# Module - 5

## NUMERICAL SOLUTION FOR SECOND ORDER DIFFERENTIAL EQUATION

Consider the differential equation of second order

$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = Q$ —①, where $a_0, a_1, a_2$ are the functions of '$x$'.

Let $\frac{dy}{dx} = y' = z = f(x, y, z)$, then

① ⇒ $\frac{dz}{dx} = g(x, y, z)$ to the initial conditions $y(x_0) = y_0$.

$y'(x_0) = y'_0$

$\Rightarrow z(x_0) = z_0$

### R-K method :-

$y(x_1) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$z(x_1) = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$, where

$k_1 = hf(x_0, y_0, z_0)$; $l_1 = hg(x_0, y_0, z_0)$

$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$, $l_2 = hg\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$

$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$, $l_3 = hg\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$

$k_4 = hf[x_0 + h, y_0 + k_3, z_0 + l_3]$, $l_4 = hg[x_0 + h, y_0 + k_3, z_0 + l_3]$

① Solve $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$, for $x = 0.1$, correct to 4 decimals using initial conditions $y(0) = 1$, $y'(0) = 0$, by using R-K method of $4^{th}$ order.

<u>Soln</u> :- Given $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$ — ①

Let $\frac{dy}{dx} = y' = z = f(x, y, z)$

① $\Rightarrow \frac{dz}{dx} - x^2 z - 2xy = 1$

$\frac{dz}{dx} = 1 + x^2 z + 2xy = g(x, y, z)$

and given

$y(0) = 1, \quad y'(0) = 0$

$\Rightarrow x_0 = 0, \quad y_0 = 1, \quad y'_0 = z_0 = 0, \quad h = 0.1$

$K_1 = hf(x_0, y_0, z_0)$

$= (0.1) f(0, 1, 0)$

$= (0.1)(0)$

$K_1 = 0$

$\therefore l_1 = hg(x_0, y_0, z_0)$

$= (0.1) g(0, 1, 0)$

$= (0.1)(1)$

$l_1 = 0.1$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= (0.1) f(0.05, 1, 0.05)$

$= (0.1)(0.05)$

$K_2 = 0.005$

$\therefore l_2 = (0.1) g(0.05, 1, 0.05)$

$= (0.1)(1.100125)$

$l_2 = 0.11$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= (0.1) f(0.05, 1.0025, 0.055)$

$= (0.1)(0.055)$

$K_3 = 0.0055$

$\therefore l_3 = (0.1) g(0.05, 1.0025, 0.055)$

$l_3 = (0.1) g(1.1003875)$

$l_3 = 0.11$

$K_4 = hf(x_0 + h, y_0 + K_3, z_0 + l_3)$

$= (0.1) f(0 + 0.1, 1 + 0.0055, 0 + 0.1)$

$= (0.1) f(0.1, 1.0055, 0.11)$

$= (0.1)(0.11)$

$K_4 = 0.011$

$\therefore y(x_1) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 1 + \frac{1}{6}[0 + 2\times0.005 + 2\times0.0055 + 0.011]$

$y(0.1) \simeq 1.00533$ //

② Using RK method, solve $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$, for $x = 0.2$. Correct to 4 decimals using initial conditions $y(0) = 1, y'(0) = 0$

**Soln**:- Given $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$ — ①

Let $\frac{dy}{dx} = y' = z = f(x, y, z)$

① $\Rightarrow \frac{dz}{dx} = xz^2 - y^2 = g(x, y, z)$

and $y'(0) = 0$, $y(0) = 1$

$\Rightarrow x_0 = 0$, $y_0 = 1$ $y'_0 = z_0 = 0$ , $h = 0.2$

$K_1 = hf(x_0, y_0, z_0)$

$= (0.2)(0)$

$K_1 = 0$

$l_1 = hg(x_0, y_0, z_0)$

$= (0.2)\, g(0, 1, 0)$

$= (0.2)(-1)$

$l_1 = -0.2$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= (0.2) f(0.1, 1, -0.1)$

$= (0.2)(-0.1)$

$K_2 = -0.02$

$\therefore l_2 = hg(x_1, y_1, z_1)$

$= (0.2)(0.1, 1, -0.1)$

$= (0.2)(-0.999)$

$l_2 = -0.2$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= (0.2)\, f\,(0.1, 0.99, -0.1)$

$= (0.2)\,(-0.1)$

$= -0.02$

$\therefore l_3 = (0.2)\, g\,(0.1, 0.99, -0.1)$

$= (0.2)\,(-0.989)$

$l_3 = -0.1978$

$K_4 = hf\,(x_0+h, y_0+K_3, z_0+l_3]$

$= (0.2)\, f\,(0.2, 0.98, -0.1978)$

$= (0.2)\,(-0.1978)$

$K_4 = -0.03956$

$\therefore y(x_1) = y_0 + \frac{1}{6}\,[K_1+2K_2+2K_3+K_4]$

$= 1 + \frac{1}{6}\,[0 - 0.04 - 0.04 - 0.03956]$

$y(0.2) \cong 0.98$

③ Find $y(0.1)$ using R-K method given that $y'' = xy' - y$, $y(0) = 3$, $y'(0) = 0$.

<u>Soln</u>:- Given: $y'' = xy' - y$ — ①

Let $\frac{dy}{dx} = y' = z = f(x, y, z)$

① $\Rightarrow \frac{dz}{dx} = xz - y = g(x, y, z)$

and $y(0) = 3$, $y'(0) = 0$

$x_0 = 0$, $y_0 = 3$, $y'_0 = z_0 = 0$, $h = 0.1$

$K_1 = hf\,(x_0, y_0, z_0)$

$= (0.1)\, f\,(0, 3, 0)$

$= (0.1)\,(0)$

$K_1 = 0$

$\therefore l_1 = hg\,(x_0, y_0, z_0)$

$= (0.1)\, g\,(0, 3, 0)$

$= (0.1)\,(-3)$

$l_1 = -0.3$

$K_2 = hf\,\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}, z_0+\frac{l_1}{2}\right]$

$= (0.1) f\,\left(0+\frac{0.1}{2}, 3+\frac{0}{2}, 0-\frac{0.3}{2}\right]$

$= (0.1)\,(-0.15)$

$K_2 = -0.015$

$l_2 = hg\,(0.05, 3, -0.15)$

$= (0.1)\, g\,(-3.015)$

$l_2 = -0.30075$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= (0.1) f\left(0.05, 3 - \frac{0.015}{2}, -\frac{0.3015}{2}\right)$

$= (0.1) f(0.05, 2.9925, -0.15075)$

$= -0.015075$

$l_3 = (0.1)\, g(0.05, 2.9925, -0.15075)$

$l_3 = -0.30000$

$K_4 = hf[x_0 + h, y_0 + K_3, z_0 + l_3]$

$= (0.1) f[0.1, 2.9849, -0.30000]$

$K_4 = -0.03$

$y(0.1) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 3 + \frac{1}{6}[0 - 0.03 - 0.03015 - 0.03]$

$y(0.1) \simeq 2.9849$

④ Using R-K method to solve $\frac{d^2y}{dx^2} = x^3\left(y + \frac{dy}{dx}\right)$, $y(0) = 1$, $y'(0) = 0.5$, find $y$ at $x = 0.1$.

Soln :- Given: $\frac{d^2y}{dx^2} = x^3\left(y + \frac{dy}{dx}\right)$ — ①

Let $\frac{dy}{dx} = y' = z = f(x, y, z)$

① $\Rightarrow \frac{dz}{dx} = x^3(y + z) = g(x, y, z)$

and $y(0) = 1$, $y'(0) = 0.5$

$x_0 = 0$, $y_0 = 1$, $y'_0 = z_0 = 0.5$

$\Rightarrow K_1 = hf(x_0, y_0, z_0)$

$= (0.1) f(0, 1, 0.5)$

$K_1 = 0.05$

$l_1 = hg(x_0, y_0, z_0)$

$= 0.1\, g(0, 1, 0.5)$

$l_1 = 0$

$K_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right]$

$= (0.1)\, f\, (0.05, 1.025, 0.5)$

$K_2 = 0.05$

$K_3 = hf\left[x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2},\ z_0 + \frac{l_2}{2}\right]$

$= (0.1)\, f\, [0.05, 1.025, 0.5)$

$= 0.05$

$K_4 = hf\, [x_0 + h,\ y_0 + K_3,\ z_0 + l_3]$

$= (0.1)\, f\, [0.1, 1.05, 0.5]$

$= 0.05$

$y(x) = y_0 + \frac{1}{6}\, [K_1 + 2K_2 + 2K_3 + K_4]$

$= 1 + \frac{1}{6}\, [0.05 + 0.1 + 0.1 + 0.05]$

$y(0.1) \simeq 1.05$

$l_2 = hg\, (x_2, y_2, z_3)$

$= 0.1 g\, (0.05, 1.025, 0.5)$

$= 0$

$l_3 = hg\, (x_3, y_3, z_3)$

$= (0.1) g\, (0.05, 1.025, 0.5)$

$l_3 = 0$

① Apply Milne's method to compute $y(0.8)$ given that $y'' = 1 - 2yy'$

| $x$ | 0 | 0.2 | 0.4 | 0.6 |
|---|---|---|---|---|
| $y$ | 0 | 0.02 | 0.0795 | 0.1762 |
| $y' = z$ | 0 | 0.1996 | 0.3972 | 0.5689 |

Sol:- Given $y'' = 1 - 2yy'$ —①

Let $y' = z = f(x, y, z)$

$\therefore$ ① $\Rightarrow \frac{dz}{dx} = 1 - 2yz = g(x, y, z)$

and given

$x_0 = 0$ $\quad y_0 = 0$ $\quad y'_0 = z_0 = 0$

$x_1 = 0.2$ $\quad y_1 = 0.02$ $\quad y'_1 = z_1 = 0.1996$

$x_2 = 0.4$ $\quad y_2 = 0.0795$ $\quad y'_2 = z_2 = 0.3972$

$x_3 = 0.6$ $\quad y_3 = 0.1762$ $\quad y'_3 = z_3 = 0.5689$

$f_0 = f(x_0, y_0, z_0) = 0$ $\qquad$ $g_0 = 1 - 2y_0 z_0 = 1$

$f_1 = f(x_1, y_1, z_1) = 0.1996$ $\qquad$ $g_1 = 1 - 2y_1 z_1 = 0.9920$

$f_2 = f(x_2, y_2, z_2) = 0.3972$ $\qquad$ $g_2 = 1 - 2y_2 z_2 = 0.93684$

$f_3 = f(x_3, y_3, z_3) = 0.5689$ $\qquad$ $g_3 = 1 - 2y_3 z_3 = 0.7995$

$$\therefore y_4^{(P)} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 0 + \frac{4 \times 0.2}{3}\left[2 \times 0.1996 - 0.3972 + 2 \times 0.5689\right]$$

$$y_4^{(P)} = 0.3039$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left[2g_1 - g_2 + 2g_3\right]$$

$$= 0 + \frac{4 \times 0.2}{3}\left[1.984 - 0.93684 + 1.599\right]$$

$$z_4^{(P)} = 0.7056$$

$$\therefore f_4^{(P)} = 0.7056$$

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4^{(P)}\right]$$

$$= 0.0795 + \frac{0.2}{3}\left[0.3972 + 4 \times 0.5689 + 0.7056\right]$$

$$y_4^{(C)} = 0.3047$$

$$y(0.8) = 0.3047$$

(2) Apply Milne's Predictor - corrector method to compute $y(0.4)$ given the differential equation $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$ and the following table of initial values.

| $x$ | 0 | 0.1 | 0.2 | 0.3 |
|---|---|---|---|---|
| $y$ | 1 | 1.1103 | 1.2427 | 1.3990 |
| $z$ | 1 | 1.2103 | 1.4427 | 1.6990 |

Soln: Given $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$ — (1)

Let $y' = \frac{dy}{dx} = z = f(x, y, z)$

① $\Rightarrow \frac{dz}{dx} = 1 + z = g(x, y, z)$

and given

$x_0 = 0 \qquad y_0 = 1 \qquad y'_0 = z_0 = 1$

$x_1 = 0.1 \qquad y_1 = 1.1103 \qquad y'_1 = z_1 = 1.2103$

$x_2 = 0.2 \qquad y_2 = 1.2427 \qquad y_2' = z_2 = 1.4427$

$x_3 = 0.3 \qquad y_3 = 1.3990 \qquad y_3' = z_3 = 1.6990$

$\therefore f_0 = f(x_0, y_0, z_0) = 1 \qquad g_0 = 1 + z_0 = 2$

$f_1 = f(x_1, y_1, z_1) = 1.2103 \qquad g_1 = 1 + z_1 = 2.2103$

$f_2 = f(x_2, y_2, z_2) = 1.4427 \qquad g_2 = 1 + z_2 = 2.4427$

$f_3 = f(x_3, y_3, z_3) = 1.6990 \qquad g_3 = 1 + z_3 = 2.6990$

$$y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 1 + \frac{4 \times 0.1}{3}[2 \times 1.2103 - 1.4427 + 2 \times 1.6990]$$

$$y_4^{(P)} = 1.5835$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}[2g_1 - g_2 + 2g_3]$$

$$= 1 + \frac{4 \times 0.1}{3}[2 \times 2.2103 - 2.4427 + 2 \times 2.6990]$$

$$z_4^{(P)} = 1.9835$$

$$\therefore f_4^{(P)} = 1.9835$$

$$y_4^{(c)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$$

$$= 1.2427 + \frac{0.1}{3}[1.4427 + 4 \times 1.6990 + 1.9835]$$

$$y_4^{(c)} = 1.5835$$

$$y(0.4) = 1.5835 //$$

③ Apply Milne's method to find $y(0.4)$ for the given D.E

$$\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6y = 0$$

| $x$ | 0 | 0.1 | 0.2 | 0.3 |
|---|---|---|---|---|
| $y$ | 1 | 1.03995 | 1.138036 | 1.29865 |
| $z$ | 0.1 | 0.6995 | 1.2580 | 1.8730 |

Soln:- Given $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6y = 0$ — ①

Let $y' = \frac{dy}{dx} = z = f(x,y,z)$

① $\Rightarrow \frac{dz}{dx} = -3xz + 6y$

$$\frac{dz}{dx} = 6y - 3xz = g(x,y,z)$$

and

$x_0 = 0 \qquad y_0 = 1 \qquad y'_0 = z_0 = 0.1$

$x_1 = 0.1 \qquad y_1 = 1.03995 \qquad y'_1 = z_1 = 0.6995$

$x_2 = 0.2 \qquad y_2 = 1.138036 \qquad y'_2 = z_2 = 1.2580$

$x_3 = 0.3 \qquad y_3 = 1.29865 \qquad y'_3 = z_3 = 1.8730$

$\therefore f_0 = f(x_0, y_0, z_0) = 1 \qquad g_0 = 6y_0 - 3x_0 z_0 = 6$

$f_1 = f(x_1, y_1, z_1) = 0.6995 \qquad g_1 = 6y_1 - 3x_1 z_1 = 6.02985$

$f_2 = f(x_2, y_2, z_2) = 1.2580 \qquad g_2 = 6y_2 - 3x_2 z_2 = 6.073416$

$f_3 = f(x_3, y_3, z_3) = 1.8730 \qquad g_3 = 6y_3 - 3x_3 z_3 = 6.1062$

$$y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 1 + \frac{4 \times 0.1}{3}[2 \times 0.6995 - 1.2580 + 2 \times 1.8730]$$

$$y_4^{(P)} = 1.5183$$

$$\therefore z_4^{(P)} = z_0 + \frac{4h}{3}\left[2g_1 - g_2 + 2g_3\right]$$

$$= 0.1 + \frac{4\times 0.1}{3}\left[2\times 6.02985 - 6.02985 + 2\times 6.073416\right]$$

$$= 2.5236$$

$$\therefore f_4^{(P)} = 2.5236$$

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4^{(P)}\right]$$

$$= 1.138036 + \frac{0.1}{3}\left[1.2580 + 4\times 1.8730 + 2.5236\right]$$

$$y_4^{(C)} = 1.5138$$

(4) Use milne's method, obtain an approximate solution at the point $x = 0.8$ of the problem $y'' = 2yy'$, given $y(0) = 0$, $y'(0) = 1$, $y(0.2) = 0.2027$, $y'(0.2) = 1.041$, $y(0.4) = 0.4228$, $y'(0.4) = 1.179$ $y(0.6) = 0.6841$ $y'(0.6) = 1.468$.

Soln :- Given $y'' = 2yy'$ —(1)

Let $y' = z = f(x, y, z)$

$$(1) \Rightarrow \frac{dz}{dx} = 2yz = g(x, y, z)$$

and

| | | |
|---|---|---|
| $x_0 = 0$ | $y_0 = 0$ | $y'_0 = z_0 = 1$ |
| $x_1 = 0.2$ | $y_1 = 0.2027$ | $y'_1 = z_1 = 1.041$ |
| $x_2 = 0.4$ | $y_2 = 0.4228$ | $y'_2 = z_2 = 1.179$ |
| $x_3 = 0.6$ | $y_3 = 0.6841$ | $y'_3 = z_3 = 1.468$ |

| | |
|---|---|
| $f_0 = f(x_0, y_0, z_0) = 1$ | $g_0 = 2y_0 z_0 = 0$ |
| $f_1 = f(x_1, y_1, z_1) = 1.041$ | $g_1 = 2y_1 z_1 = 0.4220$ |
| $f_2 = f(x_2, y_2, z_2) = 1.179$ | $g_2 = 2y_2 z_2 = 0.9969$ |
| $f_3 = f(x_3, y_3, z_3) = 1.468$ | $g_3 = 2y_3 z_3 = 2.0085$ |

$$y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 0 + \frac{4\times 0.2}{3}[2\times 1.041 - 1.179 + 2\times 1.468]$$

$$y_4^{(P)} = 1.0237$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}[2g_1 - g_2 + 2g_3]$$

$$= 1 + \frac{4\times 0.2}{3}[2\times 0.4220 - 0.9969 + 2\times 2.0085]$$

$$z_4^{(P)} = 2.0304$$

$$\therefore f_4^{(P)} = 2.0304$$

$$y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$$

$$= 0.4228 + \frac{0.2}{3}[1.179 + 4\times 1.468 + 2.0304]$$

$$= 1.02823$$

$$y(0.8) = 1.02823$$

⑤ Obtain the solution of the equation $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$, by computing the value of the dependent variable corresponding to the value 1.4 of the independent variable by applying Milne's method using the following data.

| $x$ | 1 | 1.1 | 1.2 | 1.3 |
|---|---|---|---|---|
| $y$ | 2 | 2.2156 | 2.4649 | 2.7514 |
| $z$ | 2 | 2.3178 | 2.6725 | 2.0657 |

Soln:- $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ — ①

Let $y' = \frac{dy}{dx} = z = f(x, y, z)$

① $\Rightarrow 2\frac{dz}{dx} = 4x + z$

$$\frac{dz}{dx} = 2x + \frac{z}{2} = g(x, y, z) \quad \text{— ②}$$

and

| | | |
|---|---|---|
| $x_0 = 1$ | $y_0 = 2$ | $y_0' = z_0 = 2$ |
| $x_1 = 1.1$ | $y_1 = 2.2156$ | $y_1' = z_1 = 2.3178$ |
| $x_2 = 1.2$ | $y_2 = 2.4649$ | $y_2' = z_2 = 2.6725$ |
| $x_3 = 1.3$ | $y_3 = 2.7514$ | $y_3' = z_3 = 2.0657$ |

$\therefore f_0 = f(x_0, y_0, z_0) = 2$

$f_1 = f(x_1, y_1, z_1) = 2.3178$

$f_2 = f(x_2, y_2, z_2) = 2.6725$

$f_3 = f(x_3, y_3, z_3) = 2.0657$

$g_0 = 2x_0 + \frac{z_0}{2} = 3$

$g_1 = 2x_1 + \frac{z_1}{2} = 3.3589$

$g_2 = 2x_2 + \frac{z_2}{2} = 3.73625$

$g_3 = 2x_3 + \frac{z_3}{3} = 3.63285$

$$\therefore y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 2 + \frac{4 \times 0.1}{3}[2 \times 2.3178 - 2.6725 + 2 \times 2.0657]$$

$$y_4^{(P)} = 2.8126$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}[2g_1 - g_2 + 2g_3]$$

$$= 2 + \frac{4 \times 0.1}{3}[2 \times 3 - 3.73625 + 2 \times 3.63285]$$

$$z_4^{(P)} = 3.2706$$

$$\therefore f_4^{(P)} = 3.2706$$

$$y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$$

$$= 2.4649 + \frac{0.1}{3}[2.6725 + 4 \times 3.0657 + 3.2706]$$

$$= 3.07176$$

$$y(1.4) = 3.07176 \;//$$

⑥ Apply Milne's method to compute $y(0.4)$ given that $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$, $y(0.1) = 0.995$, $y'(0.1) = -0.0995$, $y(0.2) = 0.9802$, $y'(0.2) = -0.196$, $y(0.3) = 0.956$, $y'(0.3) = -0.2863$

Soln:- $y'' + xy' + y = 0$

Let $y' = z = f(x, y, z)$

$$\frac{dz}{dx} = -[y + zx] = g(x, y, z)$$

| | | |
|---|---|---|
| $x_0 = 0$ | $y_0 = 1$ | $y'_0 = z_0 = 0$ |
| $x_1 = 0.1$ | $y_1 = 0.995$ | $y'_1 = z_1 = -0.0995$ |
| $x_2 = 0.2$ | $y_2 = 0.9802$ | $y_2' = z_2 = -0.196$ |
| $x_3 = 0.3$ | $y_3 = 0.956$ | $y_3' = z_3 = -0.2863$ |

$f_0 = f(x_0, y_0, z_0) = 0$ $\qquad g_0 = -[y_0 + z_0 x_0] = -1$

$f_1 = f(x_1, y_1, z_1) = -0.0995$ $\qquad g_1 = -[y_1 + z_1 x_1] = -0.98505$

$f_2 = f(x_2, y_2, z_2) = -0.196$ $\qquad g_2 = -[y_2 + z_2 x_2] = -0.941$

$f_3 = f(x_3, y_3, z_3) = -0.2863$ $\qquad g_3 = -[y_3 + z_3 x_3] = -0.87011$

$$y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 1 + \frac{4 \times 0.1}{3}[2(-0.0995) + 0.196 - 2 \times (-0.2863)]$$

$$= 0.92326$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}[2g_1 - g_2 + 2g_3]$$

$$= 0 + \frac{4 \times 0.1}{3}[2(-0.98505) + 0.941 - 2(0.87011)]$$

$$= -0.36923$$

$f_4^{(p)} = -0.36923$

$y_4^{(c)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4^{(p)}\right]$

$= 0.9802 + \frac{0.1}{3}\left[-0.196 + 4(-0.2863) - 0.36923\right]$

$= 0.92325$

$y(0.4) = 0.92325$ //

## Calculus of Variations:-

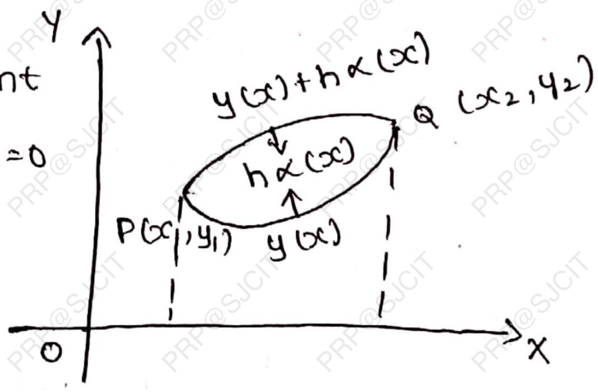
### Euler's theorem :-

A necessary condition for the integral $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$ where $y(x_1) = y_1$, $y(x_2) = y_2$, to be as extremum that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

Proof:- Let the curve $y = y(x)$ passing through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ and make $I$ as extremal. Also Let $y = y(x) + h\alpha(x)$ be the neighbouring curve passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$ be an extremal.

The curves both are coincident at P and Q $\Rightarrow \alpha(x_1) = 0$, $\alpha(x_2) = 0$




Given: $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$ —①

$$I = \int_{x_1}^{x_2} f\left(x_1, y(x) + h\alpha(x), y'(x) + h\alpha'(x)\right] dx \quad -①$$

Diff eqⁿ ① w.r.t to $h$ on both sides.

$$\Rightarrow \frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial h}\right] dx$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial x}(0) + \frac{\partial f}{\partial y}(0) + \frac{\partial f}{\partial y'}(\alpha'(x)] dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}(\alpha(x)) + \frac{\partial f}{\partial y'}(\alpha'(x))] dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \alpha(x)\frac{\partial f}{\partial y} dx + \int_{x_1}^{x_2} \alpha'(x)\frac{\partial f}{\partial y'} dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \alpha(x)\frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial y'}\int_{x_1}^{x_2} \alpha'(x) dx - \int_{x_1}^{x_2}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\int \alpha(x) dx\right] dx$$

$$= \int_{x_1}^{x_2} \alpha(x)\frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial y'}\left[\alpha(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\alpha(x) dx$$

$$= \int_{x_1}^{x_2} \alpha(x)\frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial y'}\left[\alpha(x_2) - \alpha(x_1)\right] - \int_{x_1}^{x_2} \alpha(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx]$$

$$= \int_{x_1}^{x_2} \alpha(x)\frac{\partial f}{\partial y} dx - \int_{x_1}^{x_2} \alpha(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\alpha(x) dx \quad - (2)$$

For the extremum of I, then $\frac{dI}{dh} = 0$

$(2) \Rightarrow \frac{dI}{dh} = 0$

$$\Rightarrow \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\alpha(x) dx = 0$$

$$\Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Hence proved //

① Find the extremal for the function $\int_0^{\pi/2}(y^2-y'^2-2y\sin x)\,dx$, $y(0)=0$, $y\left(\frac{\pi}{2}\right)=1$

Soln :- Given $I=\int_{x_1}^{x_2} f(x,y,y')\,dx$

$$=\int_0^{\pi/2}[y^2-y'^2-2y\sin x]\,dx$$

$f(x,y,y') = y^2-y'^2-2y\sin x$

$\therefore$ WKT for the extremum of $I$

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$$

$$\frac{\partial f}{\partial y}=2y-2y\sin x$$

$$\frac{\partial f}{\partial y'}=-2y'$$

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$$

$$2y-2\sin x-\frac{d}{dx}(-2y')=0$$

$$y-\sin x+\frac{d}{dx}(y')=0$$

$$y-\sin x+y''=0$$

$$y''+y=\sin x$$

$$(D^2+1)y=\sin x$$

Auxiliary equation is

$$m^2+1=0$$

$$m=-1$$

$$m=0\pm i$$

$$y_c=c_1\cos x+c_2\sin x$$

$$y_p = \frac{\sin x}{D^2+1}$$

$$= -\frac{x}{2(1)} \cos x$$

$$y_p = -\frac{x}{2} \cos x$$

$$y = y_c + y_p$$

$$y = c_1 \cos x + c_2 \sin x - \frac{x}{2} \cos x \quad -①$$

where $x = 0, y = 0$

$① \Rightarrow 0 = c_1(1)$

$c_1 = 0$

where $x = \frac{\pi}{2} \Rightarrow y = 1$

$① \Rightarrow 1 = c_1(0) + c_2(1) - 0$

$c_2 = 1$

$① \Rightarrow y = \sin x - \frac{x}{2} \cos x$.

② On what curves can the functional $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\, dx$. $y(0) = 0$, $y(\frac{\pi}{2}) = 0$ be extremised.

Given $\int_{x_1}^{x_2} f(x, y, y') = \int_0^{\pi/2} (y'^2 - y^2 + 2xy)\, dx$

$$f(x, y, y') = y'^2 - y^2 + 2xy$$

$$\frac{\partial f}{\partial y} = 2x - 2y$$

$$\frac{\partial f}{\partial y'} = 2y'$$

WKT

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$2x - 2y - \frac{d}{dx}(2y') = 0$$

$$x - y - \frac{d}{dx}(y') = 0$$

$$x - y - y'' = 0$$

$$y'' + y = x$$

$$(D^2+1)y = x \quad \text{—} ①$$

The A E $\quad m^2+1 = 0$

$$m = -1$$

$$m = 0 \pm i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$y_p = \frac{x}{D^2+1}$$

$$y_p = (1+D^2)^{-1} x$$

$$y_p = (1 - D^2 + D^4 - D^6 + \dots)x$$

$$y_p = x$$

$$y = y_c + y_p$$

$$y = c_1 \cos x + c_2 \sin x + x \quad \text{—} ①$$

when $x = 0$, $y = 0$

$$① \Rightarrow 0 = c_1(1) + 0 + 0$$

$$c_1 = 0$$

when $x = \frac{\pi}{2}$, $y = 0$

$$① \Rightarrow 0 = c_1(0) + c_2(1) + \frac{\pi}{2}$$

$$c_2 = -\frac{\pi}{2}$$

$$y = -\frac{\pi}{2}\sin x + x \;//$$

③ Find the extremal of the functional $\int_0^{\pi/2} (y'^2 - y^2 + 4y\cos x)\,dx$

$y(0) = 0, \; y(\pi/2) = 0$.

Given $I = \int_{x_1}^{x_2} f(x, y, y')\,dx = \int_{0}^{\pi/2} (y'^2 - y^2 + 4y\cos x)\,dx$

$f(x, y, y') = y'^2 - y^2 + 4y\cos x$

$\frac{\partial f}{\partial y} = -2y + 4\cos x$

$\frac{\partial f}{\partial y'} = 2y'$

WKT $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$-2y + 4\cos x - \frac{d}{dx}(2y') = 0$

$-y + 2\cos x - \frac{d}{dx}(\cancel{2}y') = 0$

$-y + 2\cos x - y'' = 0$

$y'' + y = 2\cos x$

$(D^2 + 1)y = 2\cos x$

The Auxillary equation is

$m^2 + 1 = 0$

$m^2 = -1$

$m = 0 \pm i$

$y_c = c_1\cos x + c_2\sin x$

$\frac{\cos ax}{D^2 + a^2} = \frac{x}{2a}\sin ax$

$\frac{\sin ax}{D^2 + a^2} = -\frac{x}{2a}\cos ax$

$y_p = \frac{2\cos x}{D^2 + 1}$

$= \frac{\cancel{2}(x)}{\cancel{2}(1)}\sin x$

$y_p = x\sin x$

$y = y_c + y_p$

$y = c_1\cos x + c_2\sin x + x\sin x$ — (1)

where $x = 0$, $y = 0$

$$(1) \Rightarrow 0 = c_1(1) + 0 + 0$$

$$c_1 = 0$$

when $x = \pi/2$, $y = 0$

$$(1) \Rightarrow 0 = c_1(0) + c_2(1) + \frac{\pi}{2}(1)$$

$$c_2 = -\frac{\pi}{2}$$

$$\therefore y = -\frac{\pi}{2}\sin x + x\sin x \,//$$

(4) Find the extremal of the function $\int_0^1 (y'^2 - y^2 - ye^{2x})\,dx$ that passes through the point $(0,0)$ $(1, \frac{1}{e})$.

$$\text{Let } I = \int_{x_1}^{x_2} f(x, y, y')\,dx = \int_0^{\pi/2} (y'^2 - y^2 - ye^{2x})\,dx$$

$$f(x, y, y') = y'^2 - y^2 - ye^{2x}$$

$$\frac{\partial f}{\partial y} = -2y - e^{2x}$$

$$\frac{\partial f}{\partial y'} = 2y'$$

WKT $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$

$$-2y - e^{2x} - \frac{d}{dx}(2y') = 0$$

$$-y - e^{2x} - y'' = 0$$

$$y'' + y = e^{-2x}$$

$$(D^2 + 1)y = e^{-2x}$$

The AE $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = 0 \pm i$$

$y_c = c_1 \cos x + c_2 \sin x$

$$y_p = \frac{-e^{2x}}{(D^2+1)}$$

$$= \frac{-e^{2x}}{2^2+1} = \frac{-e^{2x}}{5}$$

$$y_p = -\frac{1}{5} e^{2x}$$

$y = y_c + y_p$

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{5} e^{2x} \quad \text{--- (2)}$$

when $x = 0,\ y = 0$

$$0 = c_1(1) + 0 - \frac{1}{5}$$

$$c_1 = \frac{1}{5}$$

$x = 1,\ y = \frac{1}{e}$

$$\text{eq}^n (1) \Rightarrow \quad \frac{1}{e} = \frac{1}{5} \cos(1) + c_2 \sin(1) - \frac{1}{5} e^{2x}$$

$$0.3678 = \frac{1}{e} = 0.10806 + 0.8457 - 1.4775$$

$$c_2 (0.8457) = 1.8340$$

$$c_2 = \frac{1.8340}{0.8457}$$

$$c_2 = 2.1686$$

$$y = \frac{1}{5} \cos x + 2.1686 \sin x - \frac{1}{5} e^{2x} \;//$$

(5) Find the extremal of the function $\int_{x_1}^{x_2} (y'^2 - y^2 + 2y \sec x)\,dx$

Let $I = \int_{x_1}^{x_2} f(x, y, y')\,dx = \int_{x_1}^{x_2} (y'^2 - y^2 + 2y \sec x)\,dx$

$f = (y'^2 - y^2 + 2y\sec x)$

$\frac{\partial f}{\partial y} = -2y + 2\sec x$

$\frac{\partial f}{\partial y'} = 2y'$

WKT

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$-2y + 2\sec x - \frac{d}{dx}(2y') = 0$

$-y + \sec x - \frac{d}{dx}(y') = 0$

$-y + \sec x - y'' = 0$

$y'' + y = \sec x$

$(D^2 + 1)y = \sec x$

The A.E

$m^2 + 1 = 0$

$m = 0 \pm 1i$

$y_c = c_1 \cos x + c_2 \sin x$

Let $y = Ay_1 + By_2$

when $y_1 = \cos x$, $y_2 = \sin x$

$y_1' = -\sin x$, $y_2' = \cos x$

$w = y_1 y_2' - y_2 y_1'$

$= (\cos x)(\cos x) - (\sin x)(-\sin x)$

$w = 1$

$A = \int \frac{y_2 Q(x)}{w} dx + k_1$

$$= -\int \frac{\sin x \sec x}{1} dx + K_1$$

$$= -\int \frac{\sin x}{\cos x} dx + K_1$$

$$A = \log|\cos x| + K_1$$

$$B = \int \frac{y_1 Q(x)}{\omega} dx + K_2$$

$$= \int \frac{\cos x \sec x}{1} dx + K_2$$

$$= \int 1 dx + K_2$$

$$B = x + K_2$$

$$y = \log|\cos x| + K_1 \cdot \cos x + (x + K_2)\sin x \quad //$$

(6) Find the extremum for the function $\int_{x_1}^{x_2} (y^2 + y'^2 + 2ye^x)dx$

$$I = \int_{x_1}^{x_2} f(x, y, y')dx = \int_{x_1}^{x_2} (y^2 + y'^2 + 2ye^x)dx$$

$$f(x, y, y') = y^2 + y'^2 + 2ye^x$$

$$\frac{\partial f}{\partial y} = 2y + 2e^x$$

$$\frac{\partial f}{\partial y'} = 2y'$$

WKT

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$2y + 2e^x - \frac{d}{dx}(2y') = 0$$

$$y + e^x - \frac{d}{dx}(y') = 0$$

$$y + e^{x} - y'' = 0$$

$$y'' - y = e^{x}$$

$$(D^2-1)y = e^{x}$$

The AE $m^2-1=0$

$$m^2 = 1$$

$$m = \pm 1$$

$$y_c = c_1 e^{-x} + c_2 e^{x}$$

$$y_p = \frac{e^{x}}{D^2-1} = \frac{e^{x}}{(D-1)(D+1)} = \frac{x^1}{1!}\frac{e^{x}}{D+1} = \frac{x e^{x}}{2}$$

$$y = y_c + y_p$$

$$y = c_1 e^{x} + c_2 e^{x} + \frac{x e^{x}}{2}$$ //

⑦ Find the extremum for the functional $\int_0^1 (y'^2 + 2xy)\,dx$ with $y(0)=0$, $y(1)=1$

Let $\int_{x_1}^{x_2} f(x, y, y') = \int_0^1 (y'^2 + 2xy)\,dx$

$$f(x, y, y') = y'^2 + 2xy$$

$$\frac{\partial f}{\partial y} = 2x$$

$$\frac{\partial f}{\partial y'} = 2y'$$

WKT $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$2x - \frac{d}{dx}(2y') = 0$$

$$x - \frac{d}{dx}(y') = 0$$

$$x - y'' = 0$$

$$y'' = x$$

$$\frac{d^2y}{dx^2} = x$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = x$$

$$\int d\left(\frac{dy}{dx}\right) = \int x\, dx$$

$$\frac{dy}{dx} = \frac{x^2}{2} + c_1$$

$$\int y = \int \frac{x^2}{2}\, dx + c_1 \int 1\, dx$$

$$y = \frac{x^3}{6} + c_1 x + c_2 \quad \text{— (1)}$$

when $x = 0$, $y = 0$

$$0 = c_2 \Rightarrow c_2 = 0$$

$$x = 1, \quad y = 1$$

$$1 = \frac{1}{6} + c_1 + 0$$

$$c_1 = 1 - \frac{1}{6}$$

$$c_1 = \frac{5}{6}$$

$$y = \frac{x^3}{6} + \frac{5}{6}x \quad //$$

(8) Find the extremum of the functional $\int_{x_1}^{x_2} [y' + x^2 y'^2]\, dx$

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx = \int_{x_1}^{x_2} (y' + x^2 y'^2)\, dx$$

$$f(x, y, y') = y' + x^2 y'^2$$

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = 1 + 2x^2 y'$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$0 - \frac{d}{dx}\left[1 + 2x^2 y'\right] = 0$$

$$\frac{d}{dx}\left[1 + 2x^2 y'\right] = 0$$

$$0 + 2\left[2xy' + x^2 y''\right] = 0$$

$$x^2 y'' + 2xy' = 0$$

$$\log x = z \Rightarrow x = e^z$$

$$xy' = Dy$$

$$x^2 y'' = D(D-1)y \qquad D = \frac{d}{dz}$$

$$① \Rightarrow D(D-1)y + 2Dy = 0$$

$$(D^2 - D)y + 2Dy = 0$$

$$D^2 y - Dy + 2Dy = 0$$

$$(D^2 + D)y = 0$$

The AE $m^2 + m = 0$

$$m(m+1) = 0$$

$$m = 0,\ m = -1$$

$$y = c_1 + c_2 e^{-z}$$

$$= c_1 + \frac{c_2}{e^z}$$

$$y = c_1 + \frac{c_2}{x}$$

9. Prove that geodesic of a plane surface are straight line.

WKT the arclength of a Geodesic is $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

(or) $\frac{ds}{dx} = \sqrt{1+y'^2}$

Therefore the extremum of a Geodesic can be defined

as $I = \int_{x_1}^{x_2} f(x, y, y')\, dx = \int_{x_1}^{x_2} \frac{ds}{dx}\, dx$

$$= \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$$

$$\Rightarrow f(x, y, y') = \sqrt{1+y'^2}$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = \frac{1}{\cancel{2}\sqrt{1+y'^2}} (\cancel{2}y')$$

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

WKT

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0$$

$$\frac{\left(\sqrt{1+y'^2}\right) y'' - y' \frac{1}{\cancel{2}\sqrt{1+y'^2}} (\cancel{2}y'y'')}{\left(\sqrt{1+y'^2}\right)^2} = 0$$

$$\Rightarrow \left(\sqrt{1+y'^2}\right) y'' - \frac{y'^2 y''}{\sqrt{1+y'^2}} = 0$$

$$\frac{(1+y'^2)\, y'' - y'^2 y''}{\sqrt{1+y'^2}} = 0$$

$$y'' + \cancel{y'^2 y''} - \cancel{y'^2 y''} = 0$$

$$y'' = 0$$

$$D^2 y = 0 \;, \quad D = \frac{d}{dx}$$

A.E is $m^2 = 0$

$$m = 0, 0$$

$$y = c_1 + c_2 x$$

↳ straight line equation.