

Probability Theory-2

Random Variables - Introduction.

In a random experiment, if a real variable is associated with every outcome then it is called a random variable (or) stochastic variable.

Random variables are usually denoted by

$X, Y, Z, \dots$

The set of all real numbers of a random variable X is called the range of X .

Example:- $\rightarrow$ while tossing a coin, Suppose that the value '1' is associated for the outcome 'head' and '0' for the outcome 'tail'. when we've the sample space

$$S = \{H, T\}$$

and if X is the random variable

$$\text{then } X(H) = 1 \text{ and } X(T) = 0$$

$$\therefore \text{Range of } X = \{0, 1\}$$

Example 2 Suppose a coin is tossed twice, we shall associate two different random variables x, y as follows

where we've the sample space,

$$S = \{HH, HT, TH, TT\}$$

x = Number of 'heads' in the outcome.

The association of the elements in S to x is as follows.

outcome	HH	HT	TH	TT
Random Variable x	2	1	1	0

Range of $x = \{0, 1, 2\}$

Discrete and continuous random variables

Defⁿ :- If a random variable takes finite or countably infinite number of values then it is called a discrete random variable.
Here countably infinite means a sequence of real numbers. It is evident that a discrete random variable will have finite (or) countably infinite range.

Ex:- 1) Tossing a coin and observing the outcome.

2) Tossing coins and observing the no of heads turning up.

3) Throwing a 'die' and observing the no on the face.

(2)

Defⁿ: - If a random variable takes non countable infinite number of values then it is called a non-discrete (or) continuous random variable. Equivalently we can say that, if the range of random variable X is an interval of real no's then X is a continuous random variable. A continuous random variable can assume any value in the interval of real no's.

Example $\rightarrow$ weight of articles.

2) Length of nails produced by a machine.

3) observing the pointer on a speedometer / Voltmeter.

4) Conducting a survey on the life of electric bulbs.

Discrete probability distribution - Definitions.

if for each value x_i of a discrete random variable X , we assign a real number $p(x_i)$ such that

$$1) p(x_i) \geq 0$$

$$2) \sum_i p(x_i) = 1$$

then the function $p(x)$ is called a probability function.

If the probability that X take the values x_i is p_i , then

$$p(X=x_i) = p_i \quad \text{or} \quad p(x_i)$$

The set of values $[x_i, p(x_i)]$ is called a discrete (finite) probability distribution of the discrete random variable X . The function $p(x)$ is called the probability density function (p.d.f) (or) the probability mass function (p.m.f)

The distribution function $f(x)$ defined by

$$f(x) = P(X \leq x) = \sum_{i=1}^x p(x_i),$$

x being an integer is called the commulative distribution function (c.d.f)

The mean and variance of the discrete Probability distribution is defined as follows.

1) Mean (μ) = $\sum_i x_i \cdot p(x_i)$

2) Variance (V) = $\sum_i (x_i - \mu)^2 \cdot p(x_i)$

3) Standard deviation (σ) = $\sqrt{V}$

Note:- Variance can also be put in the form

$$V = \sum_i x_i^2 p(x_i) - \left[\sum_i x_i p(x_i) \right]^2$$

worked problem.

(3)

1) A coin is tossed twice. A random variable X represent the no of heads turning up. Find the discrete probability distribution for X . Also find its mean and variance.

$$S = \{HH, HT, TH, TT\}$$

The association of the elements of S to the random variable X are respectively 2, 1, 1, 0

ie $X = \{0, 1, 2\}$

$$\text{Now } p(HH) = \frac{1}{4}, p(HT) = \frac{1}{4}, p(TH) = \frac{1}{4}$$

$$p(TT) = \frac{1}{4}$$

$$p(X=0, \text{ ie no head}) = p(TT) = \frac{1}{4}$$

$$p(X=1, \text{ ie one head}) = p(HT \cup TH) = p(HT) + p(TH) \\ = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$p(X=2, \text{ ie 2 heads}) = p(HH) = \frac{1}{4}$$

The discrete probability distribution for X is as follows.

$X = x_i$	0	1	2
$p(x_i)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

we observe that $p(x_i) > 0$ and $\sum p(x_i) = 1$

Now Mean $= \mu = \sum_i x_i \cdot p(x_i)$

$$\mu = x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + x_3 \cdot p(x_3)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4}$$

$$= 0 + \frac{1}{2} + \frac{1}{2} = 1$$

$$\boxed{\mu = 1}$$

Variance $V = \sum_i (x_i - \mu)^2 \cdot p(x_i)$

$$V = (x_1 - \mu)^2 \cdot p(x_1) + (x_2 - \mu)^2 \cdot p(x_2) + (x_3 - \mu)^2 \cdot p(x_3)$$

$$= (0 - 1)^2 \cdot \frac{1}{4} + (1 - 1)^2 \cdot \frac{1}{2} + (2 - 1)^2 \cdot \frac{1}{4}$$

$$= \frac{1}{4} + 0 + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

Thus Mean = 1 and Variance = 1/2

2) S.T the foll'g distribution represents a discrete probability distribution. Find the mean and variance.

x	10	20	30	40
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(4)

>> we observe that $p(x) > 0$ for all x
and $\sum p(x) = 1$

$$\text{Mean } (\mu) = \sum_i x_i \cdot p(x_i)$$

$$\mu = x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + x_3 \cdot p(x_3) + x_4 \cdot p(x_4)$$

$$= 10 \cdot \frac{1}{8} + 20 \cdot \frac{3}{8} + 30 \cdot \frac{3}{8} + 40 \cdot \frac{1}{8}$$

$$= \frac{10}{8} + \frac{60}{8} + \frac{90}{8} + \frac{40}{8}$$

$$= \frac{10 + 60 + 90 + 40}{8} = \frac{200}{8} = \underline{\underline{25}}$$

$$\text{Variance } (V) = \sum_i (x_i - \mu)^2 \cdot p(x_i)$$

$$V = (x_1 - \mu)^2 \cdot p(x_1) + (x_2 - \mu)^2 \cdot p(x_2) + (x_3 - \mu)^2 \cdot p(x_3) + (x_4 - \mu)^2 \cdot p(x_4)$$

$$= (10 - 25)^2 \cdot \frac{1}{8} + (20 - 25)^2 \cdot \frac{3}{8} + (30 - 25)^2 \cdot \frac{3}{8} + (40 - 25)^2 \cdot \frac{1}{8}$$

$$= (-15)^2 \cdot \frac{1}{8} + (-5)^2 \cdot \frac{3}{8} + (5)^2 \cdot \frac{3}{8} + (15)^2 \cdot \frac{1}{8}$$

$$= \frac{225}{8} + \frac{25 \times 3}{8} + \frac{25 \times 3}{8} + \frac{225}{8}$$

$$= \frac{225}{8} + \frac{75}{8} + \frac{75}{8} + \frac{225}{8}$$

$$= \frac{600}{8} = \underline{\underline{75}}$$

Thus Mean = 25 Variance = 75

Standard deviation $\sigma = \sqrt{V} = \sqrt{75} = \underline{\underline{8.6603}}$

3) Find the value of k such that the following distribution represents a finite probability distribution. Hence find the mean and standard deviation. Also find $P(X \leq 1)$, $P(X > 1)$ and $P(-1 < X \leq 2)$

X	-3	-2	-1	0	1	2	3
$P(X)$	k	$2k$	$3k$	$4k$	$3k$	$2k$	k

$\Rightarrow$ we must have $P(X) \geq 0$ for all x and $\sum P(X) = 1$
 The first condition is satisfied if
 if $k \geq 0$ and the 2nd condition requires that,

$$k + 2k + 3k + 4k + 3k + 2k + k = 1$$

$$16k = 1 \Rightarrow \boxed{k = \frac{1}{16}}$$

The discrete finite probability distribution is as follows.

x	x_1 -3	x_2 -2	x_3 -1	x_4 0	x_5 1	x_6 2	x_7 3
$P(x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

$$\text{Mean } (\mu) = \sum_i x_i \cdot P(x_i)$$

$$\mu = x_1 \cdot P(x_1) + x_2 \cdot P(x_2) + \dots + x_7 \cdot P(x_7)$$

$$= -3 \cdot \frac{1}{16} + (-2) \cdot \frac{2}{16} + (-1) \cdot \frac{3}{16} + 0 \cdot \frac{4}{16} + 1 \cdot \frac{3}{16} + 2 \cdot \frac{2}{16} + 3 \cdot \frac{1}{16}$$

$$= -\frac{3}{16} - \frac{4}{16} - \frac{3}{16} + \frac{3}{16} + \frac{4}{16} + \frac{3}{16}$$

$$= \frac{-3 - 4 - 3 + 3 + 4 + 3}{16} = \underline{\underline{0}}$$

(5)

$$\text{Variance (V)} = \sum_i (x_i - m)^2 \cdot p(x_i)$$

$$V = (x_1 - m)^2 \cdot p(x_1) + (x_2 - m)^2 \cdot p(x_2) + \dots + (x_7 - m)^2 \cdot p(x_7)$$

$$= (-3-0)^2 \cdot \frac{1}{16} + (-2-0)^2 \cdot \frac{2}{16} + (-1-0)^2 \cdot \frac{3}{16} + (0-0)^2 \cdot \frac{4}{16}$$

$$+ (1-0)^2 \cdot \frac{3}{16} + (2-0)^2 \cdot \frac{2}{16} + (3-0)^2 \cdot \frac{1}{16}$$

$$= \frac{9}{16} + \frac{8}{16} + \frac{3}{16} + \frac{3}{16} + \frac{8}{16} + \frac{9}{16}$$

$$= \frac{9+8+3+3+8+9}{16} = \frac{40}{16} = \underline{\underline{\frac{5}{2}}}$$

Thus $K = \frac{1}{16}$, Mean = 0, S.D = $\sqrt{5/2}$

Also $p(x \leq 1) = p(-3) + p(-2) + p(-1) + p(0) + p(1)$

$$= \frac{1}{16} + \frac{2}{16} + \frac{3}{16} + \frac{4}{16} + \frac{3}{16}$$

$$= \underline{\underline{\frac{13}{16}}}$$

$$p(x > 1) = p(2) + p(3)$$

$$= \frac{2}{16} + \frac{1}{16}$$

$$= \underline{\underline{\frac{3}{16}}}$$

$$p(-1 < x \leq 2) = p(0) + p(1) + p(2)$$

$$= \frac{4}{16} + \frac{3}{16} + \frac{2}{16}$$

$$= \underline{\underline{\frac{9}{16}}}$$

4) The p.d.f of a variable x is given by the following table.

x	0	1	2	3	4	5	6
$p(x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

For what value of k , this represents a valid probability distribution?

Also find $P(x \geq 5)$ and $P(3 < x \leq 6)$

>> The probability distribution is valid if $p(x) \geq 0$ and $\sum p(x) = 1$

Hence we must have $k \geq 0$ and

$$k + 3k + 5k + 7k + 9k + 11k + 13k = 1$$

$$49k = 1 \Rightarrow k = \frac{1}{49}$$

The discrete probability distribution is as follows

x	0	1	2	3	4	5	6
$p(x)$	$\frac{1}{49}$	$\frac{3}{49}$	$\frac{5}{49}$	$\frac{7}{49}$	$\frac{9}{49}$	$\frac{11}{49}$	$\frac{13}{49}$

$$\text{Also, } P(x \geq 5) = P(5) + P(6) = \frac{11}{49} + \frac{13}{49} = \frac{24}{49}$$

$$P(3 < x \leq 6) = P(4) + P(5) + P(6)$$

$$= \frac{9}{49} + \frac{11}{49} + \frac{13}{49} = \frac{33}{49}$$

$\frac{11}{24}$

(6)

5) The probability distribution of a finite random variable X is given by the following table

x_i	-2	-1	0	1	2	3
$p(x_i)$	0.1	K	0.2	$2K$	0.3	K

Find the value of K , mean and variance.

>> we must have $p(x_i) \geq 0$ and $\sum p(x_i) = 1$ for a probability distribution.

$$\sum p(x_i) = 1 \text{ required } 4K + 0.6 = 1$$

$$4K = 1 - 0.6$$

$$4K = 0.4$$

$$K = \frac{0.4}{4} = \underline{\underline{0.1}}$$

Prob distribution

x_i	-2	-1	0	1	2	3
$p(x_i)$	0.1	0.1	0.2	0.2	0.3	0.1

$$\text{Mean } (M) = \sum_i x_i \cdot p(x_i)$$

$$M = x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + x_3 \cdot p(x_3) + x_4 \cdot p(x_4) + x_5 \cdot p(x_5) + x_6 \cdot p(x_6)$$

$$= -2 \cdot (0.1) + (-1) \cdot (0.1) + 0 \cdot (0.2) + 1 \cdot (0.2) + 2 \cdot (0.3) + 3 \cdot (0.1)$$

$$= \underline{\underline{0.8}}$$

$$\text{Variance } (V) = \sum_i (x_i - M)^2 \cdot p(x_i)$$

$$V = (-2 - 0.8)^2 \cdot (0.1) + (-1 - 0.8)^2 \cdot (0.1) + (0 - 0.8)^2 \cdot 0.2$$

$$+ (1 - 0.8)^2 \cdot 0.2 + (2 - 0.8)^2 \cdot 0.3 + (3 - 0.8)^2 \cdot 0.1$$

$$= 2.16$$

Thus Mean = 0.8 and Variance = 2.16

6) A random variable X has the following Probability function for various values of x

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

i) Find k ii) Evaluate $P(X < 6)$, $P(X \geq 6)$, $P(3 < X \leq 6)$
Also find the probability distribution function of X .

>> we must have $P(x) \geq 0$ and $\sum P(x) = 1$
The first condition is satisfied for $k \geq 0$ and we have to find k such that $\sum P(x) = 1$

$$\text{i.e. } 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + (7k^2 + k) = 1$$

$$\text{i.e. } 10k^2 + 9k - 1 = 0$$

$$(\text{or}) (10k - 1)(k + 1) = 0$$

$$(\text{or}) k = \frac{1}{10} \text{ and } k = -1$$

if $k = -1$ the first condition fails and hence

$$k \neq -1 \therefore k = \frac{1}{10}$$

Hence we have the following table

x	0	1	2	3	4	5	6	7
$P(x)$	0	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{1}{50}$	$\frac{17}{100}$

$$\begin{aligned} \text{Now } P(X < 6) &= P(0) + P(1) + P(2) + P(3) + P(4) + P(5) \\ &= 0 + \frac{1}{10} + \frac{1}{5} + \frac{1}{5} + \frac{3}{10} + \frac{1}{100} + \frac{1}{50} \end{aligned}$$

$$= \underline{\underline{0.81}}$$

7

$$P(X \geq 6) = P(6) + P(7)$$

$$= \frac{1}{50} + \frac{17}{100} = \frac{19}{100} = 0.19$$

$$P(3 < X \leq 6) = P(4) + P(5) + P(6)$$

$$= \frac{3}{10} + \frac{1}{100} + \frac{1}{50}$$

$$= \frac{33}{100} = \underline{\underline{0.33}}$$

The probability distribution is as follows.

x	0	1	2	3	4	5	6	7
$P(x)$	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

The distribution function of X is

$f(x) = P(X \leq x) = \sum_{i=1}^x P(x_i)$ it also called cumulative distribution funⁿ and the same is as follows:

x	0	1	2	3	4	5
$f(x)$	0	$0 + 0.1 = 0.1$	$0.1 + 0.2 = 0.3$	$0.3 + 0.2 = 0.5$	$0.5 + 0.3 = 0.8$	$0.8 + 0.01 = 0.81$

6	7
$0.81 + 0.02 = 0.83$	$0.83 + 0.17 = 1$

7) A random variable X take the values $-3, -2, -1, 0, 1, 2, 3$ such that $p(X=0) = p(X < 0)$ and $p(X=-3) = p(X=-2) = p(X=-1) = p(X=1) = p(X=2) = p(X=3)$. Find the probability distribution.

>> Let the distribution $[X, p(x)]$ be as follows.

X	-3	-2	-1	0	1	2	3
$p(x)$	p_1	p_2	p_3	p_4	p_5	p_6	p_7

By data, $p(X=0) = p(X < 0)$

$$\Rightarrow p(X=0) = p(X=-1) + p(X=-2) + p(X=-3)$$

$$\text{ie } p_4 = p_3 + p_2 + p_1 \quad \text{--- (1)}$$

Also by data

$$p_1 = p_2 = p_3 = p_5 = p_6 = p_7 \quad \text{--- (2)}$$

Further we must have,

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 = 1 \quad \text{--- (3)}$$

using (2) in (1) we get $3p_1 = p_4$

$$\text{using (2) in (3)} \quad 6p_1 + p_4 = 1$$

$$\Rightarrow 6p_1 + 3p_1 = 1 \quad \therefore p_4 = 3p_1$$

$$9p_1 = 1 \Rightarrow \boxed{p_1 = 1/9} \quad \text{and} \quad 3p_1 = p_4 \Rightarrow p_4 = \frac{3}{9} = \frac{1}{3}$$

$$\boxed{p_4 = 1/3}$$

Thus the discrete / finite probability distribution is

X	-3	-2	-1	0	1	2	3
$p(x)$	$1/9$	$1/9$	$1/9$	$1/3$	$1/9$	$1/9$	$1/9$

8) From a sealed box containing a dozen (12) apples it was found that 3 apples are perished. obtain the probability distribution of the no of perished apples when 2 apples are drawn at random. Also find the mean and variance? (8)

>> Let X be the number of perished apples, since 2 were drawn we have $X = 0, 1, 2$, then 2 out of 12 can be selected in ${}^{12}C_2$ ways, remaining 9 apples are good apples and 3 are perished apples. Hence we have

$$P(X=0) = \text{probability of getting 0 perished apples} = \frac{{}^3C_0 \cdot {}^9C_2}{{}^{12}C_2} = \frac{6}{11}$$

$$P(X=1) = \text{probability of getting 1 perished apple} = \frac{{}^3C_1 \cdot {}^9C_1}{{}^{12}C_2} = \frac{9}{22}$$

$$P(X=2) = \text{probability of getting 2 perished apples} = \frac{{}^3C_2 \cdot {}^9C_0}{{}^{12}C_2} = \frac{1}{22}$$

Thus the discrete/finite probability distribution is

$X = x_i$	0	1	2
$P(X) = p_i$	$\frac{6}{11}$	$\frac{9}{22}$	$\frac{1}{22}$

$$\text{Mean } (\mu) = \sum_i x_i \cdot p(x_i)$$

$$\mu = x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + x_3 \cdot p(x_3)$$

$$= 0 \cdot \frac{6}{11} + 1 \cdot \frac{9}{22} + 2 \cdot \frac{1}{22}$$

$$= \frac{9}{22} + \frac{2}{22} = \frac{11}{22} = \frac{1}{2} = \underline{\underline{0.5}}$$

$$\text{Variance } (V) = \sum_i (x_i - \mu)^2 \cdot p(x_i)$$

$$V = (0 - \frac{1}{2})^2 \cdot \frac{6}{11} + (1 - \frac{1}{2})^2 \cdot \frac{9}{22} + (2 - \frac{1}{2})^2 \cdot \frac{1}{22}$$

$$= (-\frac{1}{2})^2 \cdot \frac{6}{11} + (\frac{1}{2})^2 \cdot \frac{9}{22} + (\frac{3}{2})^2 \cdot \frac{1}{22}$$

$$= \frac{1}{4} \cdot \frac{6}{11} + \frac{1}{4} \cdot \frac{9}{22} + \frac{9}{4} \cdot \frac{1}{22}$$

$$= \frac{6}{44} + \frac{9}{88} + \frac{9}{88}$$

$$= \frac{12 + 9 + 9}{88} = \frac{30}{88} = \frac{15}{44}$$

$$\text{Thus S.D} = (\sigma) = \sqrt{V} = \sqrt{15/44}$$

$$\text{Thus Mean} = \frac{1}{2}$$

$$\text{Variance} = \frac{15}{44}$$

$$\text{S.D} = \sqrt{15/44}$$

Q) If the random variable X take the values 1, 2, 3, 4 such that $2p(X=1) = 3p(X=2) = p(X=3) = 5p(X=4)$, find the probability distribution funⁿ of X .

>> Let the distribution $[X, p(X)]$ be as follows.

X	1	2	3	4
$p(X)$	p_1	p_2	p_3	p_4

By data

$$2p_1 = 3p_2 = p_3 = 5p_4 \quad \text{--- (1)}$$

also we have

$$p_1 + p_2 + p_3 + p_4 = 1 \quad \text{--- (2)}$$

from (1) $2p_1 = 3p_2$, $2p_1 = p_3$, $2p_1 = 5p_4$

$$\Rightarrow \boxed{p_2 = \frac{2}{3}p_1}, \quad \boxed{p_3 = 2p_1}, \quad \boxed{p_4 = \frac{2}{5}p_1}$$

Hence (2) $\Rightarrow$

$$p_1 + \frac{2}{3}p_1 + 2p_1 + \frac{2}{5}p_1 = 1$$

$$3p_1 + \frac{2}{3}p_1 + \frac{2}{5}p_1 = 1$$

$$\frac{(3 \times 15)p_1 + 2p_1 \times 5 + (2 \times 3)p_1}{15} = 1$$

$$\frac{45p_1 + 10p_1 + 6p_1}{15} = 1$$

$$\frac{61p_1}{15} = 1$$

$$\Rightarrow \boxed{p_1 = \frac{15}{61}}$$

$$p_2 = \frac{2}{3} \cdot p_1$$

$$p_2 = \frac{2}{3} \cdot \frac{15}{61} = \frac{10}{61}$$

$$\boxed{p_2 = \frac{10}{61}}$$

$$p_3 = 2 \cdot p_1 = 2 \cdot \frac{15}{61} = \frac{30}{61}$$

$$p_4 = \frac{2}{5} p_1 = \frac{2}{5} \cdot \frac{15}{61} = \frac{6}{61}$$

$$\boxed{p_3 = \frac{30}{61}}$$

$$\boxed{p_4 = \frac{6}{61}}$$

The probability distribution $p(x)$ and the cumulative distribution $f(x) = \sum_{i=1}^x p(x_i)$ is as follows,

$X=x_i$	1	2	3	4
$p(x)$	$15/61$	$10/61$	$30/61$	$6/61$
$f(x)$	$15/61$	$25/61$	$55/61$	$61/61 = 1$

10) A random variable X has $p(x) = 2^{-x}$ where $x = 1, 2, 3, \dots$. Show that $p(x)$ is a probability function. Also find $p(X \text{ even})$, $p(X \text{ being divisible by } 3)$ and $p(X > 5)$

>> $p(x) = 2^{-x} = \frac{1}{2^x}$ Evidently $p(x) > 0$ for all x .

$$\sum_i p(x_i) = \sum_{x=1}^{\infty} \frac{1}{2^x} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$$

RHS is a geometric series of the form $a + ar + ar^2 + \dots$ whose sum to infinity is $\frac{a}{1-r}$ where we have $a = \frac{1}{2} = r$

$$\begin{aligned} \therefore \sum_i p(x_i) &= \frac{a}{1-r} \\ &= \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1 \end{aligned}$$

Hence $p(x) = 2^{-x}$ is a probability function

Case i)

$$p(x \text{ even}) = \sum_{x=2,4,6,\dots}^{\infty} p(x)$$

$$= \sum_{2,4,6}^{\infty} \frac{1}{2^x}$$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \dots$$

$$= \frac{\frac{1}{2^2}}{1 - \frac{1}{2^2}} = \frac{\frac{1}{4}}{\frac{4-1}{4}} = \frac{1/4}{3/4} = \frac{1}{3}$$

Thus $p(x \text{ even}) = \frac{1}{3}$

Case ii)

$$p(x \text{ divisible by } 3) = \sum_{x=3,6,9,\dots}^{\infty} p(x)$$

$$= \sum_{3,6,9,\dots}^{\infty} \frac{1}{2^x}$$

$$= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \dots$$

$$= \frac{\frac{1}{2^3}}{1 - \frac{1}{2^3}} = \frac{\frac{1}{8}}{\frac{8-1}{8}} = \frac{1/8}{7/8} = \frac{1}{7}$$

Thus $p(x \text{ divisible by } 3) = \frac{1}{7}$

Case iii)

$$p(x \geq 5) = 1 - p(x < 5)$$

$$= 1 - \sum_{i=1}^4 p(x_i)$$

$$= 1 - \sum_{i=1}^4 \frac{1}{2^i}$$

$$= 1 - \left[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} \right]$$

$$= 1 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \right]$$

$$= 1 - \frac{15}{16} = \frac{1}{16}$$

Thus $p(x \geq 5) = \frac{1}{16}$

11) if x is a discrete random variable taking values $1, 2, 3, \dots$ with $p(x) = \frac{1}{2} \left(\frac{2}{3}\right)^x$, find $p(x \text{ being an odd number})$ by first establishing that $p(x)$ is a probability function.

$$\gg p(x) = \frac{1}{2} \left(\frac{2}{3}\right)^x$$

$$\sum_i p(x_i) = \sum_{x=1}^{\infty} \frac{1}{2} \left(\frac{2}{3}\right)^x$$

$$= \frac{1}{2} \left[\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots \right]$$

$$= \frac{1}{2} \cdot \frac{\frac{2}{3}}{1 - \frac{2}{3}} = \frac{1}{2} \cdot \frac{\frac{2}{3}}{\frac{1}{3}} = 1$$

$\therefore p(x)$ is a probability function.

Next $p(x \text{ being an odd no.}) = \sum_{x=1,3,5,\dots}^{\infty} p(x)$

$$= \sum_{1,3,5,\dots}^{\infty} \frac{1}{2} \cdot \left(\frac{2}{3}\right)^x$$

$$= \frac{1}{2} \left[\frac{2}{3} + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^5 + \dots \right]$$

$$= \frac{1}{2} \left[a + ar + ar^2 + \dots \right] \quad a = \frac{2}{3}, \quad r = \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{2} \cdot \frac{a}{1-r} = \frac{1}{2} \cdot \frac{\frac{2}{3}}{1 - \frac{4}{9}} = \frac{1}{2} \cdot \frac{\frac{2}{3}}{\frac{5}{9}}$$

$$= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{9}{5} = \frac{3}{5}$$

Thus $p(x \text{ being an odd no.}) = \frac{3}{5}$

12) X is a discrete random variable having (11)
 $p(x)$ defined as follows.

$$p(x) = \begin{cases} x/15 & \text{if } 1 \leq x \leq 5 \\ 0 & \text{if } x > 5 \end{cases} \quad \text{show that } p(x) \text{ is a probability function.}$$

Find i) $P(X=1 \text{ or } 2)$

ii) $P(1/2 < X < 5/2) / (X > 1)$

>> The probability distribution is as follows

x	1	2	3	4	5	6, 7, ...
$p(x)$	$1/15$	$2/15$	$3/15$	$4/15$	$5/15$	0

we've $p(x) \geq 0$ and $\sum_i p(x_i) = 1$

$$\therefore \frac{1}{15} + \frac{2}{15} + \frac{3}{15} + \frac{4}{15} + \frac{5}{15} = 1$$

$$\frac{1+2+3+4+5}{15} = \frac{15}{15} = 1$$

$\therefore p(x)$ is a probability function.

$$\begin{aligned} \text{Now } P(X=1 \text{ or } 2) &= P(X=1) + P(X=2) \\ &= \frac{1}{15} + \frac{2}{15} = \frac{3}{15} = \frac{1}{5} \end{aligned}$$

$$\text{Thus } \boxed{P(X=1 \text{ or } 2) = \frac{1}{5}}$$

Bernoulli's Theorem

(12)

Stmt:- The probability of x successes in n trials is equal to $n C_x p^x q^{n-x}$ where p is the probability of success and q is the probability of failure.

Proof:- Since p is the probability of success, the probability of x successes is
 $p, p, p, \dots, x \text{ times} = p^x$

Also x successes imply $(n-x)$ failures and since q is the probability of failure, the probability of $(n-x)$ failures is q^{n-x} .

By the multiplication rule the probability of the simultaneous happening is $p^x q^{n-x}$. But x successes in n trials can occur in $n C_x$ ways and all these cases are favourable to the event.

Hence by the addition rule, the probability of x successes out of n trials is given by

$$p^x q^{n-x} + p^x q^{n-x} + \dots \dots \dots n C_x \text{ times} = n C_x p^x q^{n-x}$$

This proves Bernoulli's theorem.

Binomial Distribution.

If p is the probability of success and q is the probability of failure, the probability of x successes out of n trials is given by

$$p(x) = {}^nC_x p^x q^{n-x}$$

we form the following probability distribution of $[x, p(x)]$ where $x = 0, 1, 2, \dots, n$

x	0	1	2	...	n
$p(x)$	q^n	${}^nC_1 q^{n-1} p$	${}^nC_2 q^{n-2} p^2$	...	p^n

It may be observed that the value of $p(x)$ for different values $x = 0, 1, 2, \dots, n$ are the successive terms in the binomial expansion of $(q+p)^n$ and accordingly this distribution is called the Binomial Distribution (or)

Bernoulli Distribution.

$$\sum p(x) = q^n + {}^nC_1 q^{n-1} p + {}^nC_2 q^{n-2} p^2 + \dots + p^n = (p+q)^n = 1$$

Hence $p(x)$ is a probability function.

Mean and Standard Deviation of the Binomial Distribution.

(13)

$$\text{Mean } (\mu) = \sum_{x=0}^n x p(x)$$

$$\therefore nC_x = \frac{n!}{(n-x)! x!}$$

$$x! = x(x-1)!$$

$$\mu = \sum_{x=0}^n x \cdot nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^n x \cdot \frac{n!}{x! (n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^n x \cdot \frac{n(n-1)!}{x(x-1)! (n-x)!} p \cdot p^{x-1} \cdot q^{n-x}$$

$$= np \sum_{x=0}^n \frac{(n-1)!}{(x-1)! (n-x)!} p^{x-1} q^{n-x}$$

$$= np \sum_{x=1}^n \frac{(n-1)!}{(x-1)! [(n-1)-(x-1)]!} p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^n \frac{(n-1)!}{(x-1)!} p^{x-1} q^{(n-1)-(x-1)}$$

$$= np (2+p)^{n-1}$$

$$= np (1)^n$$

$$= np$$

Thus $\text{Mean } (\mu) = np$

$$\text{Variance (V)} = \sum_{x=0}^n x^2 p(x) - m^2 \quad \text{--- (1)}$$

$$\text{Now } \sum_{x=0}^n x^2 p(x) = \sum_{x=0}^n [x(x+1) + x] p(x)$$

$$= \sum_{x=0}^n x(x-1) \cdot p(x) + \sum_{x=0}^n x p(x)$$

$$= \sum_{x=0}^n x(x-1) \cdot n C_x p^x q^{n-x} + np$$

$$= \sum_{x=0}^n x(x-1) \cdot \frac{n!}{x!(n-x)!} p^x q^{n-x} + np$$

$$= \sum_{x=0}^n x(x-1) \cdot \frac{n(n-1)(n-2)!}{x(x-1)(n-2)!(n-x)!} p^2 p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=0}^n \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^n \frac{(n-2)!}{(x-2)![(n-2)-(x-2)]!} p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 \sum_{x=2}^n (n-2) C_{(x-2)} p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 (q+p)^{n-2} + np$$

$$= n(n-1)p^2 + np$$

$$\text{(1)} \Rightarrow V = n(n-1)p^2 + np - n^2 p^2$$

$$= n^2 p^2 - np^2 + np - n^2 p^2$$

$$= np - np^2 = np(1-p)$$

$$= np(q)$$

$$\boxed{V = npq}$$

$$\boxed{S.D(\sigma) = \sqrt{V} = \sqrt{npq}}$$

$\therefore p + q = 1$
 $p = 1 - q$
 $q = 1 - p$
 Thus we've the Binomial distribution
 Mean (M) = np and
 S.D (σ) = $\sqrt{npq}$

Poisson Distribution.

(14)

Poisson distribution is regarded as the limiting form of the binomial distribution when n is very large ($n \rightarrow \infty$) and p the probability of success is very small ($p \rightarrow 0$) so that np tends to a fixed finite constant say m .

we have in the case of binomial distribution, the Probability of x successes out of n trials.

$$\begin{aligned} P(x) &= {}^nC_x p^x q^{n-x} \\ &= \frac{n(n-1)(n-2) \dots (n-(x-1))}{x!} p^x q^{n-x} \\ &= \frac{n \cdot n(1-\frac{1}{n}) \cdot n(1-\frac{2}{n}) \dots n(1-\frac{x-1}{n})}{x!} p^x q^{n-x} \\ &= \frac{n^x (1-\frac{1}{n})(1-\frac{2}{n}) \dots (1-\frac{x-1}{n})}{x!} p^x q^{n-x} \\ &= \frac{(np)^x (1-\frac{1}{n})(1-\frac{2}{n}) \dots (1-\frac{x-1}{n}) q^n}{x! 2^x} \end{aligned}$$

But $(np)^x = m$
 $q^n = (1-p)^n = (1-\frac{m}{n})^n = \left\{ \left(1-\frac{m}{n}\right)^{\frac{n}{m}} \right\}^{-m}$

Denoting $\frac{m}{n} = p$ we have

$$Q^n = \{(1+k)^{1/k}\}^{-m} \rightarrow e^{-m} \text{ as } n \rightarrow \infty \text{ (or) } k \rightarrow 0$$

$$[\text{Note: } \lim_{k \rightarrow 0} (1+k)^{1/k} = e]$$

Further $Q^n = (1-p)^n \rightarrow 1$ for a fixed n as $p \rightarrow 0$

Also the factors $(1 - \frac{1}{n}) \cdot (1 - \frac{2}{n}) \cdots (1 - \frac{n-1}{n})$ will all tend to 1 as $n \rightarrow \infty$

Thus we get

$$p(x) = \frac{m^x e^{-m}}{x!}$$

This is known as the Poisson distribution of the random variable.

$p(x)$ is also called Poisson probability function and x is called a Poisson variate.

The distribution of Probabilities for

$x = 0, 1, 2, 3, \dots$ is as follows.

x	0	1	2	3	...
$p(x)$	e^{-m}	$\frac{m e^{-m}}{1!}$	$\frac{m^2 e^{-m}}{2!}$	$\frac{m^3 e^{-m}}{3!}$	...

we have $p(x) \geq 0$ and

$$\sum_{x=0}^{\infty} p(x) = e^{-m} + \frac{m e^{-m}}{1!} + \frac{m^2 e^{-m}}{2!} + \frac{m^3 e^{-m}}{3!} + \dots$$

$$= e^{-m} \left\{ 1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots \right\}$$

$$= e^{-m} \cdot e^m = e^{-m+m} = e^0 = e^0 = 1$$

$$\therefore \sum p(x) = 1$$

Hence $p(x)$ is a probability function

Mean and Standard Deviation of the Poisson Distribution

(15)

$$\text{Mean (M)} = \sum_{x=0}^{\infty} x \cdot p(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{m^x e^{-m}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{x! \cdot m^x e^{-m}}{x(x-1)!}$$

$$= m e^{-m} \sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

put $x=1, 2, 3, \dots$
expanding summation

$$= m e^{-m} \left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots \right]$$

$$= m e^{-m} \cdot e^m = m e^0 = m$$

$$\therefore \boxed{\text{Mean (M)} = m}$$

$$\text{Variance (V)} = \sum_{x=0}^{\infty} x^2 p(x) - m^2 \quad \text{--- (1)}$$

$$\text{Now } \sum x^2 p(x) = \sum [x(x-1) + x] \cdot p(x)$$

$$= \sum [x(x-1) + x] \cdot \frac{m^x e^{-m}}{x!}$$

$$= \sum x(x-1) \cdot \frac{m^x e^{-m}}{x!} + \sum x \cdot \frac{m^x e^{-m}}{x!}$$

$$= \sum x(x-1) \cdot \frac{m^x e^{-m}}{x(x-1)(x-2)!} + \sum x \cdot \frac{m^x e^{-m}}{x(x-1)!}$$

$$= \sum_0^{\infty} \frac{m^x e^{-m}}{(x-2)!} + \sum_0^{\infty} \frac{m^x e^{-m}}{(x-1)!}$$

$$= \sum_2^{\infty} \frac{m^2 m^{x-2} e^{-m}}{(x-2)!} + \sum_1^{\infty} \frac{m m^{x-1} e^{-m}}{(x-1)!}$$

$$= m^2 e^{-m} \sum_2^{\infty} \frac{m^{x-2}}{(x-2)!} + m$$

$$\sum x^2 p(x) = m^2 e^{-m} \sum_{x=2}^{\infty} \frac{m^{x-2}}{(x-2)!} + m$$

$$= m^2 e^{-m} \left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \dots \right] + m$$

$$= m^2 e^{-m} \cdot e^m + m$$

$$= m^2 + m$$

(1) $\Rightarrow$

$$\text{Variance} = m^2 + m - (m)^2$$

$$\boxed{V = m}$$

$$\therefore \text{S.D}(\sigma) = \sqrt{V} = \sqrt{m}$$

Thus we have for the Poisson distribution

$$\boxed{\text{Mean}(\mu) = m}$$

and

$$\boxed{\text{S.D}(\sigma) = \sqrt{m}}$$

further we can say that the mean and variance are equal for the Poisson distribution.

Problems

(16)

1) Find the binomial probability distribution which has mean 2 and variance $4/3$.

>> wkt the binomial distribution

$$\text{mean} = np \quad \text{and} \quad \text{Variance} = npq$$

$$\text{Hence } np = 2 \quad \text{and} \quad npq = \frac{4}{3}$$

$$\Rightarrow npq = \frac{4}{3} \Rightarrow 2q = \frac{4}{3} \Rightarrow q = \frac{4}{3 \cdot 2} = \frac{2}{3}$$

$$\boxed{q = \frac{2}{3}}$$

$$\text{wkt } p + q = 1$$

$$p = 1 - q$$

$$p = 1 - \frac{2}{3} = \frac{3-2}{3} = \frac{1}{3}$$

$$\boxed{p = \frac{1}{3}}$$

$$\text{and } np = 2$$

$$n\left(\frac{1}{3}\right) = 2$$

$$\boxed{n = 6}$$

The binomial probability function $p(x) = {}^nC_x p^x q^{n-x}$ becomes

$$p(x) = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$$

The distribution of Probabilities is as follows.

B.D/P.D $x = 0, 1, 2, \dots, \infty$ ($x \rightarrow$ starts with 0. --- to end with ∞)				
x	0	1	2	
$p(x)$	${}^6C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^6$	${}^6C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^5$	${}^6C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4$	
	3	4	5	6
	${}^6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3$	${}^6C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2$	${}^6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)$	${}^6C_6 \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^0$

2) when a coin is tossed 4 times, find the probability of getting

i) exactly one head

ii) at most 3 heads

iii) at least 2 heads

$$\gg p = p(H) = \frac{1}{2} = 0.5, \quad q = 0.5, \quad n = 4$$

Let $x \rightarrow$ be the prob. of heads
 wkt the probability of x succeeded out of n trials
 is given by $p(x) = {}^nC_x p^x q^{n-x}$

i) prob (exactly one head) is $p(x=1)$

$$\text{ie } p(\text{1 head}) = p(x=1)$$

$$= {}^4C_1 (0.5)^1 (0.5)^3$$

$$= 4 (0.5)^4 = \underline{\underline{0.25}}$$

At most (max) $\rightarrow \leq$
 At least (min) $\rightarrow \geq$
 more than $\rightarrow >$
 less than $\rightarrow <$
 2 or more $\rightarrow x \geq 2$
 1 or more $\rightarrow x \geq 1$

ii) p (at most 3 heads)

$$\text{ie } p(x \leq 3) = p(x=0) + p(x=1) + p(x=2) + p(x=3)$$

$$p(x \leq 3) = {}^4C_0 (0.5)^0 (0.5)^4 + {}^4C_1 (0.5)^1 (0.5)^3$$

$$+ {}^4C_2 (0.5)^2 (0.5)^2 + {}^4C_3 (0.5)^3 (0.5)^1$$

$$= {}^4C_0 (0.5)^4 + {}^4C_1 (0.5)^4 + {}^4C_2 (0.5)^4 + {}^4C_3 (0.5)^4$$

$$= (0.5)^4 [{}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3]$$

$$= (0.5)^4 [1 + 4 + 6 + 4]$$

$$= (0.5)^4 [15]$$

$$= \underline{\underline{0.9375}}$$

(17)

iii) $P(\text{at least 2 heads})$

$$\begin{aligned}
 P(X \geq 2) &= P(2) + P(3) + P(4) \\
 &= P(X=2) + P(X=3) + P(X=4) \\
 &= {}^4C_2 (0.5)^2 (0.5)^2 + {}^4C_3 (0.5)^3 (0.5)^1 + {}^4C_4 (0.5)^4 (0.5)^0 \\
 &= {}^4C_2 (0.5)^4 + {}^4C_3 (0.5)^4 + {}^4C_4 (0.5)^4 (1) \\
 &= (0.5)^4 [{}^4C_2 + {}^4C_3 + {}^4C_4] \\
 &= (0.5)^4 [6 + 4 + 1] \\
 &= (0.5)^4 [11] \\
 &= \underline{\underline{0.6875}}
 \end{aligned}$$

(OR)

$$\begin{aligned}
 P(X \geq 2) &= 1 - P(X < 2) \\
 &= 1 - [P(X=0) + P(X=1)] \\
 &= 1 - [{}^4C_0 (0.5)^0 (0.5)^4 + {}^4C_1 (0.5)^1 (0.5)^3] \\
 &= 1 - [{}^4C_0 (1) \cdot (0.5)^4 + {}^4C_1 (0.5)^4] \\
 &= 1 - (0.5)^4 [{}^4C_0 + {}^4C_1] \\
 &= 1 - (0.5)^4 [1 + 4] \\
 &= 1 - (0.5)^4 (5) \\
 &= 1 - 0.3125 \\
 &= \underline{\underline{0.6875}}
 \end{aligned}$$

Q3) The probability that a pen manufactured by a factory be defective is $\frac{1}{10}$. If 12 such pens are manufactured, what is the probability that

- i) exactly 2 are defective
- ii) at least 2 are defective
- iii) none of them are defective.

>> Let $x \rightarrow$ be defective

Probability of a defective pen is $p = \frac{1}{10}$

Probability of a non-defective pen $= q = 1 - p$
 $= 1 - \frac{1}{10} = \frac{10-1}{10} = \frac{9}{10} = 0.9$

we have $p(x) = {}^nC_x p^x q^{n-x}$ where $n = 12$

$$\boxed{p = 0.1} \quad \boxed{q = 0.9} \quad \boxed{n = 12}$$

i) prob. (exactly two defective)

$$p(x=2) = {}^{12}C_2 (0.1)^2 (0.9)^{10} = \underline{0.2301}$$

ii) prob (atleast 2 defective)

$$p(x \geq 2) = p(x=2) + p(x=3) + \dots + p(x=12)$$

$$\text{(or)}$$
$$= 1 - p(x < 2)$$

$$= 1 - [p(x=0) + p(x=1)]$$

$$= 1 - [{}^{12}C_0 (0.1)^0 (0.9)^{12} + {}^{12}C_1 (0.1)^1 (0.9)^{11}]$$

$$= 1 - 0.6590$$

$$= \underline{0.341}$$

iii) prob. (no of defective)

(18)

$$P(X=0) = {}^{12}C_0 (0.1)^0 (0.9)^{12} = \underline{0.2824}$$

4) In a consignment of electric lamp 5% are defective. if a random sample of 8 lamps are inspected what is the probability that one or more lamps are defective?

Let $x \rightarrow$ be the defective lamp
 $\Rightarrow$ probability of a defective lamp $p = \frac{5}{100} = 0.05$
 $\boxed{p = 0.05}$ and $q = 1 - p = 1 - 0.05 = 0.95$
 $\boxed{q = 0.95}$ $\boxed{n = 8}$

we have $P(x) = {}^nC_x p^x q^{n-x}$

prob. (1 or more lamps are defective)

$$P(X \geq 1) = P(X=1) + P(X=2) + \dots + P(X=8)$$

$$P(X \geq 1) = 1 - P(X < 1) \quad (\text{or})$$

$$= 1 - P(X=0)$$

$$= 1 - {}^8C_0 (0.05)^0 (0.95)^{8-0}$$

$$= 1 - {}^8C_0 (0.95)^8$$

$$= 1 - 1 \cdot (0.95)^8$$

$$= 1 - 0.6634$$

$$= \underline{0.3366}$$

Thus the required probability is 0.3366.

5) The probability that a person aged 60 years will live upto 70 is 0.65. what is the probability that out of 10 persons aged 60 atleast 7 of them will live upto 70.

>> let $x \rightarrow$ be the no. of persons aged 60 years living upto 70 years.

by data $p = 0.65$ $q = 1 - p = 1 - 0.65 = 0.35$

total no. of persons $n = 10$

$$\therefore \boxed{p = 0.65} \quad \boxed{q = 0.35} \quad \boxed{n = 10}$$

consider $p(x) = {}^n C_x p^x q^{n-x}$

$$\boxed{p(x) = {}^{10} C_x (0.65)^x (0.35)^{10-x}}$$

$$\text{To find } p(x \geq 7) = p(7) + p(8) + p(9) + p(10)$$

$$= p(x=7) + p(x=8) + p(x=9) + p(x=10)$$

$$= {}^{10} C_7 (0.65)^7 (0.35)^3 + {}^{10} C_8 (0.65)^8 (0.35)^2$$

$$+ {}^{10} C_9 (0.65)^9 (0.35)^1 + {}^{10} C_{10} (0.65)^{10} (0.35)^0$$

$$= \cancel{(0.65)^7} (0.35)^3 + {}^{10} C_8 (0.65)^8 (0.35)^2$$

$$= {}^{10} C_7 (0.65)^7 (0.35)^3 + {}^{10} C_8 (0.65)^8 (0.35)^2$$

$$+ {}^{10} C_9 (0.65)^9 (0.35) + {}^{10} C_{10} (0.65)^{10}$$

$$= \underline{\underline{0.5138}} \quad (\text{OR})$$

$$p(x \geq 7) = 1 - p(x < 7)$$

$$= 1 - [p(x=0) + p(x=1) + \dots + p(x=6)]$$

$$= \underline{\underline{0.5138}}$$

6) The no of telephone lines busy at an instant of time is a binomial variable with probability 0.1 that a line is busy. if 10 lines are chosen at random, what is the probability that

- i) no line is busy
- ii) all lines are busy
- iii) at least one line is busy
- iv) at most 2 lines are busy.

>> let x denote the no of telephone lines busy

go by the data $p = 0.1$

$$q = 0.9$$

$$n = 10$$

$$q = 1 - p = 1 - 0.1 = 0.9$$

we have $p(x) = {}^n C_x p^x q^{n-x}$

$$p(x) = {}^{10} C_x (0.1)^x (0.9)^{n-x}$$

i) prob(that no line is busy)

$$\begin{aligned} p(x=0) &= {}^{10} C_0 (0.1)^0 (0.9)^{10-0} \\ &= 1 (1) \cdot (0.9)^{10} \\ &= (0.9)^{10} = \underline{\underline{0.3487}} \end{aligned}$$

ii) prob(that all lines are busy)

$$\begin{aligned} p(x=10) &= {}^{10} C_{10} (0.1)^{10} (0.9)^0 \\ &= 1 (0.1)^{10} (1) \\ &= \underline{\underline{(0.1)^{10}}} \end{aligned}$$

iii) Prob that atleast one line is busy.

$$= 1 - \text{prob of no line is busy}$$

$$= 1 - P(X=0)$$

$$= 1 - 0.3487$$

$$= \underline{\underline{0.6513}}$$

iv) prob that atmost 2 lines are busy

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= (0.9)^{10} + {}^{10}C_1 (0.1)^1 (0.9)^9 + {}^{10}C_2 (0.1)^2 (0.9)^8$$

$$= \underline{\underline{0.9298}}$$

(or)

$$P(X \leq 2) = 1 - P(X > 2)$$

$$= 1 - [P(X=3) + P(X=4) + \dots + P(X=10)]$$

$$= \underline{\underline{0.9298}}$$

7) In a quiz context of answering 'Yes' (or) 'No' what is the probability of guessing atleast 6 answers correctly out of 10 questions asked? Also find the probability of the same if there are 4 options for a correct answer.

>> Let x denote the correct answer in the first case,

$$p = \frac{1}{2}, \text{ and } q = \frac{1}{2} \text{ and } \underline{\underline{n=10}}$$

$$P(x) = {}^nC_x p^x q^{n-x}$$

(20)

$$P(x) = {}^{10}C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x}$$

we have to find $P(x \geq 6)$

$$P(x \geq 6) = P(6) + P(7) + P(8) + P(9) + P(10)$$

$$= {}^{10}C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^4 + {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3$$

$$+ {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10}$$

$$= \frac{1}{2^{10}} [{}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}]$$

$$= \frac{1}{2^{10}} [210 + 120 + 45 + 10 + 1]$$

$$= \frac{386}{1024} = 0.377$$

Thus $\underline{P(x \geq 6) = 0.377}$

In the second case when there are 4 options

$$p = \frac{1}{4}, q = \frac{3}{4}, n = 10$$

$$P(x) = {}^{10}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{10-x}$$

Hence $P(x \geq 6) = P(6) + P(7) + P(8) + P(9) + P(10)$

$$= \frac{1}{4^{10}} [3^4 {}^{10}C_6 + 3^3 {}^{10}C_7 + 3^2 {}^{10}C_8 + 3^1 {}^{10}C_9 + {}^{10}C_{10}]$$

$$= \frac{1}{4^{10}} [81 \times 210 + 27 \times 120 + 9 \times 45 + 3 \times 10 + 1]$$

$$= 0.019$$

Thus $P(x \geq 6) = 0.019$

8) In Sampling a large number of parts manufactured by a company of defectives. in samples of 20, how many would be expected to contain atleast 3 defective parts.
the mean no. out of 1000 such samples

>> Let $x \rightarrow$ defective part

$$\text{Mean } (M) = 2 \quad \text{and } n = 20$$

$$np = 2 \Rightarrow 20 \cdot p = 2$$

$$\Rightarrow p = \frac{2}{20} = \frac{1}{10}$$

$$p = \frac{1}{10} = 0.1 \quad q = 1 - p = 0.9$$

$$\therefore \boxed{p = 0.1} \quad \boxed{q = 0.9} \quad \boxed{n = 20}$$

$$P(x) = {}^nC_x p^x q^{n-x}$$

$$P(x) = {}^{20}C_x (0.1)^x (0.9)^{20-x}$$

Probability of atleast 3 defective parts

$$P(x \geq 3) = P(3) + P(4) + P(5) + \dots + P(20)$$

$$= 1 - P(x < 3)$$

$$= 1 - [P(x=0) + P(x=1) + P(x=2)]$$

$$= 1 - [{}^{20}C_0 (0.1)^0 (0.9)^{20} + {}^{20}C_1 (0.1)^1 (0.9)^{19} + {}^{20}C_2 (0.1)^2 (0.9)^{18}]$$

$$= 1 - [(0.9)^{20} + 20 (0.1) (0.9)^{19} + 190 (0.1)^2 (0.9)^{18}]$$

$$= \underline{\underline{0.323}}$$

Thus the no. of defectives in 1000 samples

$$\text{is } 1000 \times 0.323 = \underline{\underline{323}}$$

9) If the mean and standard deviation (21)
of the no of correctly answered questions
in a test given to 4096 students are 2.5 and
 $\sqrt{1.875}$. Find an estimate of the no of candidates
answering correctly (i) 8 or more questions
ii) 2 or less iii) 5 questions.

>> we have mean (μ) = np and S.D(σ) = $\sqrt{npq}$
for a binomial distribution.

By data $np = 2.5$ and $\sqrt{npq} = \sqrt{1.875}$
(or)
 $npq = 1.875$

Hence we have

$$2.5(q) = 1.875$$

$$q = \frac{1.875}{2.5} = 0.75$$

$$p = 1 - q = 1 - 0.75 = 0.25$$

$$\text{Since } np = 2.5 \Rightarrow n(0.25) = 2.5$$

$$n = 2.5 / 0.25 = 10$$

$$\text{Thus } \boxed{p = 0.25 = \frac{1}{4}} \quad \boxed{q = \frac{3}{4}} \quad \boxed{n = 10}$$

Let $x \rightarrow$ denote the no of correctly answered
questions.

$$P(x) = {}^nC_x p^x q^{n-x} = {}^{10}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{10-x}$$

$$P(x) = \frac{1}{4^{10}} [{}^{10}C_x (3)^{10-x}] \rightarrow \underline{\underline{1 \text{ student}}}$$

Since the estimate is needed for 4096
students we have $f(x) = 4096 \times P(x)$

$$f(x) = \frac{4096}{4^{10}} [{}^{10}C_x 3^{10-x}]$$

$$f(x) = \frac{2^{12}}{2^{20}} \left[{}^{10}C_x (3)^{10-x} \right]$$

$$= \frac{1}{2^8} \left[{}^{10}C_x (3)^{10-x} \right]$$

$$f(x) = \frac{1}{256} \left[{}^{10}C_x (3)^{10-x} \right]$$

i) we have to find $f(8) + f(9) + f(10)$

$$= \frac{1}{256} \left[{}^{10}C_8 (3)^{10-8} + {}^{10}C_9 (3)^{10-9} + {}^{10}C_{10} (3)^{10-10} \right]$$

$$= \frac{1}{256} \left[{}^{10}C_8 (3)^2 + {}^{10}C_9 (3)^1 + 1 \right]$$

$$= \frac{1}{256} \left[45 \times 9 + 10 \times 3 + 1 \right] = \frac{436}{256} = 1.6992$$

$$\approx \underline{\underline{2}}$$

No of students correctly answering 8 or more questions is 2.

ii) we have to find $f(2) + f(1) + f(0)$

$$= \frac{1}{256} \left[{}^{10}C_2 (3)^8 + {}^{10}C_1 3^9 + 3^{10} \right]$$

$$= \frac{3^8}{256} \left[{}^{10}C_2 + {}^{10}C_1 \cdot 3 + 3^2 \right]$$

$$= \frac{3^8}{256} (45 + 30 + 9) = 2152.8 \approx$$

$$= 2152.8281$$

$$= 2152.8 \approx \underline{\underline{2153}}$$

No of students correctly answering 2 or less than 2 questions is 2153

iii) we have to find $f(5)$

$$= \frac{1}{256} \left[{}^{10}C_5 \cdot 3^5 \right] = 239.2 = \underline{\underline{239}}$$

No of students correctly answering 5 questions is 239

10) An air line knows that 5% of the people making reservations on a certain flight will not turn up. Consequently their policy is to sell 52 tickets for a flight that can only hold 50 people. What is the probability that there will be a seat for every passenger who turns up? (22)

>> The probability (p) that a passenger will not turn up is

$$p = 0.05 \quad \text{and} \quad q = 0.95$$

Let x denote the no of passengers who will not turn up.

$$P(x) = {}^{n}C_x p^x q^{n-x} \quad \text{where } n = 52$$

$$\therefore P(x) = {}^{52}C_x (0.05)^x (0.95)^{52-x}$$

A seat is assumed for every passenger who turns up if the no of passengers who fail to turn up is more than or equal to 2.

Hence we have to find $P(x \geq 2)$

$$\begin{aligned} P(x \geq 2) &= 1 - P(x < 2) \\ &= 1 - [P(x=0) + P(x=1)] \\ &= 1 - \left[{}^{52}C_0 (0.05)^0 (0.95)^{52} + {}^{52}C_1 (0.05)^1 (0.95)^{51} \right] \\ &= 1 - \left[(0.95)^{52} + 52 (0.05)^1 (0.95)^{51} \right] \\ &= 1 - (0.95)^{51} \left[(0.95) + 52 (0.05) \right] \\ &= 1 - 0.2595 = \underline{0.7405} \end{aligned}$$

Probability that a seat is available for every passenger is 0.7405

- 11) In 800 families with 5 children each how many families would be expected to have
- i) 3 boys ii) 5 girls iii) either 2 or 3 boys
 - iv) atleast 2 girls by assuming probabilities for boys and girls to be equal.

>> $p = \text{prob of having a boy} = \frac{1}{2}$

$q = \text{prob of having a girl} = \frac{1}{2}$

Let x denote the no of boys in the family

$P(x) = {}^nC_x p^x q^{n-x}$ where $n=5$

i.e. $P(x) = {}^5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}$

$$= \frac{1}{2^5} {}^5C_x = \frac{{}^5C_x}{32}$$

$\frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} = \frac{1}{32}$

Since we have to find the expected no in respect of 800 families we have

$f(x) = 800 P(x)$

$= 800 \frac{{}^5C_x}{32} = 25 \cdot {}^5C_x = \underline{\underline{f(x)}}$

Thus $f(x) = 25 \cdot {}^5C_x$

i) we have to $f(3)$

$f(3) = 25 \cdot {}^5C_3 = 25 \cdot 10 = \underline{\underline{250}}$

Expected no of families with 3 boys is 250

ii) we have to find $f(0)$

$$f(0) = 25 \cdot {}^5C_0 = 25 \times 1 = \underline{25}$$

Expected no. of families with 5 girls is 25

iii) we have to find $f(2) + f(3)$

$$= 25 \cdot {}^5C_2 + 25 \cdot {}^5C_3$$

$$= 25 \times 10 + 25 \times 10$$

$$= 250 + 250 = \underline{500}$$

Expected no. of families with 2 or 3 boys is 500

iv) Atmost 2 girls means that, families can have
5 boys and 0 girls (or) 4 boys and 1 girl (or)
3 boys and 2 girls

Hence we have to find $f(5) + f(4) + f(3)$

$$= 25 \cdot {}^5C_5 + 25 \cdot {}^5C_4 + 25 \cdot {}^5C_3$$

$$= 25 \cdot (1 + 5 + 10)$$

$$= 25 (16)$$

$$= \underline{400}$$

Expected no. of families with atmost
 2 girls is 400.

(*) A class of 100 students contains 10 bright students. Five students from the class are picked at random. Find the probabilities that

i) none of the picked is a bright student.

ii) all the picked are bright students.

>> since 10 out of 100 students are bright.

The probability that a student picked is bright is

$$p = \frac{10}{100} = \frac{1}{10} = 0.1$$

$$q = 0.9$$

$$n = 5$$

$x \rightarrow$ bright students

$$\therefore p(x) = {}^nC_x p^x q^{n-x}$$

$$p(x) = {}^5C_x (0.1)^x (0.9)^{5-x}$$

$$i) p(\text{none of p. b. s.}) = p(x=0)$$

$$p(x=0) = {}^5C_0 (0.1)^0 (0.9)^{5-0}$$

$$= (0.9)^5 = \underline{\underline{0.59049}}$$

$$ii) p(x=5) = {}^5C_5 (0.1)^5 (0.9)^0$$

$$= 1 (0.1)^5 (1)$$

$$= (0.1)^5$$

$$= 1 \times 10^{-5}$$

$$= \underline{\underline{0.00001}}$$

④. The probability that a man aged 60 will live to be 70 is 0.65. what is the prob that out of 10 men, now aged 60

- i) exactly 9 will live to be 70.
- ii) at most 9 will live to be 70.
- iii) atleast 7 will live to be 70?

let $x \rightarrow$ living

$$p = 0.65 \text{ and } q = 1 - p = 1 - 0.65 = 0.35, \underline{n=10}$$

$$P(x) = {}^n C_x p^x q^{n-x}$$

$$P(x) = {}^{10} C_x (0.65)^x (0.35)^{10-x}$$

$$\begin{aligned} \text{i)} \quad P(x=9) &= {}^{10} C_9 (0.65)^9 (0.35)^{10-9} \\ &= {}^{10} C_9 (0.65)^9 (0.35)^1 \end{aligned}$$

$$= 10 \cdot "$$

$$= \underline{0.0725}$$

$$\begin{aligned} \text{ii)} \quad P(x \leq 9) &= P(x=9) + P(x=10) \\ &= {}^{10} C_9 (0.65)^9 (0.35)^1 + {}^{10} C_{10} (0.65)^{10} (0.35)^0 \\ &= 0.0725 + (0.65)^{10} \end{aligned}$$

$$\begin{aligned} P(x \leq 9) &= 1 - P(x \neq 9) \\ &= 1 - P(x=10) \\ &= 1 - (0.65)^{10} \\ &= 1 - 0.01346 \\ &= \underline{0.9865} \end{aligned}$$

$$\text{iii)} \quad p(x \geq 7) = p(x=7) + p(x=8) + p(x=9) + p(x=10) \\ = \underline{\underline{0.5139}}$$

1) Fit a poisson distribution for the foll^g data (25) and calculate the theoretical frequencies:

x	0	1	2	3	4
f	122	60	15	2	1

>> we shall compute the mean (m) of the given distribution

$$m = \frac{\sum fx}{\sum f} = \frac{0 \times 122 + 1 \times 60 + 2 \times 15 + 3 \times 2 + 4 \times 1}{122 + 60 + 15 + 2 + 1}$$

$$= \frac{60 + 30 + 6 + 4}{200} = 0.5$$

we have mean (M) = m for the poisson D

The poisson distribution is

$$p(x) = \frac{m^x e^{-m}}{x!} \quad \text{and let } f(x) = 200p(x)$$

$$\text{ie } f(x) = 200 \cdot \frac{(0.5)^x e^{-0.5}}{x!} \quad \text{But } e^{-0.5} \approx 0.6065$$

$$f(x) = \frac{200 \times 0.6065 (0.5)^x}{x!}$$

$$f(x) = \frac{121.3 \cdot (0.5)^x}{x!}$$

putting $x = 0, 1, 2, 3, 4$ in $f(x)$ we obtain the theoretical frequencies. They are as follows

$$f(0) = \frac{121.3 (0.5)^0}{0!} = 121.3$$

$$f(3) = \frac{121.3 (0.5)^3}{3!} = 2.527$$

$$f(1) = \frac{121.3 (0.5)^1}{1!} = 60.65$$

$$f(4) = \frac{121.3 (0.5)^4}{4!}$$

$$f(2) = \frac{121.3 (0.5)^2}{2!} = 15.1625$$

$$= 0.3159$$

2) The no of accidents per day (x) as recorded in a textile industry over a period of 400 days is given. Fit a poisson distribution for the data and calculate the theoretical frequencies.

x	0	1	2	3	4	5
f	173	168	37	18	3	1

$\Rightarrow$ we have for the poisson distribution,

$$\text{Mean } (\mu) = m = \frac{\sum f(x)}{\sum f} = \frac{0 + 1 \times 168 + 2 \times 37 + 3 \times 18 + 4 \times 3 + 5 \times 1}{173 + 168 + 37 + 18 + 3 + 1}$$

$$= \frac{168 + 74 + 54 + 12 + 5}{400}$$

$$= 0.7825$$

The poisson distribution is $P(x) = \frac{m^x e^{-m}}{x!}$

$$\text{Let } f(x) = 400 \cdot P(x)$$

$$= 400 \cdot \frac{(0.7825)^x e^{-0.7825}}{x!}$$

$$= \frac{400 \times 0.4573 (0.7825)^x}{x!}$$

$$= \frac{182.9 (0.7825)^x}{x!}$$

Theoretical frequencies are got by substituting $x = 0, 1, 2, 3, 4, 5$ in $f(x)$ and they are as follows

$$f(0) = 182.9 \frac{(0.7825)^0}{0!} = 182.9 \approx 183 \quad (26)$$

$$f(1) = 182.9 \frac{(0.7825)^1}{1!} = 143.11925 \approx 143$$

$$f(2) = 182.9 \frac{(0.7825)^2}{2!} = 55.995 \approx 56$$

$$f(3) = 182.9 \frac{(0.7825)^3}{3!} = 14.6054 \approx 15$$

$$f(4) = 182.9 \frac{(0.7825)^4}{4!} = 2.8571 \approx 3$$

$$f(5) = 182.9 \frac{(0.7825)^5}{5!} = 0.4471 \approx 0$$

3) In a certain factory turning out razor blades there is a small probability of $1/500$ for any blade to be defective. The blades are supplied in packets of 10. Use poisson distribution to calculate the approximate no of packets containing i) no defective ii) one defective iii) two defective blades in a consignment of 10,000 packets.

>> $p =$ probability of a defective blade $= \frac{1}{500} = 0.002$

In a packet of 10, the mean no of defective blades is

$$m = np = 10 \times 0.002 = 0.02$$

poisson distribution is $p(x) = \frac{m^x e^{-m}}{x!}$

$$p(x) = \frac{(0.02)^x e^{-0.02}}{x!}$$

$$\begin{aligned}
 \text{Let } f(x) &= 10,000 p(x) \\
 &= 10,000 \frac{(0.02)^x e^{-0.02}}{x!} \\
 &= \frac{10,000 \times 0.9802 \cdot (0.02)^x}{x!} \\
 &= \frac{9802 \cdot (0.02)^x}{x!}
 \end{aligned}$$

i) prob of no defective

$$f(0) = \frac{9802 (0.02)^0}{0!} = 9802$$

ii) prob of one defective

$$f(1) = \frac{9802 (0.02)^1}{1!} = 196.04 \approx 196$$

iii) prob of two defective

$$f(2) = \frac{9802 (0.02)^2}{2!} = 1.9604 \approx 2$$

4) The no of accidents in a year to taxi drivers in a city follows a Poisson d with mean 3. out of 1000 taxi drivers find approximately the no of the drivers with

i) no accident in a year

ii) more than 3 accidents in a year.

>> By data, mean (m) = 3 and we have for the poisson distribution $m = m = 3$

The poisson distribution is given by

$$P(x) = \frac{m^x e^{-m}}{x!} = \frac{e^{-3} m^x}{x!}$$

$$\text{Let } f(x) = 1000 P(x) = \frac{1000 e^{-3} m^x}{x!}$$

$$f(x) = 1000 \times 0.04979 \times \frac{(x+3)^x}{x!}$$

$$= 49.79 \times \frac{3^x}{x!}$$

$$\underline{f(x) = 50 \frac{3^x}{x!}}$$

i) No of drivers with no accident in a year = $f(0) = 50 \frac{3^0}{0!} = 50$

ii) prob of more than 3 accident on a year
 $P(X > 3) = 1 - P(X \leq 3)$

$$\begin{aligned} &= 1 - [f(0) + f(1) + f(2) + f(3)] \\ &= 1 - \left[50 + 50 \cdot \frac{3^1}{1!} + 50 \cdot \frac{3^2}{2!} + 50 \cdot \frac{3^3}{3!} \right] \\ &= 1 - [50 + 150 + 225 + 225] \\ &= 1 - 650 \end{aligned}$$

5) 2% of the fuses manufactured by a firm are found to be defective. Find the prob that a box containing 200 fuses contains

i) no defective fuses

ii) 3 or more defective fuses.

>> $p = \text{prob of a defective fuse}$

$$= 2/100 = 0.02$$

$\therefore$ mean no of defectives $\mu = m = np$

$$= 200 \times 0.02 = 4$$

The poisson distribution is given by

$$P(x) = \frac{m^x e^{-m}}{x!} = \frac{4^x e^{-4}}{x!}$$

$$P(x) = 0.0183 \frac{4^x}{x!}$$

i) prob of no defective fuse

$$P(0) = 0.0183$$

ii) prob of 3 or more defective fuses

$$P(x \geq 3) = 1 - P(x < 3)$$

$$= 1 - [P(x=0) + P(x=1) + P(x=2)]$$

$$= 1 - \left[0.0183 + 0.0183 \frac{4^1}{1!} + 0.0183 \cdot \frac{4^2}{2!} \right]$$

$$= 1 - 0.0183 \left[1 + 4 + \frac{16}{2} \right]$$

$$= 1 - 0.0183 [1 + 4 + 8]$$

$$= \underline{\underline{0.762}}$$

6/ if the prob of a bad reaction from a certain infection is 0.001, determine the chance that out of 2000 individuals, more than two will get a bad reaction. (28)

prob of a bad reaction from a certain infection is $p = 0.001$

$$\therefore \text{mean} = m = np = 2000 \times 0.001 = 2$$

$$\therefore P(x) = \frac{m^x e^{-m}}{x!} = \frac{2^x \cdot e^{-2}}{x!} = 0.1353 \frac{2^x}{x!}$$

$$P(x) = 0.1353 \frac{2^x}{x!}$$

prob (more than two) :

$$P(x > 2) = 1 - P(x \leq 2)$$

$$= 1 - [P(0) + P(1) + P(2)]$$

$$= 1 - 0.1353 \left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} \right]$$

$$\frac{2^2 = 4}{2} = 2$$

$$= 1 - 0.1353 [1 + 2 + 2]$$

$$= 1 - 0.1353 [5]$$

$$= 1 - 0.6765$$

$$= \underline{\underline{0.3235}}$$

7) A communication channel receives independent pulses at the rate of 12 pulses per microsecond. The prob of transmission error is 0.001 for each microsecond. Compute the probabilities of

- i) no error during a micro second.
- ii) one error per micro second
- iii) atleast one error per micro second
- iv) two errors
- v) at most two errors.

>> mean no of errors in one micro second

$$\mu = np = m = 12 \times 0.001 = 0.012$$

The P.D $P(x) = \frac{m^x e^{-m}}{x!}$

$$P(x) = \frac{(0.012)^x e^{-0.012}}{x!} = 0.9881 \frac{(0.012)^x}{x!}$$

i) $p(x=0) = 0.9881$

ii) $p(x=1) = 0.9881 \times 0.012 = 0.01186$

iii) $p(x \geq 1) = 1 - p(x=0)$
 $= 1 - 0.9881 = 0.0119$

iv) $p(x=2) = 0.9881 \frac{(0.012)^2}{2!} = 7.11432 \times 10^{-5}$
 $= 0.000071$

v) $p(x \leq 2) = p(x=0) + p(x=1) + p(x=2)$
 $= 0.9881 + 0.01186 + 0.000071$
 $= \underline{\underline{1}}$

8). A shop has 4 diesel generator sets which it hires every day. The demand for a gen set on an average is a poisson variable with value $5/2$. obtain the prob that on a particular day

i) there was no demand

ii) a demand had to be refused.

>> By data $m = \text{mean demand for a generator}$
 $= 5/2 = 2.5$

$$P.D \quad P(x) = \frac{m^x e^{-m}}{x!} = \frac{e^{-2.5} (2.5)^x}{x!}$$

$$P(x) = 0.0821 \frac{(2.5)^x}{x!}$$

i) No demand for a generator
 $P(0) = 0.0821$

ii) if a demand had to be refused, then should have been a demand for more than 4 generator

we've to find $P(x > 4)$

$$P(x > 4) = 1 - P(x \leq 4)$$

$$= 1 - [P(0) + P(1) + P(2) + P(3) + P(4)]$$

$$= 1 - 0.0821 \left[\frac{(2.5)^0}{0!} + \frac{(2.5)^1}{1!} + \frac{(2.5)^2}{2!} + \frac{(2.5)^3}{3!} + \frac{(2.5)^4}{4!} \right]$$

$$= 1 - 0.0821 [1 + 2.5 + 3.125 + 2.6041 + 1.6276]$$

$$= 1 - 0.8913$$

$$= \underline{\underline{0.1087}}$$

- 9) The prob that a news reader commits no mistake in reading the news is $1/e^3$. Find the prob that on a particular news broadcast he commits
- only 2 mistakes
 - more than 3 mistakes
 - at most 3 mistakes.

>> Taking a note of the prob given in the data as $1/e^3 = e^{-3}$ we consider

$$P(x) = \frac{m^x e^{-m}}{x!} \text{ where } x \rightarrow \text{committed mistake}$$

by data $P(x=0) = e^{-3}$ and hence

$$e^{-m} = e^{-3} \Rightarrow m=3$$

$$P(x) = e^{-3} \frac{3^x}{x!}$$

i) prob of committed only 2 mistakes is $P(2)$

$$P(2) = e^{-3} \cdot \frac{3^2}{2!} = 0.224 \quad e^{-3} = 0.04979 = \underline{\underline{0.0498}}$$

ii) $P(x > 3) = 1 - P(x \leq 3)$

$$= 1 - [P(0) + P(1) + P(2) + P(3)]$$

$$= 1 - 0.0498 \left[1 + \frac{3}{1} + \frac{3^2}{2!} + \frac{3^3}{3!} \right]$$

$$= 1 - 0.0498 [1 + 3 + 4.5 + 4.5]$$

$$= 1 - 0.6474 = \underline{\underline{0.3526}}$$

iii) $P(x \leq 3) = P(0) + P(1) + P(2) + P(3)$

$$= \underline{\underline{0.6474}}$$

Continuous Probability Distributions.

(30)

Defⁿ:- If for every x belonging to the range of a continuous random variable X , we assign a real number $f(x)$ satisfying the conditions

$$1) f(x) \geq 0 \quad 2) \int_{-\infty}^{\infty} f(x) dx = 1$$

then $f(x)$ is called a continuous probability function (c.p.f) (or) probability density function (p.d.f)

Mean and Variance

$$\text{Mean (M)} = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

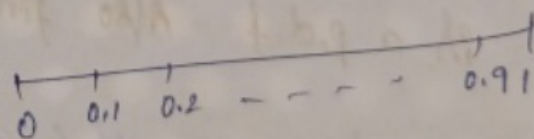
$$\text{Variance (V)} = \int_{-\infty}^{\infty} (x-M)^2 \cdot f(x) dx$$

I) Find the foll^{ng} funcⁿ is a Prob (p.d.f) or (c.d.f)

$$1) f_1(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

condⁿ for p.d.f are

$$f(x) \geq 0 \text{ and } \int_{-\infty}^{\infty} f(x) dx = 1$$



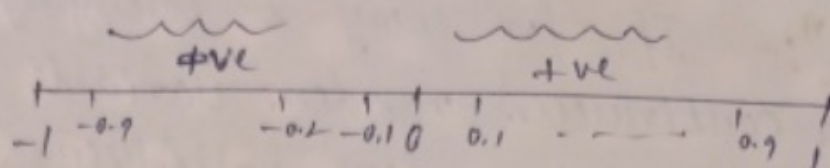
$$\left. \begin{aligned} 2x &= 2(0.1) = 0.2 \\ &= 2(0.2) = 0.4 \end{aligned} \right\} f(x) \geq 0$$

Clearly $f(x) \geq 0$

$$\int_{-\infty}^{\infty} f_1(x) dx = \int_0^1 2x dx = \frac{2x^2}{2} \Big|_0^1 = x^2 \Big|_0^1 = 1 - 0 = 1$$

$\therefore f_1(x)$ is a p.d.f / c.d.f

$$i) f_2(x) = \begin{cases} 2x, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$



clearly ~~$f_2(x) \leq 0$~~ $f_2(x)$ can rewrite as

$$f_2(x) = \begin{cases} 2x & -1 < x < 0 \\ 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

In $-1 < x < 0$, $f_2(x) = 2x < 0$ it keeps than zero.

$$\text{further } \int_{-\infty}^{\infty} f_2(x) dx = \int_{-1}^1 2x dx = [x^2]_{-1}^1 = 0$$

Both the conditions are not satisfied

$\therefore f_2(x)$ is not a p.d.f / c.d.f

2) Find the value of c such that $f(x) = \begin{cases} \frac{x}{6} + c, & 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$ is a p.d.f. Also find $P(1 \leq x \leq 2)$

$\Rightarrow f(x) \geq 0$, if $c \geq 0$; Also we must have $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\text{ie } \int_0^3 \left(\frac{x}{6} + c \right) dx = 1$$

$$\frac{x^2}{12} + cx \Big|_0^3 = 1$$

$$\frac{3^2}{12} + 3c = 1$$

$$\frac{9}{12} + 3c = 1$$

$$\frac{3}{4} + 3c = 1$$

$$3c = 1 - \frac{3}{4} = \frac{4-3}{4} = \frac{1}{4}$$

$$3c = \frac{1}{4}$$

$$c = \frac{1}{4 \cdot 3} = \frac{1}{12}$$

$$\boxed{c = \frac{1}{12}}$$

$$\text{Now } P(1 \leq x \leq 2) = \int_1^2 f(x) dx$$

$$= \int_1^2 \left(\frac{x}{6} + \frac{1}{12} \right) dx$$

$$= \left[\frac{x^2}{12} + \frac{x}{12} \right]_1^2 = \left(\frac{4}{12} + \frac{2}{12} \right) - \left(\frac{1}{12} + \frac{1}{12} \right) \\ = \frac{6}{12} - \frac{2}{12} = \frac{4}{12} = \frac{1}{3}$$

$$\text{Thus } \boxed{P(1 \leq x \leq 2) = \frac{1}{3}}$$

2) Find the constant k such that $f(x) = \begin{cases} kx^2, & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$

is a p.d.f. Also find i) $p(1 < x < 2)$

ii) $p(x \leq 1)$

iii) $p(x > 1)$

iv) Mean and variance

>> $f(x) \geq 0$ if $k \geq 0$. Also we must have $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\text{i.e. } \int_0^3 kx^2 dx = 1$$

$$k \left[\frac{x^3}{3} \right]_0^3 = 1 \Rightarrow \frac{k}{3} [3^3] = 1 \Rightarrow \frac{27k}{3} = 1 \Rightarrow$$

$$\Rightarrow k = \frac{3}{27} = \frac{1}{9} \Rightarrow \boxed{k = \frac{1}{9}}$$

$$\text{i) } p(1 < x < 2) = \int_1^2 f(x) dx$$

$$= \int_1^2 kx^2 dx$$

$$= k \left[\frac{x^3}{3} \right]_1^2$$

$$= \frac{k}{3} [8 - 1]$$

$$\frac{k}{3} [7] = \frac{7}{3} \left(\frac{1}{9} \right)$$

$$= \frac{7}{27}$$

$$\text{Thus } \boxed{p(1 < x < 2) = \frac{7}{27}}$$

$$\text{ii)} \quad p(x \leq 1) = \int_0^1 f(x) dx = \int_0^1 kx^2 dx = \int_0^1 \frac{x^2}{9} dx$$

$$= \frac{x^3}{27} \Big|_0^1 = \frac{1}{27}$$

$$\text{iii)} \quad p(x > 1) = \int_1^3 f(x) dx = \int_1^3 \frac{x^2}{9} dx = \frac{x^3}{27} \Big|_1^3 = \frac{27}{27} - \frac{1}{27} = 1 - \frac{1}{27}$$

$$= \frac{26}{27}$$

$$\text{iv)} \quad \text{Mean } (M) = \int_{-\infty}^{\infty} xf(x) dx = \int_{-\infty}^{\infty} x \cdot \frac{x^2}{9} dx = \int_0^3 x \cdot \frac{x^2}{9} dx$$

$$M = \int_0^3 \frac{x^3}{9} dx = \frac{x^4}{4 \times 9} \Big|_0^3 = \frac{1}{36} [x^4]_0^3 = \frac{1}{36} [3^4 - 0^4]$$

$$= \frac{1}{36} [81 - 0] = \frac{81}{36} = \frac{9}{4}$$

$$\text{Variance } (\sigma^2) = \int_{-\infty}^{\infty} (x - M)^2 \cdot f(x) dx$$

$$= \int_{-\infty}^{\infty} (x^2 - 2xM + M^2) \cdot f(x) dx$$

$$= \int_0^3 x^2 \cdot \frac{x^2}{9} dx - \left(\frac{9}{4}\right)^2$$

$$= \int_0^3 \frac{x^4}{9} dx - \left(\frac{9}{4}\right)^2$$

$$= \frac{x^5}{5 \times 9} \Big|_0^3 - \left(\frac{9}{4}\right)^2$$

$$= \frac{1}{45} [3^5] - \frac{81}{16}$$

$$= \frac{243}{45} - \frac{81}{16}$$

$$= \frac{81}{15} - \frac{81}{16} = \frac{81}{240} = \frac{27}{80}$$

3) Find k such that $f(x) = \begin{cases} kxe^{-x}, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(32)

is a p.d.f. Find the mean.

>> $f(x) \geq 0$ if $k \geq 0$. Also we must have $\int_{-\infty}^{\infty} f(x) dx = 1$

ie $\int_0^1 kxe^{-x} dx = 1 \Rightarrow k \int_0^1 xe^{-x} dx = 1$ Apply uo-rule

$$k \left[x \frac{e^{-x}}{(-1)} - (1) \frac{e^{-x}}{(-1)^2} \right]_0^1 = 1$$

$$k \left[-xe^{-x} - e^{-x} \right]_0^1 = 1$$

$$k \left[-1e^{-1} - e^{-1} - (0 - e^{-0}) \right] = 1$$

$$k \left[-\frac{1}{e} - \frac{1}{e} + 1 \right] = 1$$

$$k \left(1 - \frac{2}{e} \right) = 1$$

$$k = \frac{1}{1 - \frac{2}{e}} = \frac{1}{\frac{e-2}{e}} = \frac{e}{e-2}$$

$$\therefore \boxed{k = \frac{e}{e-2}}$$

$$\text{Mean (M)} = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

$$= \int_0^1 x \cdot kxe^{-x} dx$$

$$= \int_0^1 x \cdot \frac{e}{e-2} xe^{-x} dx$$

$$= \frac{e}{e-2} \int_0^1 x^2 e^{-x} dx$$

$$= \frac{e}{e-2} \left[+x^2 \frac{e^{-x}}{(-1)} - 2x \frac{e^{-x}}{(-1)^2} + 2 \frac{e^{-x}}{(-1)^3} \right]_0^1$$

$$= \frac{e}{e-2} \left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \right]_0^1$$

$$= \frac{e}{e-2} \left[-e^{-1} - 2e^{-1} - 2e^{-1} - (0 - 0 - 2e^{-0}) \right]$$

$$= \frac{e}{e-2} \left[-\frac{1}{e} - \frac{2}{e} - \frac{2}{e} + 2 \right]$$

$$= \frac{e}{e-2} \left[-\frac{5}{e} + 2 \right]$$

$$= \frac{e}{e-2} \left[2 - \frac{5}{e} \right]$$

$$= \frac{e}{e-2} \left[\frac{2e-5}{e} \right]$$

$$= \frac{2e-5}{e-2}$$

4) Is the foll^g funⁿ a density funⁿ?

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \text{also find } (1, 2) \text{ interval.}$$

>> we observe $f(x) \geq 0$. Also we must have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= 1 \\ \therefore \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx \\ &= 0 + \int_0^{\infty} e^{-x} dx = \frac{e^{-x}}{-1} \Big|_0^{\infty} = -[e^{-\infty} - e^{-0}] \\ &= -[0 - 1] = 1 \quad \because e^{-\infty} = 0 \end{aligned}$$

Hence $f(x)$ is a probability funⁿ (p.d.f) (c.d.f).

$$\begin{aligned} P(1 < x < 2) &= \int_1^2 f(x) dx \\ &= \int_1^2 e^{-x} dx \\ &= -[e^{-x}]_1^2 \\ &= -[e^{-2} - e^{-1}] \\ &= -e^{-2} + e^{-1} \\ &= \left[\frac{1}{e} - \frac{1}{e^2} \right] = 0.3679 - 0.1353 \\ &= \underline{\underline{0.2326}} \end{aligned}$$

5) A random variable x has the foll'g d.f. (33)

$$p(x) = \begin{cases} kx^2, & -3 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

Evaluate k and find i) $p(1 \leq x \leq 2)$ ii) $p(x \leq 2)$ iii) $p(x > 1)$

>> $p(x) \geq 0$ if $k \geq 0$ and also we must've $\int_{-\infty}^{\infty} p(x) dx = 1$

$$\text{ie } \int_{-3}^3 kx^2 dx = 1 \Rightarrow k \frac{x^3}{3} \Big|_{-3}^3 = 1$$

$$\frac{27}{54}$$

$$\Rightarrow \frac{k}{3} [27 - (-3)^3] = 1$$

$$\Rightarrow \frac{k}{3} [27 + 27] = 1$$

$$\frac{k}{3} [54] = 1$$

$$\boxed{k = \frac{1}{18}}$$

$$\text{i) } p(1 \leq x \leq 2) = \int_1^2 kx^2 dx = \int_1^2 \frac{x^2}{18} dx = \frac{x^3}{54} \Big|_1^2 = \frac{8}{54} - \frac{1}{54} = \frac{7}{54}$$

$$\begin{aligned} \text{ii) } p(x \leq 2) &= \int_{-3}^2 \frac{x^2}{18} dx = \frac{x^3}{54} \Big|_{-3}^2 = \frac{1}{54} [2^3 - (-3)^3] \\ &= \frac{1}{54} [8 + 27] \\ &= \frac{1}{54} [35] = \frac{35}{54} \end{aligned}$$

$$\begin{aligned} \text{iii) } p(x > 1) &= \int_1^3 \frac{x^2}{18} dx = \frac{x^3}{54} \Big|_1^3 \\ &= \frac{1}{54} [27 - 1] = \frac{26}{54} = \frac{13}{27} \end{aligned}$$

6) Find k so that the foll'g funⁿ can serve as a p.d.f of a random variable.

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ kxe^{-4x^2} & \text{for } x > 0 \end{cases}$$

>> we must have $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\text{i.e. } \int_0^{\infty} kxe^{-4x^2} dx = 1$$

$$\text{putting } 4x^2 = t \Rightarrow 8x dx = dt \\ \Rightarrow dx = \frac{dt}{8x} \Rightarrow x dx = \frac{dt}{8}$$

$$\text{put } x=0 \Rightarrow t=0$$

$$x=\infty \Rightarrow t=\infty$$

$$\therefore \int_{t=0}^{\infty} k e^{-t} \frac{dt}{8} = 1$$

$$\frac{k}{8} \int_0^{\infty} e^{-t} dt = 1 \Rightarrow \frac{k}{8} \left[\frac{e^{-t}}{-1} \right]_0^{\infty} = 1$$

$$\Rightarrow -\frac{k}{8} [0 - 1] = 1$$

$$\Rightarrow \frac{k}{8} = 1 \Rightarrow \boxed{k=8}$$

7) A random Variable x has the density funⁿ $f(x) = k/(1+x^2)$, $-\infty < x < \infty$. Determine k and hence evaluate i) $P(x \geq 0)$ ii) $P(0 < x < 1)$

>> we must have $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\text{i.e. } \int_{-\infty}^{\infty} \frac{k}{1+x^2} dx = 1$$

Since the integral is even

$$2 \int_0^{\infty} \frac{k}{1+x^2} dx = 1$$

$$2k \left[\tan^{-1} x \right]_0^{\infty} = 1$$

$$2k \left[\frac{\pi}{2} - 0 \right] = 1$$

$$2k \left[\frac{\pi}{2} \right] = 1$$

$$\boxed{k = 1/\pi}$$

$$\text{Now } p(x \geq 0) = \int_0^{\infty} f(x) dx = \frac{1}{\pi} \int_0^{\infty} \frac{1}{1+x^2} dx$$

$$= \frac{1}{\pi} [\tan^{-1} x]_0^{\infty} = \frac{1}{\pi} \left[\frac{\pi}{2} - 0 \right] = \frac{1}{2}$$

(34)

$$\text{Thus } \boxed{p(x \geq 0) = \frac{1}{2}}$$

$$\text{Also } p(0 < x < 1) = \frac{1}{\pi} \int_0^1 \frac{1}{1+x^2} dx$$

$$= \frac{1}{\pi} [\tan^{-1} x]_0^1 = \frac{1}{\pi} \left[\frac{\pi}{4} - 0 \right] = \frac{1}{4}$$

$$\text{Thus } \boxed{p(0 < x < 1) = \frac{1}{4}}$$

8) The time t years required to complete a software Project has p.d.f of the form

$$f(t) = \begin{cases} kt(1-t), & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find k and also the prob that the project will be Completed in less than 4 months.

$$>> \text{ we must have } \int_{-\infty}^{\infty} f(t) dt = 1$$

$$\text{ie } \int_0^1 kt(1-t) dt = 1$$

$$\Rightarrow k \int_0^1 (t - t^2) dt = 1$$

$$\Rightarrow k \left[\frac{t^2}{2} - \frac{t^3}{3} \right]_0^1 = 1$$

$$\Rightarrow k \left[\frac{1}{2} - \frac{1}{3} \right] = 1$$

$$\Rightarrow k \left[\frac{2-3}{6} \right] = 1$$

$$-\frac{k}{6} = 1$$

$$\Rightarrow k \int_0^1 (t - t^2) dt = 1$$

$$\Rightarrow k \left(\frac{t^2}{2} - \frac{t^3}{3} \right)_0^1 = 1$$

$$\Rightarrow k \left[\frac{1}{6} \right] = 1$$

$$\therefore \boxed{k = 6}$$

prob that the project will be completed in 4 months is equivalent to find $P(0 < t < 1/3)$

Since t is in years

$$\therefore P(0 < t < 1/3) = \int_0^{1/3} 6(t - t^2) dt$$

$$= 6 \cdot \left[\frac{t^2}{2} - \frac{t^3}{3} \right]_0^{1/3}$$

$$= 6 \left[\frac{t^2}{2} \right]_0^{1/3} - 6 \left[\frac{t^3}{3} \right]_0^{1/3}$$

$$= 3 \left[t^2 \right]_0^{1/3} - 2 \left[t^3 \right]_0^{1/3}$$

$$= \underline{\underline{\frac{7}{27}}}$$

5 apple - 100 Rs
1 apple - ?
 $\frac{1 \times 100}{5} = 20$
12 months = 1 year
4 months = ?
 $\frac{4 \times 1}{12} = \frac{1}{3} \text{ yr}$

Note:-

if X is a continuous random variable with prob (p.d.f) $f(x)$ the fun $F(x)$ defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(x) dx \quad \text{--- (1)}$$

is called the cumulative distribution function (c.d.f) of X

It is evident from (1) we've

$$F(x) = P(X \leq x) = P(-\infty < X \leq x) \text{ and } \frac{d}{dx} [F(x)] = f(x)$$

is a continuous random variable.

9) Find the c.d.f for the foll'g p.d.f of a random variable x . (35)

$$i) f(x) = \begin{cases} 6x - 6x^2, & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad ii) f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ If $f(x)$ is the p.d.f then the c.d.f $F(x) = \int_{-\infty}^x f(x) dx$

$$i) F(x) = \int_{-\infty}^x f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^x f(x) dx$$

$$= 0 + \int_0^x (6x - 6x^2) dx = \underline{3x^2 - 2x^3}$$

$$\therefore \text{c.d.f} = 3x^2 - 2x^3 \quad \text{if } 0 \leq x \leq 1$$

$$ii) F(x) = \int_0^x \frac{x}{4} e^{-x/2} dx$$

$$= \frac{1}{4} \left[\frac{x e^{-x/2}}{-1/2} - \frac{e^{-x/2}}{1/4} \right]_0^x$$

$$= \frac{1}{4} \left[-2(x e^{-x/2})_0^x - 4(e^{-x/2})_0^x \right]$$

$$= \frac{1}{4} \left[-2(x e^{-x/2} - 0) - 4(e^{-x/2} - 1) \right]$$

$$= \frac{1}{4} \left[-2x e^{-x/2} - (4e^{-x/2} - 4) \right]$$

$$= \frac{1}{4} \left[-2x e^{-x/2} - 4e^{-x/2} + 4 \right]$$

$$\text{c.d.f} = 1 - e^{-x/2} - (x/2) e^{-x/2}, \quad \text{if } 0 < x < \infty$$

10) A continuous random variable has the distribution funⁿ

$$F(x) = \begin{cases} 0 & x \leq 1 \\ c(x-1)^4 & 1 \leq x \leq 3 \\ 1 & x > 3 \end{cases} \quad \text{Find } c \text{ and also the p.d.f}$$

>> wbt the p.d.f $f(x) = \frac{d}{dx} [F(x)]$

$$\therefore f(x) = \begin{cases} 0 & x \leq 1 \\ 4c(x-1)^3 & 1 \leq x \leq 3 \\ 0 & x > 3 \end{cases}$$

$f(x) \geq 0$ for $c \geq 0$ and we must have $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\therefore \int_1^3 4c(x-1)^3 dx = 1$$

$$4c \left[\frac{(x-1)^4}{4} \right]_1^3 = 1$$

$$c [(x-1)^4]_1^3 = 1$$

$$c [(3-1)^4 - (1-1)^4] = 1$$

$$c [2^4 - 0^4] = 1$$

$$c [16] = 1$$

$$\boxed{c = \frac{1}{16}}$$

Thus p.d.f $f(x) = \frac{(x-1)^3}{4}$ where $1 \leq x \leq 3$

Exponential Distribution.

(36)

The continuous probability distribution having the proba density fun (p.d.f) $f(x)$ given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise, where } \lambda > 0 \end{cases}$$

is known as the exponential distribution.

Evidently $f(x) > 0$ and we have

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx$$

$$= 0 + \lambda \int_0^{\infty} e^{-\lambda x} dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^{\infty} = -(e^{-\infty} - 1) = 1$$

$$\text{Thus } \boxed{\int_{-\infty}^{\infty} f(x) dx = 1}$$

$f(x)$ satisfy both the condⁿs required for a continuous prob fun c.p.f | prob. density fun (p.d.f)

Mean and Standard Deviation of the Exponential

Distribution. Mean (μ) = $\int_{-\infty}^{\infty} x \cdot f(x) dx$

$$= \int_{-\infty}^0 x \cdot f(x) dx + \int_0^{\infty} x \cdot f(x) dx = \int_0^{\infty} x \cdot \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} x e^{-\lambda x} dx = \lambda \left[x \frac{e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^{\infty}$$

$$= \lambda \left[-\frac{1}{\lambda} \left[x e^{-\lambda x} \right]_0^{\infty} - \frac{1}{\lambda^2} \left[e^{-\lambda x} \right]_0^{\infty} \right]$$

$$= \lambda \left[-\frac{1}{\lambda} \left[\infty e^{-\infty} - 0 \right] - \frac{1}{\lambda^2} \left[e^{-\infty} - 1 \right] \right]$$

$$= \lambda \left[0 + \frac{1}{\lambda^2} \right] = \frac{1}{\lambda} \quad \because e^{-\infty} = 0$$

$$\text{Variance } (\sigma^2) = \int_{-\infty}^{\infty} (x-m)^2 \cdot f(x) dx$$

$$\sigma^2 = \int_{-\infty}^0 (x-m)^2 f(x) dx + \int_0^{\infty} (x-m)^2 \cdot f(x) dx$$

$$\sigma^2 = \int_0^{\infty} (x-m)^2 \cdot f(x) dx$$

$$= \alpha \int_0^{\infty} (x-m)^2 \cdot e^{-\alpha x} dx$$

$$= \alpha \left[(x-m)^2 \frac{e^{-\alpha x}}{-\alpha} - 2(x-m) \frac{e^{-\alpha x}}{\alpha^2} + 2(1-0) \frac{e^{-\alpha x}}{-\alpha^3} \right]_0^{\infty}$$

$$= \alpha \left[-\frac{1}{\alpha} \left\{ (x-m)^2 e^{-\alpha x} \right\}_0^{\infty} - \frac{2}{\alpha^2} \left\{ (x-m) e^{-\alpha x} \right\}_0^{\infty} - \frac{2}{\alpha^3} \left\{ e^{-\alpha x} \right\}_0^{\infty} \right]$$

$$= \alpha \left[-\frac{1}{\alpha} \left\{ (\infty-m)^2 e^{-\infty} - (0-m)^2 \cdot 1 \right\} - \frac{2}{\alpha^2} \left\{ (\infty-m) e^{-\infty} - (0-m) \right\} - \frac{2}{\alpha^3} \left\{ e^{-\infty} - 1 \right\} \right]$$

$$= \alpha \left[+\frac{1}{\alpha} \left\{ 0 + m^2 \right\} - \frac{2}{\alpha^2} \left\{ 0 + m \right\} - \frac{2}{\alpha^3} \left\{ 0 - 1 \right\} \right]$$

$$= \alpha \left[+\frac{m^2}{\alpha} - \frac{2m}{\alpha^2} + \frac{2}{\alpha^3} \right] \quad \because e^{-\infty} = 0$$

but $m = \frac{1}{\alpha}$

$$= \alpha \left[\frac{1}{\alpha^3} - \frac{2}{\alpha^3} + \frac{2}{\alpha^3} \right]$$

$$\sigma^2 = \alpha \left[\frac{1}{\alpha^3} \right] = \frac{1}{\alpha^2}$$

Hence $\sigma = \frac{1}{\alpha}$

Thus ED

$$\text{Mean}(m) = \frac{1}{\alpha} ; \text{S.D}(\sigma) = \frac{1}{\alpha}$$

$$\therefore \underline{m = \sigma} \quad \underline{\text{mean} = \text{S.D.}}$$

i) The kilometre run (in thousands of k.m.s) without any sort of problem in respect of a certain vehicle is a random variable having p.d.f $f(x) = \begin{cases} \frac{1}{40} e^{-x/40} & , x \geq 0 \\ 0 & , x \leq 0 \end{cases}$ (37)

Find the prob that the vehicle is trouble free

- i) atleast for 25000 k.m.s
- ii) atleast for 25000 k.m.s
- iii) between 16000 to 32000 k.m.s

>> Here x is the random variable representing kilometre in multiples of 1000 regarding trouble free run by the vehicle.

i) To find $P(x \geq 25)$

$$\begin{aligned} P(x \geq 25) &= 1 - P(x < 25) \\ &= 1 - \int_{25}^{\infty} \frac{1}{40} e^{-x/40} dx \\ &= 1 - \left[\frac{1}{40} \frac{e^{-x/40}}{-1/40} \right]_{25}^{\infty} \\ &= 1 - \left[e^{-x/40} \right]_{25}^{\infty} \\ &= 1 - \left[e^{-\infty} - e^{-25/40} \right] \\ &= 1 - \left[0 - e^{-5/8} \right] \\ &= 1 - e^{-5/8} \end{aligned}$$

$$\begin{aligned} &= \int_{25}^{\infty} \frac{1}{40} e^{-x/40} dx \\ &= \frac{1}{40} \cdot \frac{e^{-x/40}}{-1/40} \Big|_{25}^{\infty} \\ &= - (e^{-\infty} - e^{-25/40}) \\ &= - (0 - e^{-5/8}) \\ &= e^{-5/8} \\ &= 0.5353 \end{aligned}$$

Thus $P(x \geq 25) = 0.5353$

ii) To find $P(X \leq 25)$

$$\begin{aligned} P(X \leq 25) &= \int_0^{25} \frac{1}{40} e^{-x/40} dx \\ &= \frac{1}{40} \cdot \frac{e^{-x/40}}{-1/40} \Big|_0^{25} = - (e^{-x/40})_0^{25} \\ &= - (e^{-25/40} - 1) = -e^{-5/8} + 1 \\ &= 1 - e^{-5/8} \\ &= 1 - 0.5353 \end{aligned}$$

$$\therefore \boxed{P(X \leq 25) = 0.4647}$$

iii) To find $P(16 \leq X \leq 32)$

$$\begin{aligned} P(16 \leq X \leq 32) &= \int_{16}^{32} \frac{1}{40} e^{-x/40} dx \\ &= - (e^{-x/40})_{16}^{32} \\ &= - (e^{-32/40} - e^{-16/40}) \\ &= - (e^{-4/5} - e^{-2/5}) \\ &= e^{-2/5} - e^{-4/5} \\ &= 0.6703 - 0.4493 \\ &= \underline{\underline{0.221}} \end{aligned}$$

$$\begin{array}{r} 8-4 \\ 16 \\ \underline{32} \\ 40 \\ 20 \\ 10 \\ \underline{2} \\ 4 \\ \underline{16} \\ 40 \\ 10 \\ 5 \end{array}$$

2. if x is an exponential variate with mean 3 (38)
find i) $p(x > 1)$ ii) $p(x < 3)$

>> The p.d.f of the exponential distribution is given by $f(x) = \begin{cases} \alpha e^{-\alpha x}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$

The mean of this distribution is given by $\frac{1}{\alpha}$

by data $\frac{1}{\alpha} = 3 \Rightarrow \alpha = \frac{1}{3}$

Hence $f(x) = \begin{cases} \frac{1}{3} e^{-x/3}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$

$$\begin{aligned} \text{i) } p(x > 1) &= \int_1^{\infty} \frac{1}{3} e^{-x/3} dx \\ &= \frac{1}{3} \frac{e^{-x/3}}{-1/3} \Big|_1^{\infty} \\ &= - (e^{-x/3}) \Big|_1^{\infty} \\ &= - (e^{-\infty} - e^{-1/3}) \\ &= - (0 - e^{-1/3}) \\ &= e^{-1/3} \end{aligned}$$

$$p(x > 1) = e^{-1/3} = \underline{\underline{0.7165}}$$

$$\begin{aligned} \text{ii) } p(x < 3) &= \int_0^3 \frac{1}{3} e^{-x/3} dx &= - (e^{-1} - 1) \\ &= \frac{1}{3} \frac{e^{-x/3}}{-1/3} \Big|_0^3 &= 1 - \frac{1}{e} \\ &= - (e^{-x/3}) \Big|_0^3 &= 1 - 0.3679 \\ &= - (e^{-3/3} - e^{-0}) &= \underline{\underline{0.6321}} \end{aligned}$$

3) if x is an exponential variate with mean 5.
 evaluate i) $p(0 < x < 1)$ ii) $p(-\infty < x < 10)$
 iii) $p(x \leq 0 \text{ or } x \geq 1)$

>> $f(x) = \lambda e^{-\lambda x}$, $0 < x < \infty$ Mean $= \frac{1}{\lambda} = 5 \Rightarrow \lambda = \frac{1}{5}$

Thus $f(x) = \begin{cases} \frac{1}{5} e^{-x/5}, & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$

i) $p(0 < x < 1) = \int_0^1 f(x) dx = \int_0^1 \frac{1}{5} e^{-x/5} dx = \frac{1}{5} \left[\frac{e^{-x/5}}{-1/5} \right]_0^1$
 $= - (e^{-x/5})_0^1 = - (e^{-1/5} - 1) = 1 - e^{-1/5}$
 $= 1 - e^{-0.2} = 1 - 0.8187 = 0.1813$

$\therefore p(0 < x < 1) = 0.1813$

ii) $p(-\infty < x < 10) = \int_{-\infty}^{10} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{10} f(x) dx$
 $= \int_0^{10} \frac{1}{5} e^{-x/5} dx = \frac{1}{5} \left(\frac{e^{-x/5}}{-1/5} \right) \Big|_0^{10} = - (e^{-x/5}) \Big|_0^{10}$
 $= - (e^{-10/5} - 1) = - (e^{-2} - 1) = 1 - \frac{1}{e^2}$
 $= 1 - 0.1353 = \underline{0.8647}$

iii) $p(x \leq 0 \text{ or } x \geq 1) = p(x \leq 0) + p(x \geq 1)$
 $= \int_0^0 f(x) dx + \int_1^{\infty} f(x) dx = 0 + \int_1^{\infty} \frac{1}{5} e^{-x/5} dx$
 $= \frac{1}{5} \left(\frac{e^{-x/5}}{-1/5} \right) \Big|_1^{\infty} = - (e^{-x/5}) \Big|_1^{\infty} = - (e^{-\infty} - e^{-1/5})$
 $= - (0 - e^{-1/5}) = e^{-1/5}$
 $= \underline{0.8187}$

- 4) The length of telephone conversation in a booth has been an exponential distribution and found on an average to be 5 minutes. Find the prob. that a random call made from this booth
- ends less than 5 minutes.
 - between 5 and 10 minutes.

we have ED $f(x) = \alpha e^{-\alpha x}$, $x > 0$

Mean = 5 by data $\frac{1}{\alpha} = 5 \Rightarrow \alpha = \frac{1}{5}$

Hence $f(x) = \alpha e^{-\alpha x} = \frac{1}{5} e^{-x/5}$ is the p.d.f

$$\begin{aligned}
 \text{i)} \quad P(x < 5) &= \int_0^5 f(x) dx = \int_0^5 \frac{1}{5} e^{-x/5} dx \\
 &= \frac{1}{5} \left[\frac{e^{-x/5}}{-1/5} \right]_0^5 = - \left(e^{-x/5} \right)_0^5 = - (e^{-1} - 1) \\
 &= 1 - \frac{1}{e} \\
 &= 1 - 0.3679 \\
 &= \underline{\underline{0.6321}}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii)} \quad P(5 < x < 10) &= \int_5^{10} \frac{1}{5} e^{-x/5} dx \\
 &= \frac{1}{5} \left[\frac{e^{-x/5}}{-1/5} \right]_5^{10} = - \left(e^{-x/5} \right)_5^{10} \\
 &= - (e^{-10/5} - e^{-5/5}) = - (e^{-2} - e^{-1}) \\
 &= \frac{1}{e} - \frac{1}{e^2} \\
 &= 0.3679 - 0.1353 \\
 &= \underline{\underline{0.2326}}
 \end{aligned}$$

5) In a certain town the duration of a shower is exponentially distributed with mean 5 minutes. what is the prob that a shower will last for.

i) 10 minutes or more

ii) less than 10 minutes

iii) b/w 10 and 12 minutes.

>> The p.d.f of the exponential distribution is given by
 $f(x) = \lambda e^{-\lambda x}$, $\lambda > 0$ and mean $= \frac{1}{\lambda} = 5 \Rightarrow 5 = \frac{1}{\lambda}$
 $\Rightarrow \lambda = 1/5$

$$\therefore f(x) = \frac{1}{5} e^{-x/5}$$

$$\begin{aligned} \text{i) } P(x \geq 10) &= \int_{10}^{\infty} \frac{1}{5} e^{-x/5} dx = \frac{1}{5} \frac{e^{-x/5}}{-1/5} = -(e^{-x/5}) \Big|_{10}^{\infty} \\ &= -(e^{-\infty/5} - e^{-10/5}) = -(0 - e^{-2}) = e^{-2} = \frac{1}{e^2} \\ &= \underline{\underline{0.1353}} \end{aligned}$$

$$\begin{aligned} \text{ii) } P(x < 10) &= \int_0^{10} \frac{1}{5} e^{-x/5} dx = -(e^{-x/5}) \Big|_0^{10} = -(e^{-2} - 1) \\ &= 1 - \frac{1}{e^2} = \underline{\underline{0.8647}} \end{aligned}$$

$$\begin{aligned} \text{iii) } P(10 < x < 12) &= -(e^{-x/5}) \Big|_{10}^{12} \\ &= -(e^{-12/5} - e^{-2}) \\ &= e^{-2} - e^{-12/5} \\ &= \frac{1}{e^2} - \frac{1}{e^{12/5}} \\ &= \underline{\underline{0.0446}} \end{aligned}$$

Normal Distribution

(40)

The continuous Prob distribution having the Prob density funⁿ $f(x)$ given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

where $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\sigma > 0$ is known as the normal distribution.

Evidently $f(x) \geq 0$

$$\int_{-\infty}^{\infty} f(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$\text{put } t^2 = (x-\mu)^2/2\sigma^2$$

$$\Rightarrow t = (x-\mu)/\sqrt{2}\sigma$$

$$x = \mu + \sqrt{2}\sigma t$$

$$dx = \sqrt{2}\sigma dt$$

$$t \rightarrow -\infty \text{ to } \infty$$

$$\begin{aligned} \text{Hence } \int_{-\infty}^{\infty} f(x) dx &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\sigma dt \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt \\ &= \frac{1}{\sqrt{\pi}} \cdot 2 \int_0^{\infty} e^{-t^2} dt \end{aligned}$$

But $\int_0^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$ by gamma function.

$$\therefore \int_{-\infty}^{\infty} f(x) dx = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1$$

Thus $f(x)$ represents a Prob density funⁿ

Mean and Standard Deviation of the Normal D.

$$\text{Mean } (\mu) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-(x-\mu)^2/2\sigma^2} dx$$

$$\text{put } t^2 = \frac{(x-\mu)^2}{2\sigma^2} \Rightarrow t = \frac{(x-\mu)}{\sqrt{2}\sigma}$$

$$\Rightarrow x = \mu + \sqrt{2}\sigma t$$

$$dx = \sqrt{2}\sigma dt \quad t \rightarrow -\infty \text{ to } \infty$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sqrt{2}\sigma t) e^{-t^2} \sqrt{2}\sigma dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\mu + \sqrt{2}\sigma t) e^{-t^2} dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mu e^{-t^2} dt + \sqrt{2}\sigma \int_{-\infty}^{\infty} t e^{-t^2} dt$$

$$= \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} t e^{-t^2} dt$$

$$= \frac{\mu}{\sqrt{\pi}} \cdot 2 \int_0^{\infty} e^{-t^2} dt + 0$$

Here 2nd integral is zero by a standard property since $t e^{-t^2}$ is an odd funⁿ

$$= \frac{2\mu}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} \quad \text{Gamma funⁿ}$$

$$= \mu$$

Thus mean = μ

Hence we can say that the mean of the normal distribution is equal to the mean of the given distribution.

$$\text{Variance } (\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 \cdot e^{-(x-\mu)^2/2\sigma^2} dx$$

put $t^2 = \frac{(x-\mu)^2}{2\sigma^2} \Rightarrow t = \frac{(x-\mu)}{\sqrt{2}\sigma}$

$x = \mu + \sqrt{2}\sigma t \quad t \rightarrow -\infty \text{ to } \infty$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sigma^2 t^2 e^{-t^2} \sqrt{2}\sigma dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 2\sigma^2 t^2 e^{-t^2} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2} dt$$

Now $= \frac{4\sigma^2}{\sqrt{\pi}} \int_0^{\infty} t^2 e^{-t^2} dt$

Now $\int_0^{\infty} t^2 e^{-t^2} dt = \left[t^2 e^{-t^2} \right]$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot 2 \int_0^{\infty} t^2 e^{-t^2} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} t(2t e^{-t^2}) dt$$

wbv $\int u v dt = u \int v dt - \int \left(\int v dt \right) \cdot u' dt$

taking $u = t$, $v = 2t e^{-t^2}$ and noting that $\int v dt = -e^{-t^2}$ we now

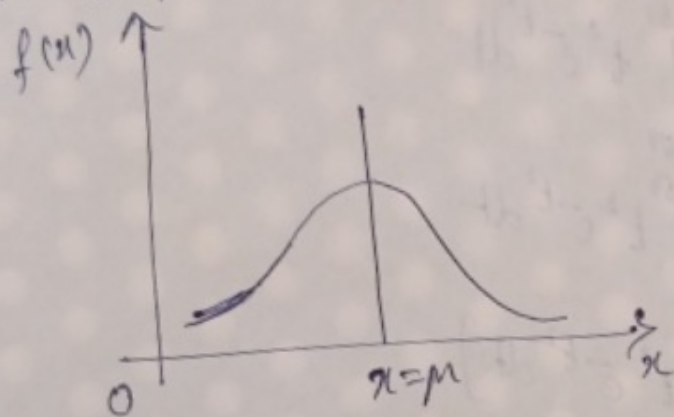
$$= \frac{2\sigma^2}{\sqrt{\pi}} \left\{ \left[t(-e^{-t^2}) \right]_0^{\infty} - \int_0^{\infty} -(e^{-t^2}) \cdot 1 \cdot dt \right\}$$

$$\text{Variance} = \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt = \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \sigma^2$$

Thus $\boxed{\text{Variance} = \sigma^2}$

Hence we can say that the Variance / s.d of the ND is equal to the Variance / s.d of the given distribution.

Note: - The graph of the prob. funⁿ $f(x)$ is a bell shaped curve symmetrical about the line $x = \mu$ and is called the normal prob. curve. The shape of the curve is as follows.



The line $x = \mu$ divides the total area under the curve which is equal to 1 into two equal parts. The area to the right as well as to the left of the line $x = \mu$ is 0.5.

Standard Normal Distribution.

(42)

* $Z = \frac{x - \mu}{\sigma}$ is called the standard normal variate (S.N.V)

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} dz$$

area under the standard normal curve from $Z=0$ to z

Note: Standard Results.

$$1) P(-\infty \leq Z \leq \infty) = 1$$

$$2) P(-\infty \leq Z \leq 0) = 1/2$$

$$3) P(0 \leq Z \leq \infty) \text{ or } P(Z \geq 0) = \frac{1}{2}$$

$$4) P(-\infty < Z < z_1) = P(-\infty < Z \leq 0) + P(0 \leq Z \leq z_1)$$

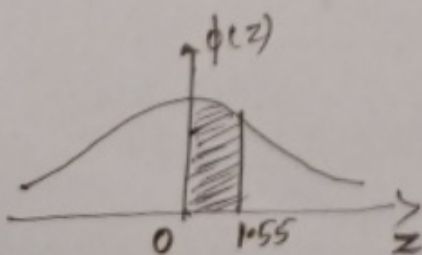
$$\Rightarrow P(Z < z_1) = 0.5 + \phi(z_1)$$

$$5) P(Z > z_2) = P(Z \geq 0) - P(0 \leq Z < z_2)$$

$$P(Z > z_2) = 0.5 - \phi(z_2)$$

Illustration - 1

To find the area under the standard normal curve b/w $Z=0$ and 1.55



$$\begin{array}{r} 1.55 \Rightarrow 1.5 \\ 0.05 \\ \hline 1.55 \end{array}$$

Theoretically the area

$$\frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} dz = \phi(z)$$

$$\frac{1}{\sqrt{2\pi}} \int_0^{1.55} e^{-z^2/2} dz = \phi(1.55)$$

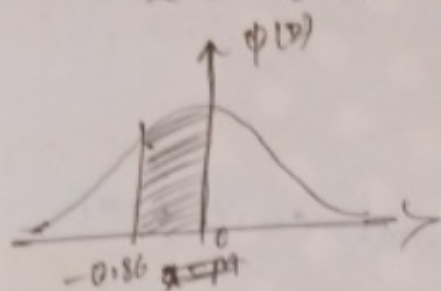
$$\Rightarrow \phi(1.55) = 0.4394$$

equivalently we have

$$P(0 \leq Z \leq 1.55) = \underline{\underline{0.4394}}$$

EX - (2)

Area of the standard normal curve b/w
 $z = -0.86$ and $z = 0$



$$\text{Area} = \frac{1}{\sqrt{2\pi}} \int_{-0.86}^0 e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{0.86} e^{-z^2/2} dz \quad \text{by Symmetry}$$

$$0.86 = \frac{0.8}{0.06} = \frac{0.86}{0.86}$$

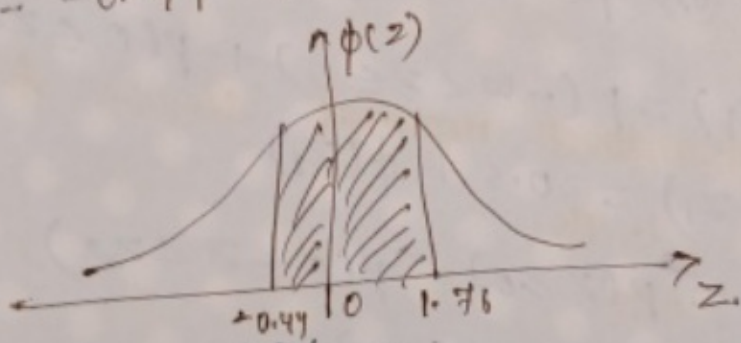
$$\therefore \text{Area} = \phi(0.86) = 0.3051$$

Equivalently

$$P(-0.86 \leq z \leq 0) = 0.3051$$

EX - (3)

Area of the standard normal curve b/w
 $z = -0.44$ and $z = 1.76$



$$\text{Area} = \frac{1}{\sqrt{2\pi}} \int_{-0.44}^{1.76} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-0.44}^0 e^{-z^2/2} dz + \int_0^{1.76} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{0.44} e^{-z^2/2} dz + \int_0^{1.76} e^{-z^2/2} dz \quad \text{by Symmetry}$$

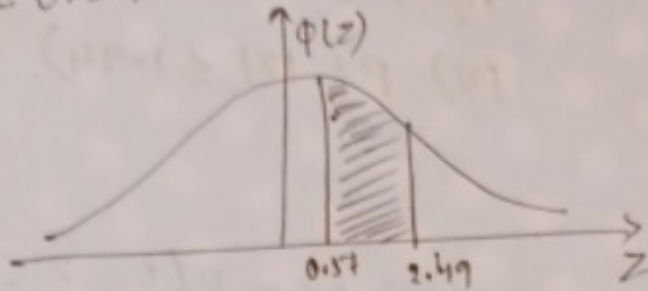
$$= \phi(0.44) + \phi(1.76)$$

$$= 0.1700 + 0.4608 = 0.6308$$

Equivalently $P(-0.44 \leq z \leq 1.76) = \underline{0.6308}$

Ex-4

Area of the standard normal curve b/w
 $z = 0.57$ to $z = 2.49$



$$\text{Required area} = (\text{Area b/w } z=0 \text{ to } 2.49) - (\text{Area b/w } z=0 \text{ to } 0.57)$$

$$= \phi(2.49) - \phi(0.57)$$

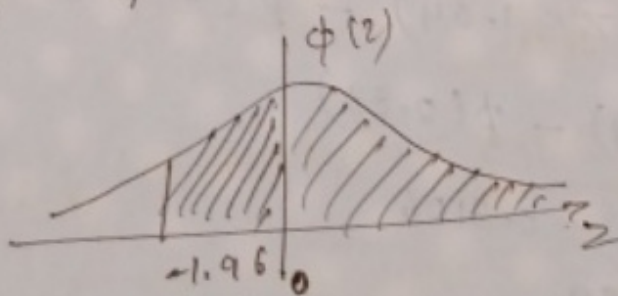
$$= 0.4936 - 0.2157$$

$$= 0.2779$$

Equivalently $P(0.57 \leq z \leq 2.49) = \underline{\underline{0.2779}}$

Ex-5

Area of the standard normal curve to the
 right of $z = -1.96$



$$\text{Required area} = (\text{Area b/w } z = -1.96 \text{ to } 0) + (\text{Area to the right of } z = 0)$$

$$= (\text{Area b/w } z=0 \text{ to } 1.96) + \text{ " by symmetry }$$

$$= \phi(1.96) + 0.5 = 0.4750 + 0.5 = 0.9750$$

Equivalently $P(z \geq -1.96) = \underline{\underline{0.975}}$

Evaluate the foll^g prob with the help of normal prob table.

i) $P(Z \geq 0.85)$

ii) $P(-1.64 \leq Z \leq -0.88)$

iii) $P(Z \leq -2.43)$

iv) $P(|Z| \leq 1.94)$

Proof i) $P(Z \geq 0.85)$

wkt $P(Z > Z_2) = P(Z > 0) - P(0 \leq Z \leq Z_2)$

$$\Rightarrow P(Z \geq 0.85) = P(Z > 0) - P(0 \leq Z \leq 0.85)$$

$$= 0.5 - \phi(0.85)$$

$$= 0.5 - 0.3023$$

$$= 0.1977$$

$$\therefore P(Z \geq 0.85) = \underline{0.1977}$$

ii) $P(-1.64 \leq Z \leq -0.88)$

By Symmetry $P(-1.64 \leq Z \leq -0.88) = P(0.88 \leq Z \leq 1.64)$

$$= P(0 \leq Z \leq 1.64) - P(0 \leq Z \leq 0.88)$$

$$= \phi(1.64) - \phi(0.88)$$

$$= 0.4495 - 0.3106$$

$$= 0.1389$$

$$\therefore P(-1.64 \leq Z \leq -0.88) = \underline{0.1389}$$

$$iii) P(Z \leq -2.43)$$

(44)

$$\text{wtr } P(Z \leq -2.43)$$

$$P(Z \leq -2.43) = P(Z \geq 2.43)$$

$$\text{wtr } P(Z \geq Z_2) = P(Z \geq 0) - P(0 \leq Z < Z_2)$$

$$= P(Z \geq 0) - P(0 \leq Z < 2.43)$$

$$= 0.5 - \phi(2.43)$$

$$= 0.5 - 0.4925$$

$$= 0.0075$$

$$\therefore P(Z \leq -2.43) = \underline{\underline{0.0075}}$$

$$iv) P(|Z| \leq 1.94) = P(-1.94 \leq Z \leq 1.94)$$

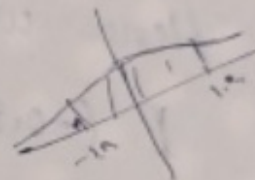
$$= 2P(0 \leq Z \leq 1.94)$$

$$= 2\phi(1.94)$$

$$= 2(0.4738)$$

$$= \underline{\underline{0.9476}}$$

$$\therefore P(|Z| \leq 1.94) = \underline{\underline{0.9476}}$$



2) if x is a normal variate with mean 30 and s.d 5 find the prob that
i) $26 \leq x \leq 40$ ii) $x \geq 45$

>> we have standard normal variate
(S.N.V) $\Rightarrow Z = \frac{x - \mu}{\sigma} = \frac{x - 30}{5}$

$$Z = \frac{x - 30}{5}$$

$$\Rightarrow x - 30 = 5Z$$

$$\Rightarrow \boxed{x = 30 + 5Z} \rightarrow \textcircled{1}$$

i) To find $P(26 \leq x \leq 40)$

if $x = 26$, $Z = -0.8$

if $x = 40$, $Z = 2$

Hence we need to find

$$P(-0.8 \leq Z \leq 2)$$

$$\begin{aligned} \textcircled{1} \Rightarrow 26 &= 30 + 5Z \\ 26 - 30 &= 5Z \\ -4 &= 5Z \\ Z &= -4/5 = -0.8 \end{aligned}$$

$$\begin{aligned} \text{Thus } P(-0.8 \leq Z \leq 2) &= P(-0.8 \leq Z \leq 0) + P(0 \leq Z \leq 2) \\ &= P(0 \leq Z \leq 0.8) + P(0 \leq Z \leq 2) \\ &= \phi(0.8) + \phi(2) \\ &= 0.2881 + 0.4772 \\ &= 0.7653 \end{aligned}$$

$$\therefore \underline{P(26 \leq x \leq 40) = 0.7653}$$

ii) To find $P(x \geq 45)$

if $x=45$, $z=3$ and hence we have to find

$$P(z \geq 3) = P(z \geq 0) - P(0 \leq z \leq 3)$$

$$= 0.5 - \phi(3)$$

$$= 0.5 - 0.4987$$

$$= 0.0013$$

$$\therefore \underline{P(x \geq 45) = 0.0013}$$

3) if x is normally distributed with mean 12 and S.D 4, find the foll^g.

i) $P(x \geq 20)$ ii) $P(x \leq 20)$

>> we've S.N.V $z = \frac{x - \mu}{\sigma} = \frac{x - 12}{4}$

$$\Rightarrow \boxed{x = 12 + 4z}$$

If $x=20$, $z=2$

we've to find $P(z \geq 2)$ and $P(z \leq 2)$

Now $P(z \geq 2) = P(z \geq 0) - P(0 \leq z \leq 2)$

$$= 0.5 - \phi(2) = 0.5 - 0.4772 = \underline{0.0228}$$

Also $P(z \leq 2) = P(-\infty \leq z \leq 0) + P(0 \leq z \leq 2)$

WbT result $= 0.5 + \phi(2)$

$$= 0.5 + 0.4772$$

$$= \underline{0.9772}$$

Thus $\underline{P(x \geq 20) = 0.0228}$ and $\underline{P(x \leq 20) = 0.9772}$

4) The marks of 1000 students in an examination follows a normal distribution with mean 70 and standard deviation 5. Find the no. of students whose marks will be

i) less than 65

ii) more than 75

iii) b/w 65 and 75

>> let x represent the marks of students.

By data $\mu = 70$, $\sigma = 5$

$$\text{Hence S.N.V. } z = \frac{x - \mu}{\sigma} = \frac{x - 70}{5}$$

$$\boxed{x = 70 + 5z}$$

i) $P(x < 65)$

if $x = 65$, $z = -1$ and hence we have to find

$$P(z < -1)$$

$$\begin{aligned} \Rightarrow P(z < -1) &= P(z > 1) \\ &= P(z \geq 0) - P(0 \leq z \leq 1) \\ &= 0.5 - \phi(1) \\ &= 0.5 - 0.3413 = 0.1587 \end{aligned}$$

$\therefore$ no. of students scoring less than 65 marks

$$= 1000 \times 0.1587 = 158.7 \approx \underline{159}$$

ii) if $x = 75$, $z = 1$ we need to find $P(z > 1)$

$$\begin{aligned} P(z > 1) &= P(z \geq 0) - P(0 < z < 1) \\ &= 0.5 - \phi(1) = 0.1587 \end{aligned}$$

$\therefore$ no. of students scoring more than 75 marks

$$= 1000 \times 0.1587 = 158.7 \approx \underline{159}$$

iii) we've to find

(46)

$$\begin{aligned}P(-1 < Z < 1) &= 2P(0 < Z < 1) \\&= 2\phi(1) \\&= 2(0.3413) \\&= 0.6826\end{aligned}$$

$\therefore$ no of students scoring marks b/w 65 and 75
 $= 1000 \times 0.6826 = 682.6 \approx \underline{683}$

5) In a normal distribution - 31% of the .

In a test on electric bulbs, it was found that the life time of a particular brand was distributed normally with an average life of 2000 hours and S.D of 60 hours.

if a firm purchased 1500 bulbs find the no of bulbs that are likely to last for

i) more than 2100 hours.

ii) less than 1960 hours

iii) b/w 1900 to 2100 hours.

>> By data $\mu = 2000$, $\sigma = 60$

$$\text{S.N.V} \quad Z = \frac{x - \mu}{\sigma} = \frac{x - 2000}{60}$$

$$\Rightarrow Z = \frac{x - 2000}{60}$$

$$\boxed{x = 2000 + 60Z}$$

i) To find $P(X > 2100)$

if $x = 2100$, $z = 1.67$

$$P(X > 2100) = P(Z > 1.67)$$

$$= P(Z \geq 0) - P(0 < Z < 1.67)$$

$$= 0.5 - \phi(1.67)$$

$$= 0.5 - 0.4525$$

$$= 0.0475$$

$\therefore$ no. of bulbs that are likely to last for more than 2100 hour is

$$2500 \times 0.0475 = 118.75 \approx 119$$

ii) To find $P(X < 1950)$

if $x = 1950$, $z = -0.83$

$$P(X < -0.83) = P(Z > 0.83)$$

$$= P(Z \geq 0) - P(0 < Z < 0.83)$$

$$= 0.5 - \phi(0.83)$$

$$= 0.5 - 0.2967$$

$$= 0.2033$$

$\therefore$ no. of bulbs that are likely to last for less than 1950 hour is

$$2600 \times 0.2033 = 508.25 \approx \underline{508}$$

$$x = 2000 + 60z$$

$$2100 - 2000 = 60z$$

$$100 = 60z$$

$$z = \frac{100}{60} = 1.67$$

iii) To find $P(1900 < x < 2100)$

(47)

i.e. if $x = 1900$, $z = -1.67$

if $x = 2100$, $z = 1.67$

$$P(-1.67 < z < 1.67) = 2 P(0 < z < 1.67)$$

$$= 2 \phi(1.67)$$

$$= 2 (0.4525)$$

$$= \underline{\underline{0.905}}$$

$\therefore$ no. of bulbs that are likely to last b/w
1900 and 2100 hours

$$= 2500 \times 0.905 = 2262.5 \approx \underline{\underline{2263}}$$

ex) In a normal distribution 31% of the items
are under 45 and 8% of the items are over 64.
Find the mean and S.D of the distribution.

>> Let μ and σ be the mean and S.D of the
normal distribution

By data $P(x < 45) = 0.31$ and $P(x > 64) = 0.08$

we have S.N.V $z = \frac{x - \mu}{\sigma}$

$$\text{when } x = 45, \quad z = \frac{45 - \mu}{\sigma} = z_1 \text{ (say)}$$

$$x = 64, \quad z = \frac{64 - \mu}{\sigma} = z_2 \text{ (say)}$$

Thus we have

$$P(Z < z_1) = 0.31 \text{ and } P(Z > z_2) = 0.08$$

$$\text{i.e. } 0.5 + \phi(z_1) = 0.31 \text{ and } 0.5 - \phi(z_2) = 0.08$$

$$\Rightarrow \phi(z_1) = -0.19 \text{ and } \phi(z_2) = 0.42$$

Referring to the normal probability tables we have

$$0.1915 (\approx 0.19) = \phi(0.5) \text{ and } 0.4992 (\approx 0.42) = \phi(1.4)$$

$$\therefore \phi(z_1) = -\phi(0.5) \text{ and } \phi(z_2) = \phi(1.4)$$

$$\Rightarrow z_1 = -0.5 \text{ and } z_2 = 1.4$$

$$\text{i.e. } \frac{45 - \mu}{\sigma} = -0.5 \text{ and } \frac{64 - \mu}{\sigma} = 1.4$$

$$\begin{array}{l|l} 45 - \mu = -0.5\sigma & 64 - \mu = 1.4\sigma \\ -\mu + 0.5\sigma = -45 & -\mu + 1.4\sigma = -64 \end{array}$$

$$\Rightarrow \begin{array}{l} -\mu + 0.5\sigma = -45 \\ -\mu + 1.4\sigma = -64 \end{array}$$

Two equations 2 unknowns
- known is given

$$\boxed{\mu = 50} \quad \boxed{\sigma = 10} \text{ on solving we get}$$

Thus mean = 50 and S.D = 10

$$\begin{array}{l} 0.19 = \phi(0.5) \\ 0.42 = \phi(1.4) \end{array}$$

ex) In a Examination 7% of student's score less than 35% marks and 89% of student's score less than 60% marks. Find the mean and SD if the marks are normally distributed, It is given that if $\frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} dz$ then $P(1.2263) = 0.39$ and $P(1.4757) = 0.43$

>> Let μ and σ be the mean and S.D of N.D
By data we have $P(X < 35) = 0.07$, $P(X < 60) = 0.89$

$$S.N.V \Rightarrow Z = \frac{X - \mu}{\sigma}$$

$$\text{when } X = 35, Z = \frac{35 - \mu}{\sigma} = Z_1 \text{ (say)}$$

$$X = 60, Z = \frac{60 - \mu}{\sigma} = Z_2 \text{ (say)}$$

Hence we have

$$P(Z < Z_1) = 0.07 \quad \text{and} \quad P(Z < Z_2) = 0.89$$

$$0.5 + \phi(Z_1) = 0.07 \quad \text{and} \quad 0.5 + \phi(Z_2) = 0.89$$

$$\phi(Z_2) = 0.39$$

$$\phi(Z_1) = -0.43$$

Using the given data in the RHS of these we have

$$\phi(Z_1) = -\phi(1.4757) \quad \text{and} \quad \phi(Z_2) = \phi(1.2263)$$

$$\Rightarrow Z_1 = -1.4757 \quad \text{and} \quad Z_2 = 1.2263$$

$$\text{ie. } \frac{35 - \mu}{\sigma} = -1.4757 \quad \text{and} \quad \frac{60 - \mu}{\sigma} = 1.2263$$

$$35 - \mu = -1.4757\sigma \quad \left| \quad 60 - \mu = 1.2263\sigma \right.$$

$$\Rightarrow -\mu + 1.4757\sigma = 35 \quad \left| \quad -\mu + 1.2263\sigma = -60 \right.$$

$$\therefore \boxed{\mu = 48.65} \quad \text{and} \quad \boxed{\sigma = 9.25} \quad \text{on solving}$$

$$\text{Thus } \boxed{\text{mean} = 48.65} \quad \text{and} \quad \boxed{\text{S.D} = 9.25}$$