0.3498$

(i) Adams method :-

$y_4^{(P)} = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$

$= 0.8061 + \frac{0.1}{24}[55(-0.3498) - 59(-0.52148) +$
$37(0.7312) - 9(-1)]$

$= 0.7789$

$f_4^{(P)} = x_4 - (y_4)^2 = -0.2066$

$y_4^{(C)} = y_3 + \frac{h}{24}[f_4^{(P)} + 19f_3 - 3f_2 + f_1]$

$= 0.8061 + \frac{0.1}{24}[-6.6294]$

$= 0.7784$

(ii) Milne's method :-

$y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$

$= 1 + \frac{4(0.1)}{3}[2(0.7312) + 0.52148 + 2(-0.3498)]$

$= \frac{1+4(0.1)}{3}[1.4624 + 0.52148 - 0.6996]$

$= 0.7784$

$f_4^{(P)} = -0.2066$

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4^{(P)}\right]$$

$$= 0.8494 + \frac{0.1}{3}\left[-0.52148 + 4(-0.3498) + (-0.2066)\right]$$

$$= 0.7784$$

$$y_{(0.4)} \cong 0.7784$$