

||Jai Sri Gurudev||

S.J.C Institute of technology, chickballapur

Department of Mathematics

Lecture notes

Complex Analysis, Probability and Statistical Methods [18MATH1]

Prepared by:

Purushotham.P

Assistant Professor

SJC Institute of technology

Email id : ppurushothamp48@gmail.com

Phone : 8197481658

MODULE -03 : RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

Random experiment:

An activity, that yield some results called the random experiment. The random variable means a real number, i.e., X associated with the outcomes of a random experiment.

Definition:

Let S be a sample space associated with a random experiment with a real value function defined and taking its values is called a Random variable.

The Random Variable are two types, they are

1. Discrete random variables
2. Continuous random variables

Discrete random Variables :

A discrete random variables is a variable which can only take a countable number of values. For example if a coin is tossed three times, the number of heads can be obtained 0, 1, 2 or 3. The probabilities of each of these probabilities can be tabulated as shown.

$S = \{HHH, HHT, HTT, HTH, THH, TTH, THT, TTT\}$

X	0	1	2	3
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Continuous Random Variable:

A Continuous random variable is a random variable where the data can take infinitely many values. For example, a random variable measuring the time taken for something to be done is continuous since there are an infinite number of possible times that can be taken. Ex: temperature of the climate, age of a person, etc.

Probability mass function:

Probability mass function is the probability distribution of a discrete random variable, and provides the possible values and their associated probabilities.

1. $P(x_i) \geq 0$

2. $\sum_{i=1}^n P(X=x_i) = 1$

3. $0 \leq P(x) \leq 1$

4. Mean $\mu = \sum_{i=1}^n x_i P(x_i)$

5. Variance $\sigma^2 = \sum_{i=1}^n x_i^2 P(x_i) - \mu^2$

Probability density function

Probability density function is the probability distribution of a continuous random variable, and provides the possible values and their associated probabilities infinitely.

1. $P(x_i) \geq 0$ or $f(x) \geq 0$

2. $\int_{-\infty}^{\infty} f(x) dx = 1$

3. Mean $\mu = \int_{-\infty}^{\infty} x f(x) dx$

4. Variance $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$

5. $P(a \leq x \leq b) = P(a < x \leq b) = P(a \leq x < b) = P(a < x < b) = \int_a^b f(x) dx$

Problem:

1) Show that the following probabilities are satisfying the properties of DRV, hence find its mean and Variance.

x	10	20	30	40
$P(x)$	$1/8$	$3/8$	$3/8$	$1/8$

Solⁿ Let X be the Random Variable for the Random Values

$$x_1 = 10, x_2 = 20, x_3 = 30, x_4 = 40$$

And given $P(X = x_1) = P(x_1) = P_1 = \frac{1}{8}$

$$P(X = x_2) = P(x_2) = P_2 = \frac{3}{8}$$

$$P(X = x_3) = P(x_3) = P_3 = \frac{3}{8} \text{ and } P(X = x_4) = P(x_4) = \frac{1}{8}$$

$$\text{Let } \sum_{i=1}^4 P(X = x_i) = P(X = x_1) + P(X = x_2) + P(X = x_3) + P(X = x_4)$$

$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$$

$$= \frac{8}{8}$$

$$= 1$$

Hence the given probabilities can satisfy the DRV property

$$\mu = \sum_{i=1}^4 x_i P(x_i)$$

$$\text{Mean} = 10 \times \frac{1}{8} + 20 \times \frac{3}{8} + 30 \times \frac{3}{8} + 40 \times \frac{1}{8}$$

$$= \frac{200}{8}$$

$$= 25$$

$$\sigma^2 = \sum_{i=1}^4 x_i^2 P(x_i) - \mu^2$$

$$= 10^2 \times \frac{1}{8} + 20^2 \times \frac{3}{8} + 30^2 \times \frac{3}{8} + 40^2 \times \frac{1}{8} - 25^2$$

$$= 700 - 625$$

$$= 75$$

$$SD = \sqrt{\text{Variance}} = \sqrt{75} = 8.66$$

2) Find the value of k , such that following distribution represents discrete probability distribution, hence find Mean, SD,

$P(x \leq 1)$, $P(x > 1)$ and $P(-1 < x \leq 2)$.

x	-3	-2	-1	0	1	2	3
$P(x)$	k	$2k$	$3k$	$4k$	$3k$	$2k$	k

Solⁿ Let X be the Random Variable for the Random Values
 $x_1 = -3, x_2 = -2, x_3 = -1, x_4 = 0, x_5 = 1, x_6 = 2, x_7 = 3$ and the given respective probabilities are

$$P(X=x_1) = P(-3) = k$$

$$P(X=x_2) = P(-2) = 2k$$

$$P(X=x_3) = P(-1) = 3k$$

$$P(X=x_4) = P(0) = 4k$$

$$P(X=x_5) = P(1) = 3k$$

$$P(X=x_6) = P(2) = 2k$$

$$P(X=x_7) = P(3) = k$$

$$\text{W.K.T } \sum_{i=1}^7 P(X=x_i) = 1$$

$$\Rightarrow k + 2k + 3k + 4k + 3k + 2k + k = 1$$

$$16k = 1$$

$$k = \frac{1}{16}$$

x	$P(x)$	$xP(x)$	x^2	$x^2 P(x)$
-3	k	$-3k$	9	$9k$
-2	$2k$	$-4k$	4	$8k$
-1	$3k$	$-3k$	1	$3k$
0	$4k$	0	0	0
1	$3k$	$3k$	1	$3k$
2	$2k$	$4k$	4	$8k$
3	k	$3k$	9	$9k$
		<u>0</u>		<u>40k</u>

$$\mu = \sum_{i=1}^n x_i P(x_i)$$

$$= 0$$

$$\text{Variance } \sigma^2 = \sum_{i=1}^7 x_i^2 P(x_i) - \mu^2 = 40k - 0 = 40k = 40 \times \frac{1}{16} = 2.5$$

$$\text{S.D} = \sqrt{2.5} = 1.5811$$

$$P(X \leq 1) = P(-3) + P(-2) + P(-1) + P(0) + P(1)$$

$$P(X \leq 1) = k + 2k + 3k + 4k + 3k$$

$$P(X \leq 1) = 13k = \frac{13}{16} = 0.8125$$

$$P(X > 1) = P(2) + P(3) = 2k + k = 3k = \frac{3}{16} = 0.1875$$

$$P(-1 < X \leq 2) = P(0) + P(1) + P(2) = 4k + 3k + 2k = 9k = \frac{9}{16} = 0.5625$$

37 Find the value of k , such that the following distribution represents discrete probability distribution, hence find Mean, SD, $P(x \geq 5)$ and $P(3 < x \leq 6)$.

x	0	1	2	3	4	5	6
$P(x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Solⁿ Let x be the random variable for the random values $x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3, x_5 = 4, x_6 = 5, x_7 = 6$ and given the respective probabilities are

$$P(x = x_1) = P(0) = k$$

$$P(x = x_2) = P(1) = 3k$$

$$P(x = x_3) = P(2) = 5k$$

$$P(x = x_4) = P(3) = 7k$$

$$P(x = x_5) = P(4) = 9k$$

$$P(x = x_6) = P(5) = 11k$$

$$P(x = x_7) = P(6) = 13k$$

$$\text{W.K.T } \sum_{i=1}^7 P(x = x_i) = 1$$

$$\Rightarrow k + 3k + 5k + 7k + 9k + 11k + 13k = 1$$

$$49k = 1$$

$$k = \frac{1}{49}$$

x	$P(x)$	$xP(x)$	x^2	$x^2P(x)$
0	k	0	0	0
1	$3k$	$3k$	1	$3k$
2	$5k$	$10k$	4	$20k$
3	$7k$	$21k$	9	$63k$
4	$9k$	$36k$	16	$144k$
5	$11k$	$55k$	25	$275k$
6	$13k$	$78k$	36	$468k$
		<u>$203k$</u>		<u>$973k$</u>

$$\mu = \sum_{i=1}^n x_i P(x_i)$$

$$= 203k$$

$$= \frac{203}{49}$$

$$= 4.1428$$

$$\sigma^2 = \sum_{i=1}^7 x_i^2 P(x_i) - \mu^2 = 978K - 4 \cdot 1428^2$$

$$\text{Variance} = \frac{978}{49} - 17.1628$$

$$= 19.8571 - 17.1628$$

$$= 2.6943$$

$$S.D = \sqrt{2.6943} = 1.6414$$

$$P(x \geq 5) = P(5) + P(6)$$

$$\Rightarrow P(x \geq 5) = 11K + 13K$$

$$P(x \geq 5) = 24K = \frac{24}{49} = 0.4898$$

$$P(3 < x \leq 6) = P(4) + P(5) + P(6) = 9K + 11K + 13K$$

$$= 33K$$

$$= \frac{33}{49}$$

$$= 0.6734$$

4) A random variable x has a probability function for various values of x

x	0	1	2	3	4	5	6	7
$P(x)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2 + K$

Find i) K ii) $P(x < 6)$ iii) $P(x \geq 6)$, $P(3 < x \leq 6)$, Also find the probability distribution and distribution function of x .

Solⁿ

Let x be the random variable for the random values $x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3, x_5 = 4, x_6 = 5, x_7 = 6, x_8 = 7$ and the given respective probabilities are

$$P(x = x_1) = P(0) = 0$$

$$P(x = x_2) = P(1) = K$$

$$P(x = x_3) = P(2) = 2K$$

$$P(x = x_4) = P(3) = 2K$$

$$P(x = x_5) = P(4) = 3K$$

$$P(x = x_6) = P(5) = K^2$$

$$P(x = x_7) = P(6) = 2K^2$$

$$P(x = x_8) = P(7) = 7K^2 + K$$

$$\text{W.K.T } \sum_{i=1}^7 P(x = x_i) = 1$$

$$\Rightarrow 0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$

$$10k^2 + 9k = 1$$

$$10k^2 + 9k - 1 = 0$$

$$(10k-1)(k+1) = 0$$

$$10k-1=0, k+1=0$$

$$k = \frac{1}{10}, k \neq -1$$

x	0	1	2	3	4	5	6	7
$P(x)$	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17
CP	0	0.1	0.3	0.5	0.8	0.81	0.83	1

i) $P(x < 6) = 1 - P(x \geq 6)$
 $\Rightarrow P(x < 6) = 1 - \{P(6) + P(7)\}$
 $\Rightarrow P(x < 6) = 1 - (0.02 + 0.17)$
 $\Rightarrow P(x < 6) = 0.81$

ii) $P(x \geq 6) = P(6) + P(7) = 0.02 + 0.17 = 0.19$

$$P(3 < x \leq 6) = P(4) + P(5) + P(6) = 0.3 + 0.01 + 0.02 = 0.33$$

5) A random variable has the following probability function for the various values of $X = x$

x	-2	-1	0	1	2	3
$P(x)$	0.1	k	0.2	$2k$	0.3	k

Find i) Value of k

ii) $P(x < 1)$

iii) $P(x \geq -1)$

Solⁿ Let X be the random variable for the random values $x_1 = -2, x_2 = -1, x_3 = 0, x_4 = 1, x_5 = 2, x_6 = 3$ and given the respective probabilities are

$$P(X = x_1) = P(-2) = 0.1$$

$$P(X = x_2) = P(-1) = k$$

$$P(X = x_3) = P(0) = 0.2$$

$$P(X = x_4) = P(1) = 2k$$

$$P(X = x_5) = P(2) = 0.3$$

$$P(X = x_6) = P(3) = k$$

$$\sum_{i=1}^6 P(X=x_i) = 1$$

$$\Rightarrow 0.1 + k + 0.2 + 2k + 0.3 + k = 1$$

$$\Rightarrow 4k + 0.6 = 1$$

$$4k = 0.4$$

$$k = \frac{0.4}{4} = 0.1$$

$$P(X < 1) = P(-2) + P(-1) + P(0)$$

$$\Rightarrow P(X < 1) = 0.1 + k + 0.2$$

$$\Rightarrow P(X < 1) = k + 0.3 = 0.1 + 0.3 = 0.4$$

$$P(X \geq -1) = P(-1) + P(0) + P(1) + P(2)$$

$$= k + 0.2 + 2k + 0.3 + k$$

$$P(X \geq -1) = 4k + 0.5$$

$$P(X \geq -1) = 4(0.1) + 0.5$$

$$= 0.4 + 0.5 = 0.9$$

6) A random variable has the following probability function for the various values of $X=x$:

x	0	1	2	3	4	5
$P(x)$	k	$5k$	$10k$	$10k$	$5k$	k

Find i) Value of k

ii) $P(X \leq 1)$

iii) $P(0 \leq X < 3)$

Solⁿ Let X be the random variable for the random values $x_1=0, x_2=1, x_3=2, x_4=3, x_5=4, x_6=5$ and the given respective probabilities are

$$P(X=x_1) = P(0) = k$$

$$P(X=x_2) = P(1) = 5k$$

$$P(X=x_3) = P(2) = 10k$$

$$P(X=x_4) = P(3) = 10k$$

$$P(X=x_5) = P(4) = 5k$$

$$P(X=x_6) = P(5) = k$$

$$\sum_{i=1}^6 P(X=x_i) = 1$$

$$\Rightarrow k + 5k + 10k + 10k + 5k + k = 1$$

$$\Rightarrow 32k = 1$$

$$\Rightarrow k = \frac{1}{32}$$

$$P(x \leq 1) = P(0) + P(1)$$

$$\Rightarrow P(x \leq 1) = k + 5k$$

$$\Rightarrow P(x \leq 1) = 6k = \frac{6}{32} = 0.1875$$

$$P(0 \leq x < 3) = P(0) + P(1) + P(2)$$

$$\Rightarrow P(0 \leq x < 3) = k + 5k + 10k$$

$$P(0 \leq x < 3) = 16k$$

$$P(0 \leq x < 3) = \frac{16}{32} = 0.5$$

7) Show that the function $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is pdf, hence find $P(1.5 < x < 2.5)$.

Solⁿ Given probability function $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$$\begin{aligned} \text{Let } \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx \\ &= \int_{-\infty}^0 0 dx + \int_0^{\infty} e^{-x} dx \\ &= 0 + \left[\frac{e^{-x}}{-1} \right]_0^{\infty} \\ &= -[e^{-\infty} - e^0] \\ &= -[0 - 1] \\ &= 1 \end{aligned}$$

Hence the given probability function is pdf.

$$P(1.5 < x < 2.5) = \int_{1.5}^{2.5} f(x) dx$$

$$\Rightarrow P(1.5 < x < 2.5) = \int_{1.5}^{2.5} e^{-x} dx$$

$$\Rightarrow P(1.5 < x < 2.5) = -[e^{-x}]_{1.5}^{2.5}$$

$$P(1.5 < x < 2.5) = -[e^{-2.5} - e^{-1.5}]$$

$$P(1.5 < x < 2.5) = \frac{1}{e^{1.5}} - \frac{1}{e^{2.5}}$$

8) A random variable x has a probability density function

$$f(x) = \begin{cases} kx^2, & 0 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases} \quad \text{Evaluate i) } k \quad \text{ii) } P(x \leq 1) \quad \text{iii) } P(x > 1)$$

$$\text{iv) } P(1 \leq x \leq 2) \quad \text{v) } P(x \leq 2) \quad \text{vi) } P(x \geq 2).$$

Soln Given probability density function $f(x) = \begin{cases} kx^2, & 0 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^3 f(x) dx + \int_3^{\infty} f(x) dx = 1$$

$$k \cdot k \cdot T \Rightarrow 0 + \int_0^3 kx^2 dx + 0 = 1$$

$$\Rightarrow k \left[\frac{x^3}{3} \right]_0^3 = 1$$

$$\Rightarrow 9k = 1$$

$$\Rightarrow k = \frac{1}{9}$$

$$\text{ii)} \quad P(x \leq 1) = \int_{-\infty}^1 f(x) dx$$

$$\Rightarrow P(x \leq 1) = \int_{-\infty}^0 f(x) dx + \int_0^1 f(x) dx$$

$$P(x \leq 1) = 0 + \int_0^1 kx^2 dx$$

$$P(x \leq 1) = k \left[\frac{x^3}{3} \right]_0^1$$

$$P(x \leq 1) = \frac{k}{3} = \frac{1}{27}$$

$$\text{iii)} \quad P(x > 1) = \int_1^3 f(x) dx + \int_3^{\infty} f(x) dx$$

$$P(x > 1) = 0 + \int_1^3 kx^2 dx$$

$$P(x > 1) = k \left[\frac{x^3}{3} \right]_1^3$$

$$P(x > 1) = \frac{26k}{3} = \frac{26}{27}$$

$$\text{iv)} \quad P(1 \leq x \leq 2) = \int_1^2 f(x) dx$$

$$= \int_1^2 kx^2 dx$$

$$= k \left[\frac{x^3}{3} \right]_1^2$$

$$= \frac{7k}{3} = \frac{7}{27}$$

$$\text{v)} \quad P(x \leq 2) = \int_{-\infty}^2 f(x) dx$$

$$= \int_{-\infty}^0 f(x) dx + \int_0^2 f(x) dx$$

$$= k \int_0^2 f(x) dx$$

$$= k \int_0^2 x^2 dx$$

$$= k \left[\frac{x^3}{3} \right]_0^2$$

$$= \frac{8}{27}$$

$$\begin{aligned} \text{vi)} \quad P(x \geq 2) &= \int_2^{\infty} f(x) dx \\ &= \int_2^3 f(x) dx + \int_3^{\infty} f(x) dx \\ &= 0 + \int_2^3 kx^2 dx \\ &= k \left[\frac{x^3}{3} \right]_2^3 \end{aligned}$$

$$\Rightarrow \frac{19k}{3} = \frac{19}{27}$$

Binomial distribution: Let x be discrete random variable, p be the probability of success and let q be the probability of failure, then the probability mass function of the Binomial distribution can be defined as (n , the number of trials is finite)

$$P(X=x) = b(n, p, x) = \begin{cases} nC_x p^x q^{n-x} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

where n, p are called the parameters

Note: 1. $P(X=\infty) = b(n, p, x) \geq 0$

$$2. p + q = 1$$

$$3. \sum_{x=0}^n nC_x p^x q^{n-x} = 1$$

4. The mean of B.D. $\mu = np$, Variance $\sigma^2 = npq$, and S.D is $\sigma = \sqrt{npq}$

Problems:

1. Let x be a binomially distributed random variable based on 6 repetitions of an experiment. If $p=0.3$, evaluate the following probabilities i) $P(x \leq 3)$ ii) $P(x > 4)$

Solⁿ

Given $p = 0.3$ and $n = 6$ Hence $q = 1 - p = 0.7$ and $P(X=x) = {}^6C_x (0.3)^x (0.7)^{6-x}$

$$i) P(X \leq 3) = P(0) + P(1) + P(2) + P(3)$$

$$= {}^6C_0 (0.3)^0 (0.7)^{6-0} + {}^6C_1 (0.3)^1 (0.7)^{6-1} + {}^6C_2 (0.3)^2 (0.7)^{6-2} + {}^6C_3 (0.3)^3 (0.7)^{6-3}$$

$$ii) P(X > 4) = P(5) + P(6)$$

$$= {}^6C_5 (0.3)^5 (0.7)^{6-5} + {}^6C_6 (0.3)^6 (0.7)^{6-6}$$

2) The probability that a pen manufactured by a company will be defective is 0.1. If 12 such pens are selected at random, find the probability that

i) Exactly two pens will be defective

ii) At most two pens will be defective

iii) None will be defective.

Solⁿ let the probability that a pen manufactured is defective = p

Then $p = 0.1$, $q = 1 - p = 0.9$ and $n = 12$

Hence $b(n, p, x) = {}^{12}C_x (0.1)^x (0.9)^{12-x} = P(x)$ say

$$i) \text{ Probability that exactly two pens will be defective} = P(X=2) \\ = {}^{12}C_2 (0.1)^2 (0.9)^{12-2} = 0.2301$$

$$ii) \text{ Probability that at most 2 pens will be defective} = P(X \leq 2) = \\ P(0) + P(1) + P(2)$$

$$= {}^{12}C_0 (0.1)^0 (0.9)^{12-0} + {}^{12}C_1 (0.1)^1 (0.9)^{12-1} + {}^{12}C_2 (0.1)^2 (0.9)^{12-2}$$

$$iii) \text{ Probability that none of the pens will be defective} = P(X=0) \\ = P(0) = {}^{12}C_0 (0.1)^0 (0.9)^{12-0} \\ = 0.2824295$$

$$iv) \text{ Probability that at least 2 pens will be defective} = P(X \geq 2) \\ = 1 - \{P(0) + P(1)\}$$

$$= 1 - \{ {}^{12}C_0 (0.1)^0 (0.9)^{12-0} + {}^{12}C_1 (0.1)^1 (0.9)^{12-1} \}$$

$$= 0.3411$$

3) The number of telephonic lines busy at an instant line is a binomial variate with a probability 0.1. If 10 lines are chosen at random, what is the probability that i) No line is busy ii) All lines are busy iii) At least one line is busy iv) At most two lines are busy.

Soln Let the probability that a telephonic line is busy $p = 0.1$

$$q = 1 - p = 1 - 0.1 = 0.9$$

And the total number of lines chosen $n = 10$

Let x be the binomial variate; $P(X=x) = {}^{10}C_x (0.1)^x (0.9)^{10-x}$

i) The probability of getting no line is busy is $P(X=0) = {}^{10}C_0 (0.1)^0 (0.9)^{10-0}$

$$= 0.3486$$

ii) The probability of getting all the lines are busy is $P(X=10)$

$$= {}^{10}C_{10} (0.1)^{10} (0.9)^{10-10}$$

$$= (0.1)^{10}$$

iii) The probability of getting at least one line is busy

$$P(X \geq 1) = 1 - P(X < 1)$$

$$\Rightarrow P(X \geq 1) = 1 - P(0)$$

$$\Rightarrow P(X \geq 1) = 1 - 0.3486$$

$$P(X \geq 1) = 0.6514$$

iv) The probability of getting at most two lines are busy

$$P(X \leq 2) = P(0) + P(1) + P(2)$$

$$P(X \leq 2) = {}^{10}C_0 (0.1)^0 (0.9)^{10-0} + {}^{10}C_1 (0.1)^1 (0.9)^{10-1} + {}^{10}C_2 (0.1)^2 (0.9)^{10-2}$$

$$P(X \leq 2) = 0.3486 + 0.3874 + 0.1937$$

$$P(X \leq 2) = 0.9297$$

4) When a coin is tossed 4 times, find the probability of getting

i) Exactly one head ii) At most three heads iii) At least two heads

Soln The number of times tossed a coin $n = 4$

Let x be the binomial variate getting head $p = 0.5$

The probability of getting tail $q = 0.5$

$$\therefore P(X=x) = {}^4C_x (0.5)^x (0.5)^{4-x}$$

i) The probability of getting exactly one head $P(X=1) = {}^4C_1 (0.5)^1 (0.5)^{4-1}$
 $= 0.25$

ii) The probability of getting at most three heads $P(X \leq 3) = 1 - P(X=4)$
 $= 1 - {}^4C_4 (0.5)^4 (0.5)^{4-4}$
 $= 0.9375$

iii) The probability of getting at least two heads

$$\Rightarrow P(X \geq 2) = 1 - P(X=0) - P(X=1)$$

$$\Rightarrow P(X \geq 2) = 1 - {}^4C_0 (0.5)^0 (0.5)^{4-0} - {}^4C_1 (0.5)^1 (0.5)^{4-1}$$

$$\Rightarrow P(X \geq 2) = 1 - 0.0625 - 0.25 = 0.6875$$

5) The probability of germination of a seed in a packet of seeds is found to be 0.7. If 10 seeds are taken for experimenting on germination in a laboratory, find the probability that

i) 8 seeds germinate

ii) at least 8 seeds germinate

iii) at most 8 seeds germinate.

Solⁿ Let X be the binomial variate of the seed germination

Given the number of seeds packets taken for the experiment in the laboratory $n=10$

The probability of germination of a seed in a packet of seeds

is $p = 0.7$, $q = 1 - p \Rightarrow q = 1 - 0.7 = 0.3$

The probability mass function of Binomial distribution is

$$\therefore P(X=x) = {}^{10}C_x (0.7)^x (0.3)^{10-x}$$

i) The probability of 8 seeds germinate $= P(X=8)$

$$\Rightarrow P(X=8) = {}^{10}C_8 (0.7)^8 (0.3)^{10-8} = 90 \times 0.05764 \times 0.09 = 0.4670$$

ii) The probability of at least 8 seeds germinate $= P(X \geq 8) = P(8) + P(9) + P(10)$

$$\Rightarrow P(X \geq 8) = {}^{10}C_8 (0.7)^8 (0.3)^{10-8} + {}^{10}C_9 (0.7)^9 (0.3)^{10-9} + {}^{10}C_{10} (0.7)^{10} (0.3)^{10-10}$$

$$\Rightarrow P(X \geq 8) = 0.4670 + 0.12106 + 0.02824 = 0.6163$$

iii) The probability of at most 8 seeds germinate $P(X \leq 8) = 1 - P(X > 8)$

$$\Rightarrow P(X \leq 8) = 1 - P(9) - P(10)$$

$$\Rightarrow P(X \leq 8) = 1 - 0.12106 - 0.02824$$

$$= 0.8507.$$

Poisson Distribution :

Let x be the discrete random variable, for any real value λ (parameter), such that the probability mass function of poisson distribution can be defined as, $P(X=x) = \begin{cases} \frac{e^{-\lambda} \lambda^x}{x!}, & x=0, 1, 2, 3, \dots, n \\ 0, & \text{otherwise} \end{cases}$

Note: 1. $P(X=x) = P(x) \geq 0$

$$2. \sum_{x=0}^n P(x) = \sum_{x=0}^n \frac{e^{-\lambda} \lambda^x}{x!} = 1.$$

$$3. \text{Mean } \mu = np = \lambda$$

$$4. \text{Variance } \sigma^2 = \lambda, \text{ S.D.} = \sqrt{\lambda}$$

Problems:

1. The number of accidents in a year to a taxi drivers in a city follows a poisson distribution with mean 3. Out of 1000 taxi drivers find approximately the number of drivers with i) No accident in a year ii) More than three accidents in a year.

Solⁿ Let x be the poisson variate follows accident in the year of the poisson distribution.

The probability mass function of the poisson distribution is

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

Given the mean of poisson distribution is $\mu = \lambda = 3$

$$\therefore P(X=x) = \frac{e^{-3} 3^x}{x!} \rightarrow \textcircled{1}$$

i) NO accident in the year out of 1000 taxi drivers = $1000 \times P(x=0)$

$$= 1000 \times \frac{e^{-3} 3^0}{0!}$$

$$= 1000 \times 0.05$$

$$= 50$$

$\therefore$ 50 members of the drivers out of 1000 drivers having no accident in the year.

ii) More than three accidents in the year out of 1000 drivers

$$= 1000 \times P(X > 3)$$

$$= 1000 \times [1 - P(X \leq 3)]$$

$$= 1000 \times [1 - P(0) - P(1) - P(2) - P(3)]$$

$$= 1000 \times \left[1 - \frac{e^{-3} 3^0}{0!} - \frac{e^{-3} 3^1}{1!} - \frac{e^{-3} 3^2}{2!} - \frac{e^{-3} 3^3}{3!} \right]$$

$$= 1000 \times [1 - 0.05 - 0.15 - 0.225 - 0.225]$$

$$= 1000 \times 0.5$$

$$= 500$$

$\therefore$ 500 members done accident of more than three accidents in the year out of 1000 drivers.

2) In a certain factory turning out razor blades there is a small probability of $\frac{1}{500}$ for any blade to be defective. The blades are supplied in a packets of 10. Use poisson distribution to calculate approximate number of packets containing i) No defective ii) Two defective iii) Three defective in Consignment of 10000 packets.

Soln Let x be the poison variate follows the blades to be defective of the poison distribution

The probability mass function of the poisson distribution is

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

Given the $n = 10$, $p = \frac{1}{500} = 0.002$

$$\Rightarrow \mu = np = 10 \times 0.002 = 0.02 = \lambda$$

$$\therefore P(X = x) = \frac{e^{-0.02} (0.02)^x}{x!} \rightarrow (1)$$

$$\begin{aligned} \text{i) NO blades are defective out of 10000 packets} &= 10000 \times P(X = 0) \\ &= \frac{10000 \times e^{-0.02} (0.02)^0}{0!} \\ &= 10000 \times 0.9802 \\ &\approx 9802 \end{aligned}$$

$\therefore$ 9802 blades are no defective out of 10000 packets.

$$\begin{aligned} \text{ii) Two defective blades out of 10000 packets} &= 10000 \times P(X = 2) \\ &= \frac{10000 \times e^{-0.02} (0.02)^2}{2!} \\ &= 10000 \times 0.0002 \approx 2 \end{aligned}$$

∴ 2 blades are defective out of 10000 packets of only two defective.

iii) Three defective blades out of 10000 packets = $10000 \times P(X=3)$

$$= 10000 \times \frac{e^{-0.02} (0.02)^3}{3!}$$

$$= 10000 \times 0.0000$$

$$= 0$$

∴ No blades are defective out of 10000 packets of only three defective

3) If the probability of bad reaction from a certain injection is 0.001, determine the probability that out of 2000 individuals more than two will get a bad reaction.

Solⁿ Let X be the poisson variate follows the bad reaction of the injection.

W.K.T The probability mass function of the poisson distribution is

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

Given the $n=2000$, $p=0.001$

$$\Rightarrow \mu = np = 2000 \times 0.001 = 2 = \lambda$$

$$\therefore P(X=x) = \frac{e^{-2} (2)^x}{x!} \rightarrow (1)$$

The probability that of more than two individuals will get bad reaction = $P(X > 2)$

$$= 1 - P(X \leq 2)$$

$$= 1 - [P(0) + P(1) + P(2)]$$

$$= 1 - \frac{e^{-2} (2)^0}{0!} - \frac{e^{-2} (2)^1}{1!} - \frac{e^{-2} (2)^2}{2!}$$

$$= 1 - \frac{5}{e^2}$$

$$= 0.32.$$

4) The probability that a news reader commits no mistakes in reading the news is $\frac{1}{e^3}$. Find the probability on a particular news broadcast he commits only i) Two mistakes ii) More than three mistakes iii) At most three mistakes.

Soln Let X be the poisson follows the news reader do mistakes of the poisson distribution
WKT, The probability mass function of the poisson distribution is $P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$

$$\text{Given } P(X=0) = \frac{1}{e^3}$$

$$\Rightarrow \frac{e^{-\lambda} \lambda^0}{0!} = \frac{1}{e^3}$$

$$\Rightarrow \frac{1}{e^\lambda} = \frac{1}{e^3} \Rightarrow \lambda = 3$$

$$\therefore P(X=x) = \frac{e^{-3} 3^x}{x!}$$

i) The probability that the news reader can do two mistakes
 $= P(X=2) = \frac{e^{-3} 3^2}{2!} = 0.2240$

ii) The probability that the news reader can do mistakes of more than three = $P(X > 3)$

$$\Rightarrow P(X > 3) = 1 - P(X \leq 3)$$

$$\Rightarrow P(X > 3) = 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$$

$$\Rightarrow P(X > 3) = 1 - e^{-3} \left[\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} \right]$$

$$\Rightarrow P(X > 3) = 1 - 0.05 (1 + 3 + 4.5 + 4.5)$$

$$\Rightarrow P(X > 3) = 1 - 0.65$$

$$\Rightarrow P(X > 3) = 0.3500$$

iii) The probability that the news reader can do at most three mistakes = $P(X \leq 3)$

$$\Rightarrow P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$\Rightarrow P(X \leq 3) = e^{-3} \left[\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} \right]$$

$$\Rightarrow P(X \leq 3) = (0.05) (1 + 3 + 4.5 + 4.5)$$

$$\Rightarrow P(X \leq 3) = 0.6500$$

5) Suppose 300 misprints are randomly distributed throughout a book of 500 pages. find the probability that a given page contains i) Exactly three misprints ii) less than three misprints iii) Four or more misprints.

Solⁿ Let X be the poisson variate of misprints throughout a book of 500 pages

Given Suppose 300 misprints are randomly distributed throughout a book of 500 pages.

$$\therefore \text{Mean} = \lambda = \frac{300}{500} = 0.6$$

W.K.T The probability mass function of the poisson distribution

$$\text{is } P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$\begin{aligned} \text{The probability of exactly three misprints} &= P(X=3) = \frac{e^{-0.6} (0.6)^3}{3!} \\ &= 0.01975 \end{aligned}$$

The probability of less than three misprints $= P(X < 3) = P(0) + P(1) + P(2)$

$$\Rightarrow P(X < 3) = \frac{e^{-0.6} (0.6)^0}{0!} + \frac{e^{-0.6} (0.6)^1}{1!} + \frac{e^{-0.6} (0.6)^2}{2!}$$

$$\Rightarrow P(X < 3) = e^{-0.6} [1 + 0.6 + 0.18]$$

$$\Rightarrow P(X < 3) = 0.5488 \times 1.78$$

$$\Rightarrow P(X < 3) = 0.9788$$

The probability of four and more misprints $= P(X \geq 4) = 1 - P(X < 4)$

$$\Rightarrow P(X \geq 4) = 1 - P(0) - P(1) - P(2) - P(3)$$

$$\Rightarrow P(X \geq 4) = 1 - \frac{e^{-0.6} (0.6)^0}{0!} - \frac{e^{-0.6} (0.6)^1}{1!} - \frac{e^{-0.6} (0.6)^2}{2!} - \frac{e^{-0.6} (0.6)^3}{3!}$$

$$\Rightarrow P(X \geq 4) = 1 - e^{-0.6} (1 + 0.6 + 0.18 + 0.036)$$

$$= 1 - 0.5488 \times 1.816$$

$$= 0.00338$$

6) A certain screw making machine produces an average 2 defective out of 100 and packs them in boxes of 500. Find the probability that the box contains i) 3 defective

ii) Atleast one defective iii) Between 2 and 4 defective.

Solⁿ Let $P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$

Given 2 are defective out of 100 $p = \frac{2}{100} = 0.02$, $n = 500$

$$\lambda = np = 0.02 \times 500 = 10$$

i) 3 defective = $P(X=3)$

$$P(X=3) = \frac{e^{-10} (10)^3}{3!} = 0.007566$$

ii) Atleast one is defective

$$\begin{aligned} P(X \geq 1) &= 1 - P(X < 1) \\ &= 1 - P(0) \\ &= 1 - \frac{e^{-10} (10)^0}{0!} \end{aligned}$$

$$= 0.9999 = 1$$

iii) Between 2 and 4 i.e. $P(2 \leq X \leq 4)$

$$\begin{aligned} &= P(2) + P(3) + P(4) \\ &= e^{-10} \left[\frac{10^2}{2!} + \frac{10^3}{3!} + \frac{10^4}{4!} \right] \\ &= e^{-10} [50 + 166.666 + 416.666] \\ &= 0.02875 \end{aligned}$$

Exponential distribution

Let x be a Continuous Random Variable, then the probability density function of an exponential distribution is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Note: 1. $f(x) \geq 0$

$$2. \int_{-\infty}^{\infty} f(x) dx = 1$$

3. Mean of exponential distribution $\mu = 1/\lambda$

4. Variance of exponential distribution $\sigma^2 = 1/\lambda^2$

5. S.D of exponential distribution $\sigma = 1/\lambda$

Problems:

1) If x is an exponential Variate with mean 3, then find $P(x > 1)$ and $P(x < 3)$.

Solⁿ Given x be a Continuous random Variable of an exponential distribution is

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

And given the mean of exponential distribution is 3

$$\Rightarrow 1/\alpha = 3 \Rightarrow \alpha = \frac{1}{3}$$

$$\therefore f(x) = \begin{cases} \frac{1}{3} e^{-x/3} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{i) } P(x > 1) &= \int_1^{\infty} f(x) dx = \frac{1}{3} \int_1^{\infty} e^{-\frac{x}{3}} dx \\ &= -\left[e^{-\frac{x}{3}} \right]_1^{\infty} \\ &= -[e^{-\infty} - e^{-1/3}] \\ &= -[0 - e^{-1/3}] \\ &= e^{-1/3} \end{aligned}$$

$$\begin{aligned} \text{ii) } P(x < 3) &= \int_{-\infty}^3 f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^3 f(x) dx \\ &= 0 + \frac{1}{3} \int_0^3 e^{-\frac{x}{3}} dx \\ &= -\left[e^{-\frac{x}{3}} \right]_0^3 \\ &= -[e^{-1} - e^0] \\ &= 1 - \frac{1}{e} \end{aligned}$$

2) If x is an exponential Variate with mean 4, then find $P(0 < x < 1)$, $P(x > 2)$ and $P(-\infty < x < 10)$.

Solⁿ Given x be a Continuous random Variable of an exponential distribution is $f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

And given the mean of exponential distribution is 3

$$\Rightarrow \frac{1}{\alpha} = 4 \Rightarrow \alpha = \frac{1}{4}$$

$$\therefore f(x) = \begin{cases} \frac{1}{4} e^{-x/4} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P(0 < x < 1) = \int_0^1 f(x) dx = \frac{1}{4} \int_0^1 e^{-x/4} dx = -[e^{-x/4}]_0^1 = 1 - \frac{1}{e^{1/4}}$$

$$P(x > 2) = \int_2^{\infty} f(x) dx = \frac{1}{4} \int_2^{\infty} e^{-x/4} dx = -[e^{-x/4}]_2^{\infty} = -[e^{-\infty} - e^{-2/4}] = e^{-1/2}$$

$$P(-\infty < x < 10) = \int_{-\infty}^{10} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{10} f(x) dx \\ = 0 + \frac{1}{4} \int_0^{10} e^{-x/4} dx$$

$$\Rightarrow P(-\infty < x < 10) = -[e^{-x/4}]_0^{10} = -[e^{-10/4} - e^0] = 1 - \frac{1}{e^{5/2}}$$

3) In a certain town the duration of shower has mean 5 minutes what is the probability that shower will last for

i) 10 minutes and more

ii) less than 10 minutes

iii) Between 10 and 12 minutes.

Solⁿ Given X be a continuous random variable of an exponential distribution is $f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

And given the mean of exponential distribution is 3

$$\Rightarrow \frac{1}{\alpha} = 5 \Rightarrow \alpha = \frac{1}{5}$$

$$\therefore f(x) = \begin{cases} \frac{1}{5} e^{-x/5} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

i) The probability that the shower will last 10 minutes and more is

$$P(x \geq 10) = \int_{10}^{\infty} f(x) dx = \int_{10}^{\infty} \frac{1}{5} e^{-x/5} dx = \frac{1}{5} \int_{10}^{\infty} e^{-x/5} dx = -[e^{-x/5}]_{10}^{\infty} \\ = -[0 - e^{-2}] = \frac{1}{e^2}$$

ii) The probability that the shower will last less than 10 minutes

$$P(x < 10) = \int_{-\infty}^{10} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{10} f(x) dx$$

$$= 0 + \int_0^{10} \frac{1}{5} e^{-x/5} dx = \frac{1}{5} \int_0^{10} e^{-x/5} dx = -[e^{-x/5}]_0^{10} = -[e^{-2} - 1] = 1 - \frac{1}{e^2}$$

iii) The probability that the shower will last between 10 and 12 minutes is

$$P(10 < x < 12) = \int_{10}^{12} f(x) dx = \int_{10}^{12} \frac{1}{5} e^{-x/5} dx = \frac{1}{5} \int_{10}^{12} e^{-x/5} dx$$

$$\Rightarrow P(10 < x < 12) = -[e^{-x/5}]_{10}^{12} = -[e^{-12/5} - e^{-2}] = \frac{1}{e^{12/5}} - \frac{1}{e^2}$$

4) The length of a telephone conversation is an exponential variate with mean 3 minutes. Find the probability that a call i) ends in less than 3 minutes ii) Between 3 and 5 minutes.

Solⁿ Let 'x' be the continuous random variable of an exponential distribution $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Given, the mean of exponential distribution = 3 mins

$$\Rightarrow \frac{1}{\lambda} = 3 \Rightarrow \lambda = \frac{1}{3}$$

$$\text{M.K.T } f(x) = \begin{cases} \frac{1}{3} e^{-x/3}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

i) The call which ends in less than 3 mins.

$$\Rightarrow P(x \leq 3) = \int_{-\infty}^3 f(x) dx$$

$$= \int_{-\infty}^0 f(x) dx + \int_0^3 f(x) dx$$

$$= \int_0^3 \frac{1}{3} e^{-x/3} dx$$

$$= \frac{1}{3} \left[\frac{e^{-x/3}}{-1/3} \right]_0^3$$

$$= -[e^{-x/3}]_0^3 = -[e^{-1} - e^0] = 1 - \frac{1}{e}$$

ii) Call ends between 3 and 5 mins

$$P(3 \leq x \leq 5) = \int_3^5 f(x) dx$$

$$= \int_3^5 \frac{1}{3} e^{-x/3} dx$$

$$\begin{aligned}
 &= \frac{1}{3} \left[\frac{e^{-x/3}}{-1/3} \right]_3^5 \\
 &= -[e^{-5/3} - e^{-1}] \\
 &= e^{-1} - e^{-5/3}
 \end{aligned}$$

Normal distribution:

Let X be Continuous random variable, for any μ, σ^2 the probability density function of the normal distribution is defined as $f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, where $-\infty < x < \infty, -\infty < \mu < \infty$ and $\sigma^2 > 0$.

whereas $f(x) \geq 0, \int_{-\infty}^{\infty} f(x) dx = 1$

and μ is called mean and σ is called standard deviation of the Normal distribution.

Note:

The graph of the probability density function $f(x)$ is always a bell shaped curve, symmetric about the line $x = \mu$ and it is called the normal probability curve and the line $x = \mu$ divides the curve into two equal parts with the area of 0.5 sq. units, and its total area of the normal probability curve is 1 sq. units.

Standard Normal Variate:

By the definition of Normal distribution we have,

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$P(a \leq x \leq b) = \int_a^b f(x) dx$$

$$\Rightarrow P(a \leq x \leq b) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{z^2}{2}} dz = P(z_1 \leq z \leq z_2)$$

$$\text{where } z = \frac{x-\mu}{\sigma}, z_1 = \frac{a-\mu}{\sigma}, z_2 = \frac{b-\mu}{\sigma}$$

and $F(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is called the standard normal function

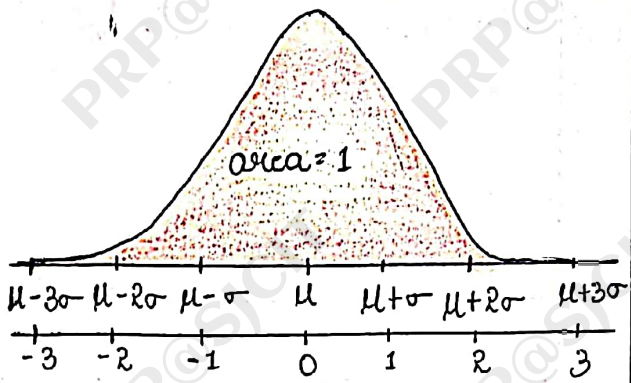
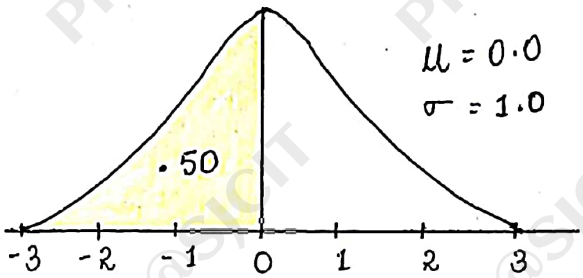
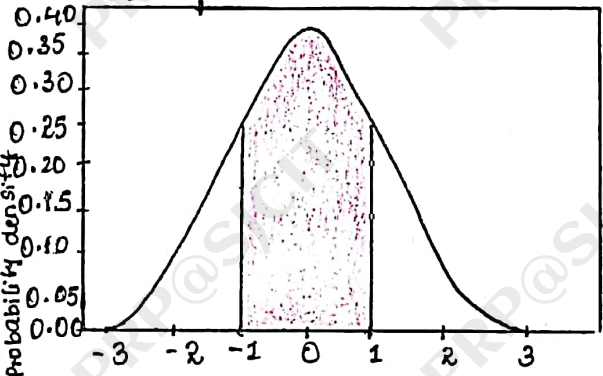
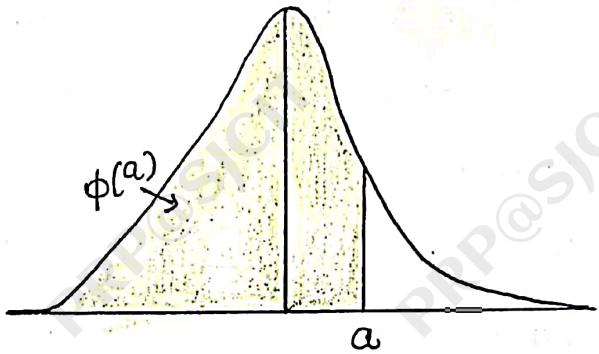
and $z = \frac{x-\mu}{\sigma}$ is called the standard normal variate

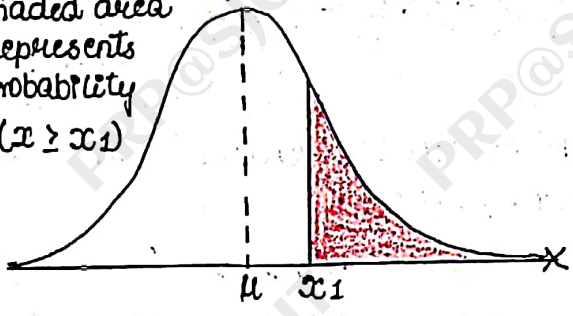
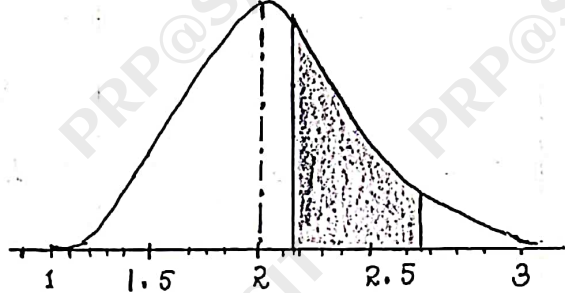
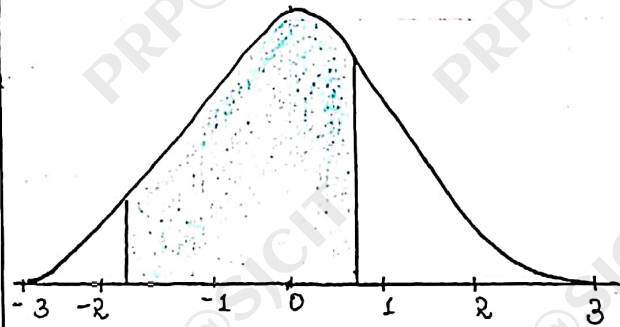
when $z_1 = 0, z_2 = z$, then the normal curve over 0 to z is defined

as $A(z) = \phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} dz$

where these values will be taken from Area table of normal distribution.

Some important results:

Sl.No	Probability Range	Result	Graph.
1)	$P(-\infty < z < \infty)$	1	
2)	$P(-\infty < z < 0) = P(0 < z < \infty)$	0.5	
3)	$P(-z_1 < z < z_1) = 2P(0 < z < z_1)$	$2A(z_1)$	68% of values are within 1 STD 
4)	$P(-\infty < z < z_1) = 0.5 + P(0 < z < z_1)$	$0.5 + A(z_1)$	

5)	$P(z_1 < z < \infty) = P(0 < z < \infty) - P(0 < z < z_1)$	$0.5 - A(z_1)$	Shaded area represents probability $P(z \geq z_1)$ 
6)	$P(z_1 < z < z_2) = P(0 < z < z_2) - P(0 < z < z_1)$	$A(z_2) - A(z_1)$	
7)	$P(-z_1 < z < z_2) = P(0 < z < z_2) + P(0 < z < z_1)$	$A(z_2) + A(z_1)$	Area chart shading under x-y chart's line 

Problems :

1) The marks of 1000 students in an examination follows normal distribution with mean 70 and standard deviation 5. Find the number of students whose marks will be

i) less than 65

ii) More than 75

iii) Between 65 and 75. [$A(z) = 0.3413$]

Solⁿ Let x be the Continuous random Variable

Given Mean of the Normal distribution $\mu = 70$

Standard deviation of the Normal distribution $\sigma = 5$

$\therefore$ The standard normal Variate $z = \frac{x - \mu}{\sigma} \Rightarrow z = \frac{x - 70}{5}$

when $x = 65$ then $z = \frac{65 - 70}{5} = -1$

when $x = 75$ then $z = \frac{75 - 70}{5} = 1$

i) No of students scored less than 65 marks $= P(X < 65) = P(Z < -1) =$
 $P(Z > 1) = 0.5 - A(1) = 0.5 - 0.3413 = 0.1587$

No of students scored less than 65 marks out of 1000 students =
 $= 1000 \times 0.1587 = 158.7 = 159$

ii) No of students scored more than 75 marks $= P(X > 75) = P(Z > 1) =$
 $P(Z > 1) = 0.5 - A(1) = 0.5 - 0.3413 = 0.1587$

No of students scored more than 75 marks out of 1000 students =
 $= 1000 \times 0.1587 = 158.7 = 159$

iii) No of students scored marks between 65 and 75 $= P(65 < X < 75) =$
 $P(-1 < Z < 1) = 2P(0 < Z < 1)$

$$= 2A(1)$$

$$= 2 \times 0.3413$$

$$= 0.6826$$

No of students scored between 65 and 75 marks out of 1000 students =
 1000×0.6826
 $= 682.6 = 683$

2) 200 students appeared in the examination, distribution of marks is assumed to be normal with mean 30 and standard deviation 6.25, how many students are expected to get marks.

i) Between 20 and 40

ii) Less than 35 [$A(1.6) = 0.4452$, $A(0.8) = 0.2881$]

Solⁿ Let X be the Continuous Random Variable

Given Mean of the Normal distribution $\mu = 30$

Standard deviation of the Normal distribution $\sigma = 6.25$

$\therefore$ The standard normal variate $z = \frac{x - \mu}{\sigma} \Rightarrow z = \frac{x - 30}{6.25}$

When $x = 20$ then $z = \frac{20 - 30}{6.25} = -1.6$

When $x = 40$ then $z = \frac{40 - 30}{6.25} = 1.6$

When $x = 35$ then $z = \frac{35 - 30}{6.25} = 0.8$

The probability that number of students expected to score between 20 and 40 marks : $P(20 < X < 40) = P(-1.6 < Z < 1.6)$

$$\Rightarrow P(20 < X < 40) = 2 \times P(0 < Z < 1.6)$$

$$\Rightarrow P(20 < x < 40) = 2 \times A(1.6)$$

$$\Rightarrow P(20 < x < 40) = 2 \times 0.4452 = 0.8904 = 0.9$$

The number of students expected to score between 20 and 40 marks out of 200:

The probability that number of students expected to score less than 35 : $= P(x < 35) = P(Z < 0.8)$

$$= A(0.8)$$

$$= 0.2881 = 0.3$$

The number of students expected to score less than 35 marks out of 200 : $= 200 \times 0.3 = 60$

3) The weekly wages of workers in a Company are normally distributed with mean of Rs. 700 and S.D of Rs. 50. Find the probability that the weekly wage of randomly chosen workers is i) Between Rs. 650 and Rs. 750 ii) More than Rs. 750.

Solⁿ Let x be the Continuous Random Variable

Given, Mean of the normal distribution $\mu = 700$

Standard deviation of the normal distribution $\sigma = 50$

$$\therefore \text{The standard normal variate } Z = \frac{x - \mu}{\sigma} \Rightarrow Z = \frac{x - 700}{50}$$

$$\text{When } x = 650, \text{ then } Z = \frac{650 - 700}{50} = -1$$

$$\text{When } x = 750, \text{ then } Z = \frac{750 - 700}{50} = 1$$

The probability of the weekly wages between Rs. 650 and Rs. 750 is

$$P(650 < x < 750) = P(-1 < Z < 1)$$

$$\Rightarrow P(650 < x < 750) = 2 \times P(0 < Z < 1)$$

$$\Rightarrow P(650 < x < 750) = 2 \times A(1)$$

$$\Rightarrow P(650 < x < 750) = 2 \times 0.3413 = 0.6826$$

The probability of the weekly wages of more than Rs. 750 is

$$= P(x > 750) = P(Z > 1)$$

$$= 0.5 - P(Z < 1)$$

$$= 0.5 - 0.3413$$

$$= 0.1587$$

4) In a test of 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours and SD of 60 hours. Estimate the number of bulbs likely to burn for

i) More than 2150 hours.

ii) Less than 1950 hours

iii) Between 1920 and 2160 hours.

Solⁿ Let X be the continuous random variable

Given Mean of the Normal distribution $\mu = 2040$

Standard deviation of the normal distribution $\sigma = 60$

$\therefore$ The standard normal variate $Z = \frac{X - \mu}{\sigma} \Rightarrow Z = \frac{X - 2040}{60}$

$$\text{when } X = 2150 \text{ then } Z = \frac{2150 - 2040}{60} = 1.83$$

$$\text{when } X = 1950 \text{ then } Z = \frac{1950 - 2040}{60} = -1.5$$

$$\text{when } X = 1920 \text{ then } Z = \frac{1920 - 2040}{60} = -2$$

$$\text{when } X = 2160 \text{ then } Z = \frac{2160 - 2040}{60} = 2$$

i) The probability that the number of bulbs likely to burn of more than 2150 hours: $P(X > 2150) = P(Z > 1.83)$

$$= 0.5 - A(1.83) = 0.5 - 0.4664$$

$$= 0.0336$$

The number of bulbs likely to burn of more than 2150 hours out of 2000 bulbs: 2000×0.0336

$$= 67.2 = 67$$

ii) The probability that the number of bulbs likely to burn of less than 1950 hours: $P(X < 1950) = P(Z < -1.5)$

$$P(Z < -1.5) = 0.5 - A(1.5) = 0.5 - 0.4332$$

$$= 0.0668$$

The number of bulbs likely to burn less than 1950 hours out of 2000 bulbs = 2000×0.0668

$$= 133.6 = 134$$

iii) The probability that the number of bulbs likely to burn between 1920 and 2160 hours = $P(1920 < X < 2160) = P(-2 < Z < 2)$

$$= 2P(0 < Z < 2)$$

$$\begin{aligned}
 &= 2A(z) \\
 &= 2 \times 0.4772 \\
 &= 0.9544
 \end{aligned}$$

The number of bulbs likely to burn between 1920 and 2160 hours out of 2000 bulbs: 2000×0.9544
 $= 1908.8 \approx 1909$

5) In a normal distribution, 7% of items are under 35 and 89% of the items are under 63. Find the mean and standard deviation of the distribution.

Soln Let x be the continuous random variable

Given let μ and σ be the Mean and Standard deviation of the distribution

$$\therefore \text{The Standard normal Variate } z = \frac{x - \mu}{\sigma} \rightarrow (1)$$

when $x = 35$ the standard normal variate $z = \frac{35 - \mu}{\sigma} = z_1$ (say)

when $x = 63$ the standard normal variate $z = \frac{63 - \mu}{\sigma} = z_2$ (say)

Given,

$$P(x < 35) = P(z < z_1) = 0.07$$

$$\Rightarrow P(z < z_1) = P(-\infty < z < 0) - P(0 < z < z_1) = 0.07$$

$$\Rightarrow 0.5 - A(z_1) = 0.07$$

$$\Rightarrow A(z_1) = 0.5 - 0.07$$

$$\Rightarrow A(z_1) = A(-1.47)$$

$$\Rightarrow z_1 = -1.47$$

$$\Rightarrow \frac{35 - \mu}{\sigma} = -1.47$$

$$\mu - 1.47\sigma = 35 \rightarrow (2)$$

$$\text{And } P(x < 63) = P(z < z_2) = 0.89$$

$$= P(z < z_2) = P(-\infty < z < 0) + P(0 < z < z_2) = 0.89$$

$$\Rightarrow 0.5 + A(z_2) = 0.89$$

$$\Rightarrow A(z_2) = 0.89 - 0.5 = 0.39$$

$$\Rightarrow A(z_2) = A(1.23)$$

$$z_2 = 1.23$$

$$\Rightarrow \frac{63 - \mu}{\sigma} = 1.23$$

$$\Rightarrow \mu + 1.23\sigma = 63 \rightarrow (3)$$

Solving eqⁿ (2) and (3) we get

$$\mu = 50.2915 \quad \sigma = 10.332.$$

6) In a normal distribution, 31% of items are under 45 and 8% of the items are over 64. Find the mean and standard deviation of the distribution.

Solⁿ Let x be the continuous random variable

Given, let μ and σ be the Mean and standard deviation of the distribution.

$\therefore$ The standard normal variate $z = \frac{x - \mu}{\sigma} \rightarrow (1)$

when $x = 45$ the standard normal variate $z = \frac{45 - \mu}{\sigma} = z_1$ (say)

when $x = 64$ the standard normal variate $z = \frac{64 - \mu}{\sigma} = z_2$ (say)

Given

$$P(x < 45) = P(z < z_1) = 0.31$$

$$\Rightarrow P(z < z_1) = P(-\infty < z < 0) - P(0 < z < z_1) = 0.31$$

$$\Rightarrow 0.5 - A(z_1) = 0.31$$

$$\Rightarrow A(z_1) = 0.5 - 0.31 = 0.19$$

$$\Rightarrow A(z_1) = A(0.5)$$

$$\Rightarrow z_1 = 0.5$$

$$\Rightarrow \frac{45 - \mu}{\sigma} = 0.5$$

$$\Rightarrow \mu + 0.5 \sigma = 45 \rightarrow (2)$$

And $P(x > 64) = P(z > z_2) = 0.08$

$$\Rightarrow P(z > z_2) = 0.5 - P(0 < z < z_2) = 0.08$$

$$\Rightarrow 0.5 - A(z_2) = 0.08$$

$$\Rightarrow A(z_2) = 0.08 - 0.5 = -0.42$$

$$\Rightarrow A(z_2) = A(1.4)$$

$$\Rightarrow z_2 = 1.4$$

$$\Rightarrow \frac{64 - \mu}{\sigma} = 1.4$$

$$\Rightarrow \mu + 1.4 \sigma = 64 \rightarrow (3)$$

Solving eqⁿ (2) and (3) we get

$$\mu = 50$$

$$\sigma = 10$$

7) In an examination 7% of the students scored less than 35% of the marks and 89% of the students scored less than 60% of the marks. Find the mean and standard deviation if marks are normally distributed.

Solⁿ Let x be the Continuous random Variable.

Given Let μ and σ be the Mean and standard deviation of the distribution

$\therefore$ The standard normal variate $z = \frac{x - \mu}{\sigma} \rightarrow (1)$

when $x = 35$ the standard normal variate $z = \frac{35 - \mu}{\sigma} = z_1$ (say)

when $x = 60$ the standard normal variate $z = \frac{60 - \mu}{\sigma} = z_2$ (say)

Given,

$$P(x < 35) = P(z < z_1) = 0.07$$

$$\Rightarrow P(z < z_1) = P(-\infty < z < 0) - P(0 < z < z_1) = 0.07$$

$$\Rightarrow 0.5 - A(z_1) = 0.07$$

$$\Rightarrow A(z_1) = 0.5 - 0.07$$

$$\Rightarrow A(z_1) = A(-1.47)$$

$$\Rightarrow z_1 = -1.47$$

$$\Rightarrow \frac{35 - \mu}{\sigma} = -1.47$$

$$\Rightarrow \mu - 1.47\sigma = 35 \rightarrow (2)$$

$$\text{And } P(x < 60) = P(z < z_2) = 0.89$$

$$\Rightarrow P(z < z_2) = P(-\infty < z < 0) + P(0 < z < z_2) = 0.89$$

$$\Rightarrow 0.5 + A(z_2) = 0.89$$

$$\Rightarrow A(z_2) = 0.89 - 0.5 = 0.39$$

$$\Rightarrow A(z_2) = A(1.23)$$

$$\Rightarrow z_2 = 1.23$$

$$\Rightarrow \frac{60 - \mu}{\sigma} = 1.23$$

$$\mu + 1.23\sigma = 60 \rightarrow (3)$$

Solving eqⁿ (2) and (3) we get

$$\mu = 48.65$$

$$\sigma = 9.25$$

- 8) The IQ of students in a certain college is assumed to be normally distributed with mean 100 and variance 25. If two students are selected at random, find the probability that
- both of them have IQ between 102 and 110
 - at least one of them have IQ between 102 and 110
 - at most one of them have IQ between 102 and 110.

Solⁿ Let X be the Continuous random Variable

Given Let μ and σ be the Mean and standard deviation of the distribution

$$\therefore \text{The standard normal Variate } Z = \frac{X - \mu}{\sigma} \rightarrow (1)$$

And given $\mu = 100, \sigma^2 = 25 \Rightarrow \sigma = 5$

$$\therefore Z = \frac{X - 100}{5} \rightarrow (2)$$

when $x = 102$, the standard normal Variate $Z = \frac{102 - 100}{5} = 0.4$

when $x = 110$, the standard normal Variate $Z = \frac{110 - 100}{5} = 2$

i) The probability of both of them have IQ between 102 and 110's is

$$P(102 < X < 110) = P(0.4 < Z < 2)$$

$$\Rightarrow P(0.4 < Z < 2) = P(0 < Z < 2) - P(0 < Z < 0.4)$$

$$\Rightarrow P(0.4 < Z < 2) = A(2) - A(0.4)$$

$$\Rightarrow P(0.4 < Z < 2) = 0.4772 - 0.1554$$

$$\Rightarrow P(0.4 < Z < 2) = 0.3218 = P \Rightarrow q = 1 - P = 1 - 0.3218 = 0.6782$$

$\therefore$ The probability that both the students have IQ between 102 and 110 is $(0.3218)^2 = 0.1036$

ii) The probability that at least one of two students have IQ b/w 102 and 110 is (By using Binomial distribution) $= P(X \geq 1) = 1 - P(X < 1)$

$$\Rightarrow P(X \geq 1) = 1 - P(X = 0)$$

$$\Rightarrow P(X \geq 1) = 1 - [{}^2C_0 p^0 q^{2-0}]$$

$$\Rightarrow P(X \geq 1) = 1 - [1 \times 1 \times (0.6782)^2]$$

$$\Rightarrow P(X \geq 1) = 1 - 0.4600$$

$$\Rightarrow P(X \geq 1) = 0.5400$$

iii) The probability that at most one of them have IQ between 102 and 110 is $P(X \leq 1) = 1 - P(X > 1)$

$$\Rightarrow P(X \leq 1) = 1 - P(X = 2)$$

$$\Rightarrow P(X \leq 1) = 1 - {}^2C_2 p^2 q^{2-2}$$

$$\Rightarrow P(X \leq 1) = 1 - [{}^2C_2 (0.3218)^2 (0.6782)^{2-2}]$$

$$\Rightarrow P(X \leq 1) = 1 - [1 \times 0.10306 \times 1]$$

$$\Rightarrow P(x \leq 1) = 1 - 0.1036$$

$$\Rightarrow P(x \leq 1) = 0.8964$$

q) A sample of 100 battery cells tested to find the length of life produced the following results. $\bar{x} = 12$ hours and $\sigma = 3$ hours. Assuming the data to be normally distributed. What percentage of batteries are expected to have life

i) more than 15 hrs

ii) Less than 6 hrs

iii) B/w 10 and 14 hrs.

$$[A(1) = 0.3413, A(2) = 0.4772, A(0.67) = 0.2487]$$

Solⁿ Let x be the Random Variable

Given $\mu = 12$, SD $\sigma = 3$ hrs

$$Z = \frac{x - \mu}{\sigma} = \frac{x - 12}{3}$$

$$\text{when } x = 15, Z = \frac{15 - 12}{3} = 1$$

$$\text{when } x = 6, Z = \frac{6 - 12}{3} = -2$$

$$\text{when } x = 10, Z = \frac{10 - 12}{3} = -2/3$$

$$\text{when } x = 14, Z = \frac{14 - 12}{3} = 2/3$$

i) more than 15 hrs

$$\begin{aligned} P(x > 15) &= P(Z > 1) \\ &= 0.5 - A(1) \\ &= 0.5 - 0.3413 = 0.1587 \end{aligned}$$

$$\% = 0.1587 \times 100 = 15.87\%$$

ii) Less than 6 hrs i.e; $P(x < 6) = P(Z < -2)$

$$\begin{aligned} &= P(Z > 2) \\ &= 0.5 - A(2) = 0.5 - 0.4772 \\ &= 0.0228 \end{aligned}$$

$$\% = 0.0228 \times 100 = 2.28\%$$

iii) B/w 10 and 14 hrs

$$\begin{aligned} P(10 < x < 14) &= P(-2/3 < Z < 2/3) \\ &= 2 \times P(0 < Z < 2/3) \\ &= 2 \times A(2/3) \\ &= 2 \times A(0.67) = 2 \times 0.2487 = 0.4974 \end{aligned}$$

$$\% = 0.4974 \times 100 = 49.74\%$$

By: Purushotham P