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S.J.C Institute of Technology, Chickballapur

Department of Mathematics

Lecture notes

Complex Analysis, Probability and Statistical Methods (18MATH41)

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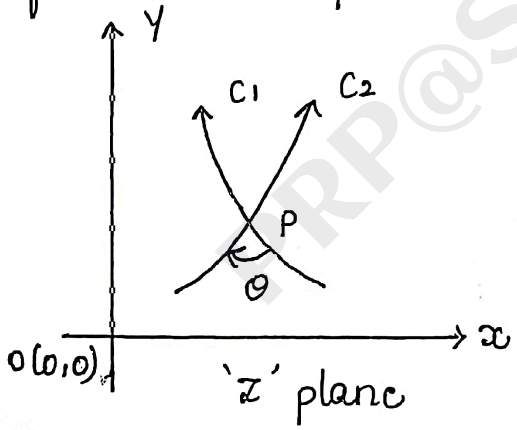
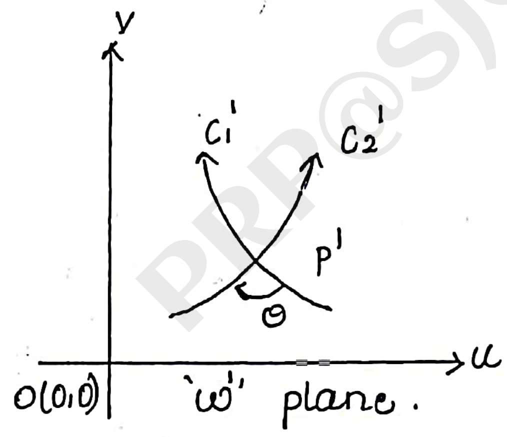
MODULE-02

CONFORMAL MAPPING (OR) TRANSFORMATION

Definition:

If the angle between any 2 curves in the magnitude and its direction is called conformal transformation (or)
Suppose 2 curves C_1, C_2 in the 'z' plane intersect at the point 'p' and the corresponding curves C_1', C_2' in the 'w' plane intersect at 'p'' and if the angle of intersection of two curves 'p' is the same as the angle of intersection of the curves at 'p'' in magnitude and direction, then this Transformation is called the Conformal transformation.





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Condition for $w=f(z)$ to represent a conformal transformation

1) Necessary Condition :

If $w=f(z)$ represents a conformal transformation of a domain (D) in the 'z' plane into a domain (D') of the 'w' plane, then $f(z)$ is analytic function of z in D.

2) Sufficient condition :

Let $w=f(z)$ be an analytic function of z in a region (D) of the 'z' plane and let $f'(z) \neq 0$ inside 'D' then the mapping $w=f(z)$ is conformal at the points of D.

1) Discussion of transformation $w=e^z$.

Given, $w=f(z)=e^z$ is analytic

$\therefore f(z)$ is differentiable

$\Rightarrow f'(z)=e^z, \; z \neq 0$

we have, $w=f(z)=u+iv=e^z \rightarrow (1)$

Let $z=x+iy$

$\therefore (1) \Rightarrow u+iv=e^{x+iy}$

$u+iv=e^x . e^{iy}$

$u+iv=e^x(\cos y+i\sin y)$

$u+iv=e^x\cos y+i\, e^x \sin y$.

$u=e^x\cos y \rightarrow (2) \qquad v=e^x \sin y \rightarrow (3)$

By eliminating 'y' we have

$(2)^2+(3)^2 = u^2+v^2 = (e^x\cos y)^2+(e^x . \sin y)^2$

$u^2+v^2=e^{2x}(\cos^2 y+\sin^2 y)$

$u^2+v^2=e^{2x} \rightarrow (4)$

By eliminating 'x' we have

$$\frac{(3)}{(2)} \Rightarrow \frac{v}{u} = \frac{e^x \sin y}{e^x \cos y} = \frac{\sin y}{\cos y} = \tan y$$

$\Rightarrow v=\tan y . u \rightarrow (5)$

Case (1) :- Let $x=C_1$ = Constant, we have

eqn (4) $\Rightarrow u^2+v^2=e^{2C_1}=r_1$

$u^2+v^2=r^2 \longrightarrow (6)$

⑥ ⇒ represents a circle having the centre as origin with radius 'r'.

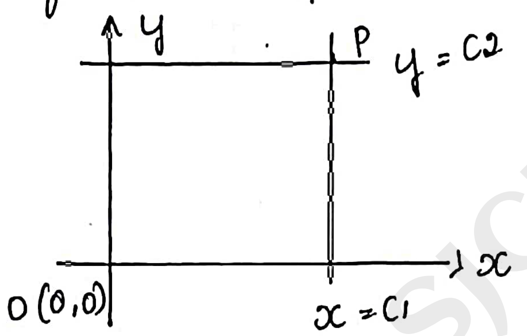
Case ② :- Let $y = C_2 =$ Constant, we have

$$eq^n ⑤ \Rightarrow V = \tan y \cdot u$$

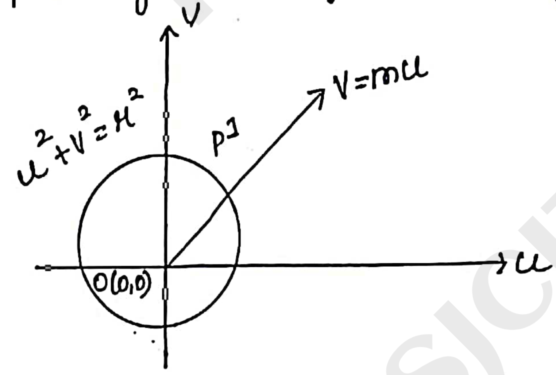
$$V = \tan C_2 \cdot u$$

$$V = m \cdot u \rightarrow ⑦ \text{ where } m = \tan C_2 = \text{Slope}$$

represents a straight line passing through the origin having the slope 'm'.



'z' plane.



'w' plane.

2) Discussion of Transformation $w = z^2$

Given, $w = f(z) = z^2$ is analytic

$\therefore f(z)$ is differentiable

$\Rightarrow f'(z) = 2z, \; z \neq 0$

we have, $w = f(z) = u + iv = z^2 \rightarrow ①$

Let $z = x + iy$

$① \Rightarrow u + iv = z^2$

$u + iv = (x+iy)^2 = x^2 + i^2y^2 + 2ixy$

$\Rightarrow u + iv = (x^2 - y^2) + i(2xy) \rightarrow *$

$u = x^2 - y^2 \rightarrow ②$ $\quad v = 2xy \rightarrow ③$

By eliminating (u, v, x, y) continously we have,

Case ① : Let $u = C_1 =$ Constant

$② \Rightarrow x^2 - y^2 = C_1 \rightarrow ④$

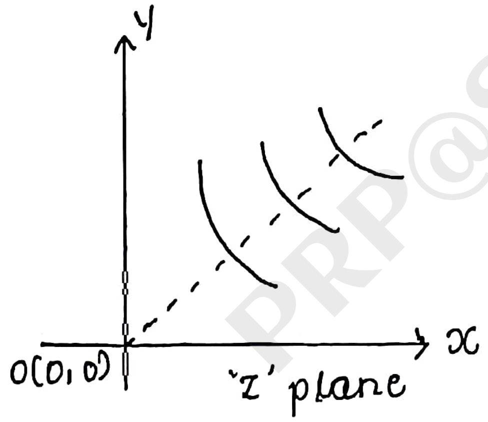
eq^n ④ represents a hyperbola.

Case ② : Let $V = C_2$

$③ \Rightarrow 2xy = C_2$

$$xy = \frac{C_2}{2} \rightarrow (5)$$

eqⁿ (5) represents a rectangular hyperbola.



Case (3) : Let $x = C_3$

$$(2) \Rightarrow u = C_3^2 - y^2$$

$$y^2 = C_3^2 - u \rightarrow (6)$$

$$(3) \Rightarrow V = 2xy$$

$$2C_3 y = V$$

$$y = \frac{V}{2C_3} \Rightarrow y^2 = \frac{V^2}{4C_3^2} \rightarrow (7)$$

From eqⁿ (6) and (7)

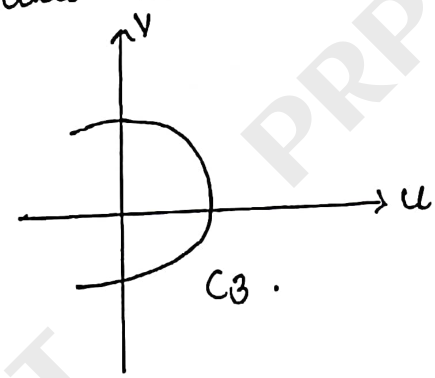
$$\frac{V^2}{4C_3^2} = C_3^2 - u$$

$$V^2 = 4C_3^2 (C_3^2 - u)$$

$$V^2 = -4C_3^2 (u - C_3^2)$$

$$(V-0)^2 = -4C_3^2 (u - C_3^2) \longrightarrow (8)$$

$\therefore$ eqⁿ (8) represents a parabola, symmetric about 'u' axis in the -ve direction having the vertex at $(C_3^2, 0)$.



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Case (4) :- Let $y = C_4$

$\therefore$ (2) $\Rightarrow u = x^2 - C_4^2$

$x^2 = u + C_4^2 \rightarrow$ (9)

$\therefore$ (3) $\Rightarrow V = 2xy$

$V = 2xC_4$

$x = \frac{V}{2C_4} \Rightarrow x^2 = \frac{V^2}{4C_4^2} \rightarrow$ (10)

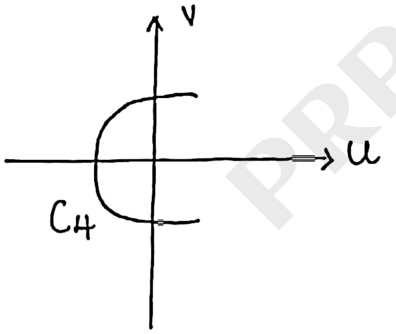
$\therefore$ From eqⁿ (9) and (10)

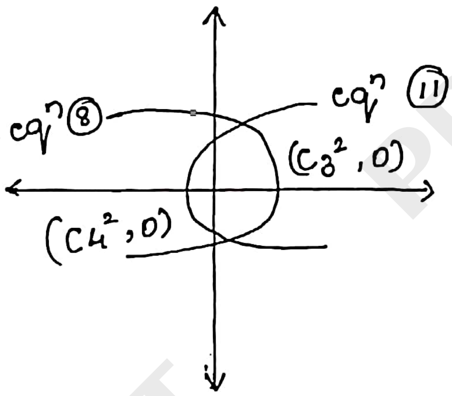
$$\frac{V^2}{4C_4^2} = u + C_4^2$$

$$V^2 = 4C_4^2 (u + C_4^2)$$

$$(V-0)^2 = 4C_4^2 (u - (-C_4^2)) \rightarrow (11)$$

$\therefore$ eqⁿ (11) represents a parabola symmetric about 'u' axis in the +ve direction having the vertex at $(-C_4^2, 0)$





3) Discussion of the transformation $w = z + \frac{1}{z}$, $z \neq 0$.

Given, $w = f(z) = z + \frac{1}{z}$ is analytic

$\therefore f(z)$ is differentiable

$$\Rightarrow f'(z) = z + \frac{1}{z}, \; z \neq 0$$

$$= 1 - \frac{1}{z^2}, \; z \neq 1.$$

Let $z = r \cdot e^{i\theta}$

$\therefore$ WKT $\quad w = f(z) = u + iV = z + \frac{1}{z}$

$$\Rightarrow u + iV = r \cdot e^{i\theta} + \frac{1}{re^{i\theta}}$$

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$$\Rightarrow u+iv = re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$\Rightarrow u+iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= (r\cos\theta + ir\sin\theta) + \left(\frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta\right)$$

$$u+iv = \left(r+\frac{1}{r}\right)\cos\theta + i\left(r-\frac{1}{r}\right)\sin\theta$$

$$\therefore\ u = \left(r+\frac{1}{r}\right)\cos\theta \rightarrow (1) \qquad v = \left(r-\frac{1}{r}\right)\sin\theta \rightarrow (2)$$

$$(1) \Rightarrow \frac{u}{\left(r+\frac{1}{r}\right)} = \cos\theta \longrightarrow (3) \qquad (2) \Rightarrow \frac{v}{\left(r-\frac{1}{r}\right)} = \sin\theta \rightarrow (4)$$

$$(3)^2 + (4)^2 = \frac{u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\left(r-\frac{1}{r}\right)^2} = \cos^2\theta + \sin^2\theta$$

$$\frac{u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\left(r-\frac{1}{r}\right)^2} = 1 \longrightarrow (5)$$

From eqn (1) $\Rightarrow \dfrac{u}{\cos\theta} = r+\dfrac{1}{r} \rightarrow (6) \qquad \dfrac{v}{\sin\theta} = r-\dfrac{1}{r} \rightarrow (7)$

$$\therefore\ (6)^2 - (7)^2 \quad \frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r+\frac{1}{r}\right)^2 - \left(r-\frac{1}{r}\right)^2$$

$$\Rightarrow \frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4r\left(\frac{1}{r}\right)$$

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4$$

$$\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1$$

$$\frac{u^2}{(2\cos\theta)^2} - \frac{v^2}{(2\sin\theta)^2} = 1 \longrightarrow (8)$$

Case (1) : when $r = C_1$, we can have eqn (5) $\Rightarrow \dfrac{u^2}{A^2} + \dfrac{v^2}{B^2} = 1$
represents ellipse at centre origin and vertex at $(\pm 2, 0)$

Case (2) : If $\theta = C_2$ is a constant, we have

$$\text{eqn } (8) \Rightarrow \frac{u^2}{A^2} - \frac{v^2}{B^2} = 1$$

represents a hyperbola at the points $(\pm 2, 0)$

Also the ellipse in the 'w' plane, we have $|z| = r$

$$\sqrt{x^2+y^2} = r$$

$$\Rightarrow x^2+y^2 = r^2$$

represents a circle in the 'z' plane and for the hyperbola in the 'w' plane, we have θ = amplitude of z.

$\tan\theta = m = \text{slope}$

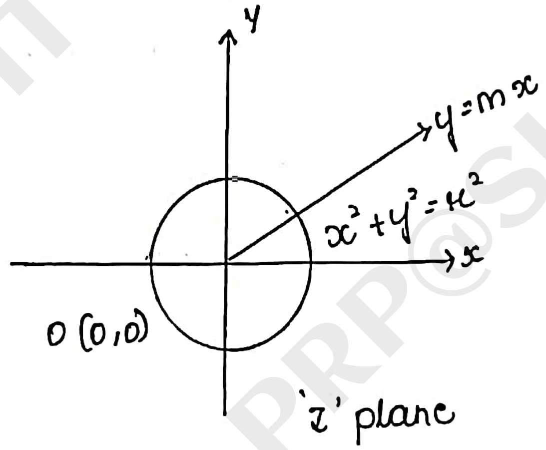
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

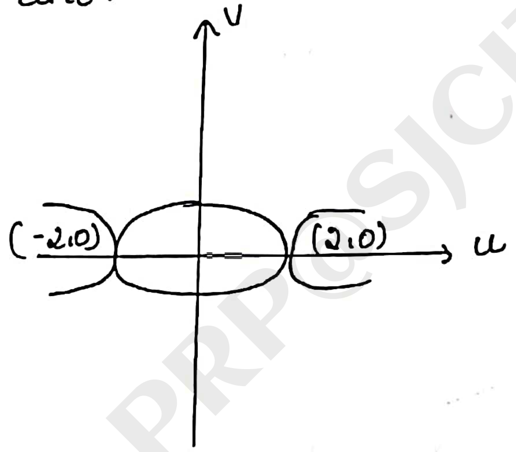
$$y/x = \tan\theta$$

$$y = x\tan\theta$$

$$\boxed{y = mx}$$

represents a straight line.





Bilinear transformation

A transformation of the form $w = \frac{az+b}{cz+d}$, where a, b, c, d are real or complex constants, such that $ad-bc \neq 0$ is called a bilinear transformation (or) linear fractional transformation (or) mobius transformation

The ratio $w = \frac{az+b}{cz+d}$ can also be expressed as $cwz + dw - az - b = 0$ is linear both in 'z' and 'w', hence it is called a bilinear transformation and if $\frac{dw}{dz} \neq 0$, hence it is the conformal.

Properties of bilinear transformation:

1) The transformation $w = \frac{az+b}{cz+d}$ sets of a one-two-one corresponds between the points of closed 'z' plane and the closed 'w' plane.

2) If $ad-bc=0$, then 'w' is either a Constant (or) meaningless.

3) Invariant points (or) fixed points:

If a point 'z' maps into itself i.e; $w=z$ under the bilinear transformation then the point is called invarient point (or) fixed points of bilinear transformation.

Ex: Suppose invarient of $w=z^2$ are the solutions of equations

$$\Rightarrow z^2-z=0$$

$$z(z-1)=0$$

$$\therefore z=0, z=1$$

4) Cross ratio: If z_1, z_2, z_3, z_4 are four distinct points then the ratio is called the Cross ratio of these points and its denoted by $(z_1, z_2, z_3, z_4) = \dfrac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$

5) A bilinear transformation preserves the Cross ratio of the 4 points or let w_1, w_2, w_3, w_4 be the images of 4 distinct points of z_1, z_2, z_3, z_4 in the z-plane under a bilinear transformation, then

$$\Rightarrow (w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$$

$$\Rightarrow \frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

Problems:

1) Find the bilinear transformation which map the points $z=1, i, -1$ into $w=2, i, -2$. Also find the invariant points of transformation.

WKT, Bilinear transformation $w=\dfrac{az+b}{cz+d} \longrightarrow (1)$

$\therefore$ when $z=1, w=2$

$$(1) \Rightarrow 2=\frac{a+b}{c+d}$$

$$a+b=2c+2d$$

$$a+b-2c-2d=0 \longrightarrow (2)$$

when $z = i,\ \omega = i$

$$i = \frac{ai+b}{ci+d}$$

$$ai+b = ci^2 + di$$

$$ai + b + c - di = 0 \rightarrow (3)$$

when $z = -1,\ \omega = -2$

$$(1) \Rightarrow -2 = \frac{a(-1)+b}{c(-1)+d}$$

$$-2 = \frac{-a+b}{-c+d}$$

$$-a+b-2c+2d=0 \longrightarrow (4)$$

Now eqn (2) - (3)

$$(a+b-2c-2a) - (ai+b+c-id) = 0$$

$$\Rightarrow a(1-i) - 3c + (i-2)d = 0$$

$$(3) - (4) \Rightarrow (ai+b+c-id) - (-a+b-2c+2d) = 0$$

$$a(1+i) + 3c - d(i+2) = 0 \longrightarrow (5)$$

$\frac{a}{}$	$\frac{c}{}$	$\frac{d}{}$	$\frac{a}{}$
$1-i$	-3	$i-2$	$1-i$
$1+i$	$+3$	$-(i+2)$	$1+i$

$$\frac{a}{3(i+2)-3(i-2)} = \frac{c}{(i-2)(1+i)+(1-i)(i+2)} = \frac{d}{3(1-i)+3(1+i)}$$

$$\frac{a}{3(i+2-i+2)} = \frac{c}{i-2-1+2i+i+2+1-2i} = \frac{d}{3(1-i+1+i)}$$

$$= \frac{a}{12} = \frac{c}{-2i} = \frac{d}{6} = k$$

$a = 12k$, $c = -2ik$, $d = 6k$

eqn (5) $\Rightarrow 12k + b + 4ik - 12k = 0$

$$b = 4ik$$

$$(1) \Rightarrow \omega = \frac{12kz - 4ik}{-2ikz + 6k}$$

$$\omega = \frac{12z - 4i}{-2iz + 6}$$

$$\omega = \frac{6z - 2i}{-iz + 3} \rightarrow (6)$$

when $\omega = z$

$$(6) \Rightarrow z = \frac{6z - 2i}{-iz+3}$$

$$z(-iz + 3) = 6z - 2i$$

$$-iz^2 + 3z - 6z + 2i = 0 \quad \times^{ly} \text{ by } i$$

$$-i^2 z^2 + 3iz - 6iz + 2i^2 = 0$$

$$z^2 - 3zi - 2 = 0$$

$$z = \frac{3i \pm \sqrt{9i^2 - 4(1)(2)}}{2(1)}$$

$$z = \frac{3i \pm \sqrt{-9+8}}{2} = \frac{3i \pm \sqrt{-1}}{2} = \frac{3i \pm \sqrt{i^2}}{2} = \frac{3i \pm i}{2}$$

Case (1) :- $\frac{3i+i}{2} = 2i$

Case (2) :- $\frac{3i-i}{2} = i$

$$\boxed{z = 2i, i}$$

2) Find the bilinear transformation which maps the points from $z = 1, i, -1$ to $\omega = i, 0, -i$

W.K.T
$$\omega = \frac{az+b}{cz+d} \rightarrow (1)$$

when $z = 1$, $\omega = i$

$$i = \frac{a+b}{c+d}$$

$$a + b - ci - di = 0 \rightarrow (2)$$

when $z = i$, $\omega = 0$

$$0 = \frac{ai+b}{ci+d}$$

$$ai + b = 0 \rightarrow (3)$$

when $z=-1,\ w=-i$

$$-i=\frac{-a+b}{-c+d}$$

$$-a+b-ci+di=0 \rightarrow (4)$$

$$(2)-(3) \Rightarrow (1-i)a-ci-di=0 \rightarrow (5)$$

$$(3)-(4) \Rightarrow (1+i)a+ci-di=0 \rightarrow (6)$$

a	c	d	a
$1-i$	$-i$	$-i$	$1-i$
$1+i$	$+i$	$-i$	$1+i$

$$\frac{a}{i^2+i^2}=\frac{c}{-i(1+i)+i(1-i)}=\frac{d}{i(1-i)+i(1+i)}$$

$$\frac{a}{-2}=\frac{c}{i(-1-i+1-i)}=\frac{d}{i(1-i+1+i)}$$

$$\frac{a}{-2}=\frac{c}{-2i^2}=\frac{d}{2i}$$

$$\frac{a}{-2}=\frac{c}{2}=\frac{d}{2i}=k$$

$$a=-2k,\quad c=2k,\quad d=2ik$$

$$(2) \Rightarrow a+b-ci-di=0$$

$$-2k+b-2ki-2i^2k=0$$

$$-2k+b-2ki+2k=0$$

$$b-2ki=0$$

$$b=2ki$$

$$(1) \Rightarrow w=\frac{-2k^2+2ik}{2k^2+2ik}$$

$$w=\frac{-z+i}{z+i}$$

$$w=\frac{-z+i}{z+i}\times\frac{i}{i}=\frac{-iz+i^2}{iz+i^2}$$

$$w=-\frac{iz-1}{iz-1}.$$

3) Find the bilinear transformation which map $z = \infty, i, 0$ into $w = -1, i, 1$. Also find the fixed points of the transformation

W.K.T the bilinear transformation $\frac{az+b}{cz+d} \rightarrow (1)$

(1) $w = \frac{a + b/z}{c + d/z} \rightarrow (2)$

when $z = \infty$, $w = -1$

(2) $\Rightarrow -1 = \frac{a + b(0)}{c + d(0)}$

$-1 = a/c$

$a + c = 0 \rightarrow (3)$

when $z = i$, $w = i$

(1) $\Rightarrow i = \frac{ai+b}{ci+d}$

$ai + b = ci^2 + di$

$ai + b + c - di = 0 \rightarrow (4)$

when $z = 0$, $w = 1$

(1) $\Rightarrow 1 = \frac{b}{d}$

$b - d = 0 \rightarrow (5)$

$a + c = 0 \rightarrow (3)$

$ai + b + c - di = 0 \rightarrow (4)$

$b - d = 0 \rightarrow (5)$

(3) - (4) $\Rightarrow (1-i)a - b + di = 0 \rightarrow (6)$

(5) $\Rightarrow 0a + b - d = 0 \rightarrow (7)$

$$\begin{array}{cccc} a & b & d & a \\ 1-i & -1 & i & 1-i \\ 0 & 1 & -1 & 0 \end{array}$$

$\Rightarrow \frac{a}{1-i} = \frac{b}{0+1-i} = \frac{d}{(1-i)-0}$

$$\Rightarrow \frac{a}{1-i} = \frac{b}{1-i} = \frac{d}{1-i}$$

$$\frac{a}{1} = \frac{b}{1} = \frac{d}{1} = K$$

$a = K$, $b = K$, $d = K$

$$③ \Rightarrow K + C = 0$$

$$C = -K$$

$$\therefore \omega = \frac{Kz+K}{-Kz+K} = \frac{z+1}{-z+1} \rightarrow ⑧$$

Let $\omega = z$ $\quad ⑧ \Rightarrow z = \frac{1+z}{1-z}$

$$z(1-z) = 1+z$$

$$z - z^2 = 1+z$$

$$-z^2 = 1$$

$$z^2 = -1$$

$$z = \sqrt{-1}$$

$$z = \sqrt{i^2}$$

$$\boxed{z = \pm i}$$

4) Find the bilinear transformation, which map $z = \infty, i, 0$ into $\omega = -1, -i, 1$. Also find the fixed points of the transformation.

Let the bilinear transformation

$$\omega = \frac{az+b}{cz+d} \rightarrow ①$$

$$\omega = \frac{a + b/z}{c + d/z} \rightarrow ②$$

when $z = \infty$, $\omega = -1$

$$① \Rightarrow -1 = \frac{a}{c}$$

$$a + c = 0 \rightarrow ③$$

when $z = i$, $\omega = -i$

$$-i = \frac{ai+b}{ci+d}$$

$$ai+b-c+di=0 \rightarrow (4)$$

when $z=0,\ \omega=1$

$$(1) \Rightarrow 1=\frac{b}{d}$$

$$b-d=0 \rightarrow (5)$$

$(3)+(4)$

$$a(1+i)+b+di=0 \rightarrow (6)$$

$$(5)\quad 0\cdot a+b-d=0 \rightarrow (7)$$

$$\begin{array}{cccc} \frac{a}{1+i} & \frac{b}{1} & \frac{c}{i} & \frac{a}{1+i} \\ 0 & 1 & -1 & 0 \end{array}$$

$$\frac{a}{(-1-i)}=\frac{b}{1+i}=\frac{d}{1+i}$$

$$\frac{a}{-(1+i)}=\frac{b}{(1+i)}=\frac{d}{(1+i)}=k$$

$$\frac{a}{-1}=\frac{b}{1}=\frac{d}{1}=k$$

$a=-k,\ b=k,\ d=k$

$$eq^n\ (3) \Rightarrow a+c=0$$

$$c=-a=-(-k)=k$$

WKT $\quad \omega=\dfrac{az+b}{cz+d}=\dfrac{-kz+k}{kz+k}$

$$\omega=\frac{1-z}{1+z} \rightarrow (8)$$

To find fixed points Let $\omega=z$

$$(8) \Rightarrow z=\frac{1-z}{1+z}$$

$$z(z+1)=1-z$$

$$z^2+z=1-z$$

$$z^2+2z-1=0$$

$$z=\frac{-2\pm\sqrt{4-4(1)(-1)}}{2(1)}$$

$$z = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm \sqrt{8}}{2}$$

$$z = \frac{-2 \pm 2\sqrt{2}}{2} = \frac{-2(-1 \pm \sqrt{2})}{2}$$

$$= -1 \pm \sqrt{2}$$

$$z = -1 + \sqrt{2}, \quad z = -1 - \sqrt{2}$$

5) Find the bilinear transformation which maps $z = 0, 1, \infty$ into $\omega = -5, -1, 3$. Also find the invarient points of transformation.

Let the bilinear equation $\omega = \dfrac{az+b}{cz+d} \rightarrow (1)$

$$\omega = \frac{a + b/z}{c + d/z} \longrightarrow (2)$$

when $z = 0, \omega = -5$

$$(1) \Rightarrow -5 = \frac{b}{d}$$

$$b + 5d = 0 \rightarrow (3)$$

when $z = 1, \omega = -1$

$$(1) \Rightarrow -1 = \frac{a+b}{c+d}$$

$$a + b + c + d = 0 \rightarrow (4)$$

when $z = \infty, \omega = 3$

$$(2) \Rightarrow 3 = \frac{a}{c}$$

$$a - 3c = 0 \longrightarrow (5)$$

$$(4) - (3) \Rightarrow a + c - 4d = 0 \rightarrow (6)$$

$$(5) \Rightarrow a - 3c + 0.d = 0 \longrightarrow (7)$$

a	c	d	a
1	1	-4	1
1	-3	0	1

$$\Rightarrow \frac{a}{0-12} = \frac{c}{-4-0} = \frac{d}{-3-1}$$

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$\frac{a}{-12} = \frac{c}{-4} = \frac{d}{-4}$ $\quad \times^{ly}$ by -3

$\frac{a}{3} = \frac{c}{1} = \frac{d}{1} = k$

$a = 3k$, $c = k$, $d = k$

(3) $\Rightarrow$ $b = -5k$

$$\omega = \frac{3kz - 5k}{kz + k}$$

$$\omega = \frac{3z-5}{z+1} \rightarrow (8)$$

Let $\omega = z$

$$z = \frac{3z-5}{z+1}$$

$z(z+1) = 3z-5$

$z^2 + z = 3z - 5$

$z^2 - 2z + 5 = 0$

$$z = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2(1)}$$

$z = \frac{2 \pm \sqrt{-16}}{2} = z = \frac{2 \pm 4i}{2}$ $\quad z = 1 \pm 2i$

$z = 1 + 2i$, $z = 1 - 2i$.

6) Find the bilinear transformation which map the points $z = 1, i, -1$ into $\omega = 0, 1, \infty$.

Let the bilinear equation $\omega = \frac{az+b}{cz+d} \rightarrow (1)$

$(1) \Rightarrow \frac{1}{\omega} = \frac{cz+d}{az+b} \rightarrow (2)$

when $z = 1$, $\omega = 0$

$(1) \Rightarrow 0 = \frac{a+b}{c+d}$

$a + b = 0 \rightarrow (3)$

when $z = i,\ w = 1$

$$① \Rightarrow 1 = \frac{ai+b}{ci+d}$$

$$ai+b-ci-d=0 \rightarrow ④$$

when $z=-1,\ w=\infty$

$$② \Rightarrow \frac{1}{\infty} = \frac{c(-1)+d}{a(-1)+b} = \frac{-c+d}{-a+b}$$

$$0 = -c+d \rightarrow ⑤$$

$$④ - ③ \Rightarrow (i-1)a - ci - d = 0 \rightarrow ⑥$$

$$⑤ \Rightarrow 0 \cdot a - c + d = 0 \rightarrow ⑦$$

a	c	d	a
$i-1$	$-i$	-1	$i-1$
0	-1	1	0

$$\frac{a}{-i-1} = \frac{c}{-(i-1)} = \frac{d}{-(i-1)} = k$$

$a = -k(1+i)$ $\quad$ $c = -k(i-1)$, $c = k(1-i)$ $\quad$ $d = -k(i-1) = k(1-i)$

eqn ③ $\Rightarrow$ $b = -a$

$$b = -(-k(1+i))$$

$$b = k(1+i)$$

$$w = \frac{-k(1+i)z + k(1+i)}{k(1-i)z + k(1-i)}$$

$$w = \frac{-(1+i)z + (1+i)}{(1-i)z + (1-i)}$$

Complex Integration :

Suppose $w = f(z) = u+iv$ is a complex valued function for the complex variable $z = x+iy$ over the region R, such that the integral of the complex valued function about the curve 'c' is defined as $I = \int_c f(z)\,dz$.

where C is the either opened or closed curve

$$I = \int_C f(z)\,dz$$

$$I = \int_C (u+iv)(dx+idy)$$

$$I = \int_C udx + iudy + ivdx + i^2vdy$$

$$I = \int_C udx + iudy + ivdx - vdy$$

$$I = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

where $u = u(x,y)$, $v = V(x,y)$

Complex line integral:

1) Evaluate $\int_C |z|^2 dz$ where 'C' is a square with the vertices $(0,0)$ $(1,0)$ $(1,1)$ $(0,1)$

Given, $\int_C |z|^2 dz \rightarrow ①$

where $z = x+iy$

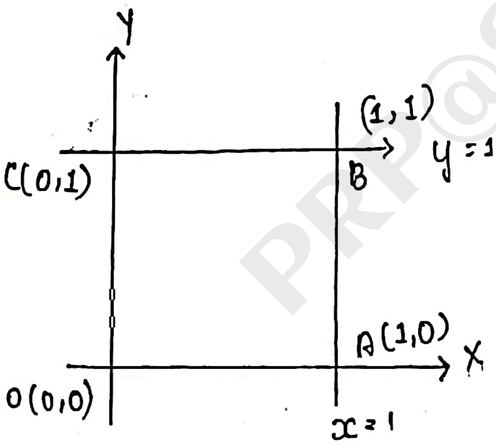
$dz = dx+idy$

$|z| = \sqrt{x^2+y^2}$

$|z|^2 = x^2+y^2$

$① \Rightarrow I = \int_C (x^2+y^2)(dx+idy) \rightarrow ②$

where 'C' is a square with vertices $(0,0)(1,0)(1,1)$ $(0,1)$



$② \Rightarrow I = \int_{OA} (x^2+y^2)(dx+idy) + \int_{AB} (x^2+y^2)(dx+idy) + \int_{BC} (x^2+y^2)(dx+idy) + \int_{CO} (x^2+y^2)(dx+idy)$

Along OA $\Rightarrow y=0 \Rightarrow dy=0$

$$\int_{OA} (x^2+y^2)(dx+idy) = \int_0^1 x^2 dx = \left|\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

About AB $\Rightarrow x = 1 \Rightarrow dx = 0$

$$\int_{AB} (x^2+y^2)(dx+idy) = \int_0^1 (1+y^2)(idy)$$
$$= i\int_{y=0}^{1} 1+y^2 dy$$
$$= i\left[1+\frac{y^3}{3}\right]_0^1$$
$$= \frac{4i}{3}$$

About BC $\Rightarrow y = 1 \Rightarrow dy = 0$

$$\int_{BC} (x^2+y^2)(dx+idy) = \int_1^0 (x^2+1)dx$$
$$= \left[\frac{x^3}{3}+x\right]_1^0$$
$$= 0-\left(\frac{1}{3}+1\right) = -\frac{4}{3}$$

About CO $\Rightarrow x = 0 \Rightarrow dx = 0$

$$\int_{CO} (x^2+y^2)(dx+idy) = \int_{y=1}^{0} y^2 \cdot idy = i\left[\frac{y^3}{3}\right]_1^0$$
$$= 0-\frac{i}{3} = -\frac{i}{3}$$

$$I = \frac{1}{3}+\frac{4i}{3}-\frac{4}{3}-\frac{i}{3}$$
$$= -1+i$$

3) Evaluate $\int_0^{2+i} |\bar{z}|^2 dz$ along the line $x = 2y$.

$$I = \int_0^{2+i} |\bar{z}|^2 \rightarrow (1)$$

where $z = x+iy \Rightarrow dz = dx+idy$

$$\bar{z} = x-iy$$
$$|\bar{z}| = \sqrt{x^2-y^2}$$
$$|\bar{z}|^2 = x^2-y^2$$

$$\therefore I = \int_{0+i(0)}^{2+i(1)} (x^2-y^2)(dx+idy) \longrightarrow (2)$$

But $x = 2y$

$dx = 2dy$

$$I = \int_{y=0}^{1} (2y)^2 - y^2 \, (2dy + i dy)$$

$$I = \int_{y=0}^{1} 3y^2 (2+i) dy$$

$$I = 3(2+i) \int_0^1 y^2 \, dy$$

$$I = 3(2+i) \left[\frac{y^3}{3} \right]_0^1$$

$$I = 2+i.$$

3) Evaluate $\int_{(0,3)}^{(2,4)} (2y+x^2) dx + (3y-x) dy$ along the parabola $x = 2t$, $y = t^2 + 3$.

Let $I = \int_{(0,3)}^{(2,4)} (2y+x^2) dx + (3y-x) dy \rightarrow (1)$

where $x = 2t$, $y = t^2 + 3$

$dx = 2.dt \quad dy = 2t \, dt$

$$(1) \Rightarrow I = \int_0^1 (2(t^2+3) + 4t^2) 2dt + (3(t^2+3) - 2t) \, 2t dt.$$

$$I = \int_{t=0}^{1} (2t^2 + 6 + 4t^2) 2dt + (3t^2 + 9 - 2t) 2t \, dt$$

$$I = \int_0^1 (12t^2 + 12 + 6t^3 + 18t - 4t^2) dt$$

$$I = \int_0^1 (6t^3 + 8t^2 + 18t + 12) \, dt$$

$$I = \left[6\frac{t^4}{4} + 8\frac{t^3}{3} + 18\frac{t^2}{2} + 12t \right]_0^1$$

$$I = \frac{6}{4} + \frac{8}{3} + 9 + 12$$

$$I = \frac{18 + 32 + 252}{12}$$

$$I = \frac{151}{6}$$

4) Evaluate $\int_{(0,3)}^{(2,4)} (2y+x^2)dx + (3x-y)dy$ along the parabola $x=2t$, $y=t^2+3$.

Let $I = \int_{(0,3)}^{(2,4)} (2y+x^2)dx + (3x-y)dy \rightarrow$ ①

Let $x=2t$, $y=t^2+3$

$dx = 2dt$ $dy = 2tdt$

$I = \int_0^1 (2(t^2+3)+4t^2)2dt + (3(2t)-(t^2+3))2t\,dt$

$I = \int_0^1 (2t^2+6+4t^2)2dt + (6t-t^2-3)2t\,dt$

$I = \int_0^1 12t^2+12+12t^2-2t^3-6t\ dt$

$I = \int_0^1 -2t^3+24t^2-6t+12\ dt$

$I = \left[-\frac{2t^4}{4}+24\frac{t^3}{3}-6\frac{t^2}{2}+12t\right]_0^1$

$I = -\frac{1}{2}+8-3+12$

$I = \frac{-1+34}{2} = \frac{33}{2}.$

5) Evaluate $\int_0^{1+i} (x^2-iy)dz$ along the line i) $y=x$ ii) $y=x^2$

Let $\int_0^{1+i} (x^2-iy)dz \rightarrow$ ①

where $z = x+iy$

$dz = dx+idy$

① $\Rightarrow I = \int_{0+i(0)}^{1+i(1)} (x^2-iy)(dx+idy) \longrightarrow$ ②

Case ① :- About the line $y=x \Rightarrow dy = dx$

$I = \int_0^1 (x^2-ix)(dx+idx)$

$I = \int_0^1 (1+i)(x^2-ix)dx$

$= (1+i)\left[\frac{x^3}{3}-i\frac{x^2}{2}\right]_0^1$

$= (1+i)\left[\frac{1}{3}-\frac{i}{2}\right]$

$= 1+i\left(\frac{2-3i}{6}\right)$

$= \frac{1}{6}(2-3i+2i+3)$

$= \frac{1}{6}(5-i)$

Case ②: about the line $y = x^2 \Rightarrow dy = 2x\,dx$

$$I = \int_0^1 (x^2 - ix^2)(dx + i2x\,dx)$$

$$I = \int_{x=0}^1 (1-i)x^2(1+2ix)\,dx$$

$$I = (1-i)\int_0^1 (x^2 + 2ix^3)\,dx$$

$$I = (1-i)\left[\frac{x^3}{3} + 2i\frac{x^4}{4}\right]_0^1$$

$$I = (1-i)\left[\frac{1}{3} + \frac{2i}{4}\right]$$

$$I = (1-i)\left(\frac{1}{3} + \frac{i}{2}\right)$$

$$I = (1-i)\left(\frac{2+3i}{6}\right)$$

$$I = \frac{1}{6}(2+3i-2i-3i^2)$$

$$I = \frac{1}{6}(5+i)$$

Cauchy's Integral theorem (or) fundamental theorem:

Statement: If a function $f(z)$ is analytic at all the points within and on a closed curve, then $\int_C f(z)\,dz = 0$

Proof: Given $w = f(z) = u+iv$ is an analytic function at all the points of z within and on a closed curve 'C'.

$\Rightarrow f(z)$ is differentiable

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$\int_C f(z)\,dz = \int_C u\,dx + ui\,dy + iv\,dx - v\,dy$$

$$\int_C f(z)\,dz = \int_C (u\,dx - V\,dy) + i\,(u\,dy + v\,dx)$$

$$\int_C f(z)\,dz = I_1 + I_2$$

W.K.T from green's theorem

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$I_1 = \iint_R \left(-\frac{\partial V}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$$I_2 = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial V}{\partial y}\right) dx\,dy$$

$$\int_C f(z)\,dz = \iint_R \left(-\frac{\partial V}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial V}{\partial y}\right) dx\,dy$$

But $f(z)$ is analytic

$$\frac{\partial V}{\partial x} = -\frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial x} = \frac{\partial V}{\partial y}$$

$$① \rightarrow \int_C f(z)\,dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial V}{\partial y}\right) dx\,dy$$

$$\int_C f(z)\,dz = 0.$$

Problem:

1) Verifying the Cauchy's integral theorem of $f(z) = z^2$ about the closed curve 'c' is a square with the vertices $(0,0)$ $(1,0)$ $(1,1)$ $(0,1)$.

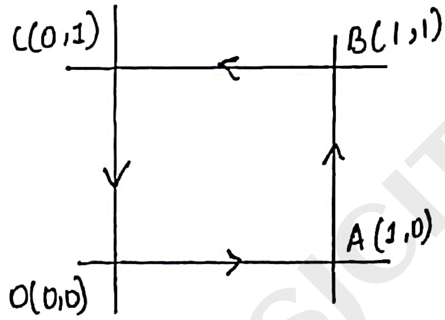
Given $f(z) = z^2$

where $z = x + iy$

$dz = dx + i\,dy$

$$\int_C f(z)\,dz = \int_C z^2\,dz = \int_C (x+iy)^2 (dx + i\,dy) \rightarrow ①$$

where 'c' is a square with vertices $(0,0)$ $(1,0)$ $(1,1)$ $(0,1)$



$$\int_C f(z)\,dz = \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CO} f(z)\,dz$$

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About $OA \Rightarrow y=0 \Rightarrow dy=0$

$$\int_{OA} f(z)dz = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

About $AB \Rightarrow x=1 \Rightarrow dx=0$

$$\int_{AB} f(z)dz = \int (1+iy)^2 \ (0+idy)$$

$$= i\int_0^1 (1+iy)^2 dy$$

$$= i\int_0^1 (1-y^2+2iy)\,dy$$

$$= i\left[y-\frac{y^3}{3}+iy^2\right]_0^1$$

$$= i\left[1-\frac{1}{3}+i\right]$$

$$= i\left[\frac{2}{3}+i\right]$$

$$\int_{AB} f(z)dz = \frac{2}{3}i-1.$$

About BC $y=1 \Rightarrow dy=0$

$$\int_{BC} f(z)dz = \int (x+i)^2 dx$$

$$= \int_1^0 (x^2-1+2ix)dx$$

$$= \left[\frac{x^3}{3}-x+ix^2\right]_1^0$$

$$= 0-\left[\frac{1}{3}-1+i\right]$$

$$= -\left[-\frac{2}{3}+i\right]$$

$$\int_{BC} f(z)dz = \frac{2}{3}-i$$

About CO, $x=0 \Rightarrow dx=0$

$$\int_{CO} f(z)dz = \int_{y=1}^{0} (iy)^2 (idy)$$

$$= \int_1^0 iy^2 \cdot idy$$

$$= -i\left[\frac{y^3}{3}\right]_1^0$$

$$= -i\left[0-\frac{1}{3}\right]$$

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$$\int_{C_0} f(z)\,dz = \frac{i}{3}$$

$$① \rightarrow \int_C f(z)\,dz = \frac{1}{3} + \frac{2}{3}i - 1 + \frac{2}{3} - i + \frac{i}{3}$$

$$\int_C f(z)\,dz = 0$$

hence the Cauchy's integral theorem is verified

<u>Theorem</u>: <u>Cauchy's Integral formula</u>

<u>Statement</u>: If $f(z)$ is analytic inside and on a simple closed curve 'c' and if 'a' is anypoint within 'c' then

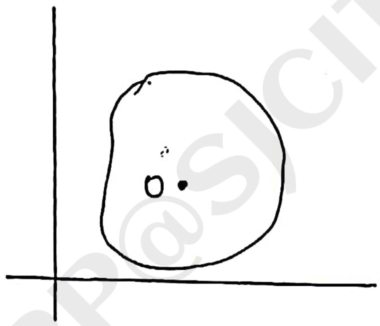
$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}\,dz.$$

<u>Proof</u>:- Given $f(z)$ is an analytic function inside and on a simple closed curve 'c'.

Since 'a' is a point within 'c' we shall enclose it by a circle 'C_1' and with $z=a$ as a center and 'r' as a radius, such that C_1 lies entirely within 'c'.

Therefore the function $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the region between 'C' and 'C_1'.

we have $\int_C \frac{f(z)}{(z-a)}\,dz = \int_{C_1} \frac{f(z)}{(z-a)}\,dz \rightarrow ①$



WKT,

$$|z-a| = r$$

$$z-a = r\cdot e^{i\theta}.$$

$$z = a + re^{i\theta}$$

$$\frac{dz}{d\theta} (0 + rie^{i\theta})$$

$$\Rightarrow dz = r.i.e^{i\theta}\,d\theta$$

$$① \rightarrow \int_C \frac{f(z)}{z-a}\,dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{r.e^{i\theta}}\, rie^{i\theta}\,d\theta$$

$$\Rightarrow \int_C \frac{f(z)}{z-a}.dz = i\int_0^{2\pi} f(a+re^{i\theta})d\theta \rightarrow ②$$

when $r=0$

$$② \Rightarrow \int_C \frac{f(z)}{z-a}dz = i\int_{\theta=0}^{2\pi} f(a).d\theta$$

$$\int_C \frac{f(z)}{z-a}dz = i.f(a)\int_0^{2\pi} 1.d\theta$$

$$= i\,f(a).|\theta|_0^{2\pi}$$

$$= i\,f(a)\,(2\pi)$$

$$\int_C \frac{f(z)}{(z-a)}dz = i.f(a).2\pi$$

$$\therefore\ f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz.$$

Generalized Cauchy's integral formula:

If $f(z)$ is analytic inside and on a simple closed curve 'c' and if 'a' is any point within 'c', then

$$f^{(n)}a = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}.dz$$

also $$f^{n}(a).\frac{2\pi i}{n!} = \int_C \frac{f(z)}{(z-a)^{n+1}}.dz$$

Problems:

1) Evaluate $\int_C \frac{e^{2z}}{(z+1)(z+2)}.dz$ where 'C' is a circle $|z|=3$.

Let $$I = \int_C \frac{e^{2z}}{(z+1)(z+2)}dz \rightarrow ①$$

Since $C: |z|=3$ is a circle with the center 'o' (zero) and radius '3'.

$z=-1$ and $z=-2$ are inside of the circle

$\therefore f(z) = e^{2z}$ is analytic at all the points except $z=-1$ & $z=-2$

$$\therefore \frac{1}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$1 = A(z+2) + B(z+1) \longrightarrow (2)$$

when $z = -2$, when $z = -1$

$(2) \Rightarrow 1 = -B$ $\qquad$ $(2) \Rightarrow 1 = A$

$\boxed{B = -1}$ $\qquad$ $\boxed{A = 1}$

$$\therefore \frac{1}{(z+1)\ (z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$$

$$\frac{e^{2z}}{(z+1)\ (z+2)} = \frac{e^{2z}}{z+1} - \frac{e^{2z}}{z+2}$$

$$\therefore \int_c \frac{e^{2z}}{(z+1)\ (z+2)} dz = \int_c \frac{e^{2z}}{z-(-1)} dz - \int_c \frac{e^{2z}}{z-(-2)} dz$$

$$= 2\pi i\ e^{2(-1)} - 2\pi i\ e^{2(-2)}$$

$$= 2\pi i e^{-2} - 2\pi i e^{-4}$$

$$= 2\pi i\ (e^{-2} - e^{-4})$$

$$= 2\pi i\ \left(\frac{1}{e^2} - \frac{1}{e^4}\right) = 2\pi i \left(\frac{e^4 - e^2}{e^4}\right)$$

$$= 2\pi i\ \left(\frac{e^2 - 1}{e^4}\right)$$

2) Evaluate $\int_c \frac{e^{2z}}{(z+1)\ (z-2)} dz$, where 'c' is a circle $|z| = 3$.

Let $I = \int_c \frac{e^{2z}}{(z+1)\ (z-2)} dz \rightarrow (1)$

Since $c : |z| = 3$ is a circle with the center 'o' (zero) & radius 3.

$z = -1$ and $z = 2$ are inside of the circle

$\therefore f(z) = e^{2z}$ is analytic at all points except $z = -1$ and $z = 2$

$$\therefore \frac{1}{(z+1)\ (z-2)} = \frac{A}{z+1} + \frac{B}{z-2} \rightarrow (2)$$

$$1 = A(z-2) + B(z+1)$$

when $z=-1$ when $z=2$

$1 = A(-3)$ $1 = B(3)$

$A = -1/3$ $B = \frac{1}{3}$

$$\frac{1}{(z+1)(z-2)} = \frac{1}{3(z-2)} - \frac{1}{3(z+1)}$$

$$\frac{e^{2z}}{(z+1)(z-2)} = \frac{e^{2z}}{3(z-2)} - \frac{e^{2z}}{3(z+1)}$$

$$\int_C \frac{e^{2z}}{(z+1)(z-2)} dz = \int_C \frac{e^{2z}}{3(z-2)} dz - \int_C \frac{e^{2z}}{3(z+1)} dz$$

$$= \int_C \frac{e^{2z}}{3(z-(2))} dz - \int_C \frac{e^{2z}}{3(z-(-1))} dz$$

$$= \frac{2\pi i}{3} e^{2(2)} - \frac{2\pi i}{3} e^{2(-1)}$$

$$= \frac{2\pi i}{3} e^{4} - \frac{2\pi i}{3} e^{-2}$$

$$= \frac{2\pi i}{3} (e^{4} - e^{-2})$$

$$= \frac{2\pi i}{3} \left(e^{4} - \frac{1}{e^{2}}\right)$$

3) Evaluate $\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz$ where 'c' is a circle $|z|=3$

Let $I = \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz \rightarrow (1)$

Since $C: |z| = 3$ is a circle with a circle '0' and radius 3 $z=1$ and $z=2$ are inside of the circle.

$\therefore f(z) = \sin(\pi z^2) + \cos(\pi z^2)$ is analytic at all points except $z=1$ and $z=2$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \rightarrow (2)$$

$1 = A(z-2) + B(z-1)$

when $z = 1$, when $z = 2$

$1 = A(-1)$ $\quad$ $1 = B(1)$

$\boxed{A = -1}$ $\quad$ $\boxed{B = 1}$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\therefore \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz = \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-2)} dz - \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)} dz$$

where substitute $z = 2$ and $z = 1$

$$= 2\pi i (\sin 4\pi + \cos 4\pi) - 2\pi i (\sin \pi + \cos \pi)$$

$$= 2\pi i (0+1) - 2\pi i (0-1)$$

$$= 2\pi i + 2\pi i$$

$$= 4\pi i$$

4) Evaluate $\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)^2 (z-2)} dz$, where $C : |z| = 3$.

Let $I = \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)^2 (z-2)} dz \rightarrow (1)$

Since $C : |z| = 3$ is a circle with centre '0' and radius '3'

$z = 2$ and $z = 1$ are inside of the circle

$\therefore f(z) = \sin(\pi z^2) + \cos(\pi z^2)$ is a analytic function at all the points except $z = 1$ and $z = 2$.

$$\therefore \frac{1}{(z-1)^2 (z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z-2)}$$

$$1 = A(z-1)(z-2) + B(z-2) + C(z-1)^2 \rightarrow (2)$$

when $z = 1$, when $z = 2$

$1 = B(-1)$ $\quad$ $1 = C$

$\boxed{B = -1}$ $\quad$ $\boxed{C = 1}$

$\therefore A + C = 0 \Rightarrow A = -C = -1$

$\boxed{A = -1}$

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$$\therefore \frac{1}{(z-1)^2(z-2)} = \frac{-1}{(z-1)} - \frac{1}{(z-1)^2} + \frac{1}{(z-2)}$$

$$\therefore \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)^2(z-2)}dz = \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-2)}dz - \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-1}dz - \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)^2}dz.$$

$$= 2\pi i(\sin 4\pi+\cos 4\pi) - 2\pi i(\sin\pi+\cos\pi) - \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)^{1+1}}dz$$

$$= 2\pi i(0+1) - 2\pi i(0-1) - \frac{2\pi i}{1!}f'(1) \rightarrow ③$$

where $f(z) = \sin\pi z^2+\cos\pi z^2$

$f'(z) = (\cos\pi z^2 - \sin\pi z^2)2\pi z$

$\therefore f'(1) = (\cos\pi - \sin\pi)\,2\pi$

$= (-1-0)\,2\pi$

$= -2\pi$

$\therefore$ from eqⁿ ③

$$\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z-2)}dz = 2\pi i + 2\pi i - \frac{2\pi i}{1!}(-2\pi)$$

$$= 4\pi i + \frac{4\pi^2 i}{1}$$

$$= 4\pi i(1+\pi)$$

5) Evaluate $\int_C \frac{z^2-z+1}{z-1}\cdot dz$ where 'c' is a circle (i) $|z|=1$ (ii) $|z|=1/2$

$$I = \int_C \frac{z^2-z+1}{z-1}\cdot dz \rightarrow ①$$

i) Since $C: |z|=1$ is a circle with the Centre 'O' (or) origin and radius 1.

$z=1$ is on the circle $|z|=1$

Hence $f(z) = z^2-z+1$ is analytic except $z=1$

$$\therefore \int_C \frac{z^2-z+1}{z-1}dz = 2\pi i(1^2-1+1)$$

$$= 2\pi i$$

ii) Since $C: |z| = \frac{1}{2}$ can represent the circle with centre as origin and radius $\frac{1}{2}$.

$z = 1$ is outside the circle $|z| = 1/2$

$$\therefore \int_C \frac{z^2 - z + 1}{z - 1} \, dz = 0 \quad (\because \text{outside - No values})$$

6) Evaluate $\int_C \frac{dz}{z^2 - 4}$ over the curve $C: |z+2| = 1$

$$\text{Let } I = \int_C \frac{dz}{z^2 - 4} = \int_C \frac{1}{(z-2)(z+2)} \, dz$$

Since $C: |z+2| = 1$ represents a circle with centre -2 and radius 1.

$\therefore f(z) = 1$ is analytic at all the points except $z = -2$ and $z = 2$

$$\therefore \int_C \frac{1}{(z-2)(z+2)} \, dz = \frac{1}{4}\left[\int_C \frac{1}{z-2} \, dz - \int_C \frac{1}{z+2} \, dz\right] \rightarrow (1)$$

$z = 2$ is outside the circle C $\therefore \int_C \frac{1}{z-2} \, dz = 0$

$$(1) \Rightarrow \int_C \frac{1}{(z-2)(z+2)} \, dz = \frac{1}{4}\left[0 - 2\pi i \cdot 1\right] = \frac{-2\pi i}{4} = -\frac{\pi i}{2}.$$