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LECTURE NOTES

TRANSFORM CALCULUS , FOURIER SERIES & NUMERICAL TECHNIQUES

(21MAT31)

Module-1

The Laplace Transformation & Applications

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INTRODUCTION :-

A function can be shifted from one domain to another domain without changing its actual Value is Called laplace transformation.

Mathematically Suppose let the $F(t)$ be a real Value function defined over the interval $[-\infty, +\infty]$, then for any Kernal of a transformation. $K(s,t)$ Such that, the transformation of $F(t)$ can be defined as,

$$T[f(t)] = \int_{-\infty}^{\infty} K(s,t)\, F(t)\,dt = f(s) \longrightarrow ①$$

Where 'T' denotes a transformation operator at any time 't' and 's' be a parameters, which may be real or Complex,

Where, the Kernal of transformation, we have

$$K[s,t] = \begin{cases} e^{-st}, & t > 0 \\ 0, & t < 0 \end{cases}$$

$$(1) \Rightarrow T[F(t)] = \int_{-\infty}^{0} K(s,t).F(t)\,dt + \int_{0}^{\infty} K(s,t)\,F(t)\,dt$$

$$= \int_{-\infty}^{0} 0.\,F(t)\,dt + \int_{0}^{\infty} K(s,t)\,F(t)\,dt$$

$$\therefore T[F(t)] = \int_{0}^{\infty} e^{-st}\,F(t)\,dt$$

$$\therefore \boxed{L\,F(t) = \int_{0}^{\infty} e^{-st}\,F(t)\,dt = f(s)}$$

is Called the Laplace transformation of F(t) and it is defined as. where L, indicates the Laplace transformation. whereas $K(s,t) = e^{ist}$, $-\infty < t < \infty$

$$\therefore T[F(t)] = \int_{-\infty}^{\infty} e^{ist}\,F(t)\,dt = F[F(t)]$$

is Called fourier transformation.

⇒ Notations:-

1] Suppose F(t), g(t), H(t).... be the functions, their Laplace transforms can be denoted by f(s), g(s), h(s).....

2] The function are f(t), g(t), h(t).... then their Laplace transforms can be notated by $\bar{f}(s)$, $\bar{g}(s)$, $\bar{h}(s)$.......
(or) F(s), g(s), h(s).

Some Important result of Laplace transform:-

(1) $F(t) = 1$

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt$$

$$L[1] = \int_0^\infty e^{-st} \cdot 1\,dt$$

$$L[1] = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[e^{-st}\right]_0^\infty$$

$$L[1] = -\frac{1}{s}\left[e^{-\infty} - e^{0}\right]$$

$$= -\frac{1}{s}[0-1] = \frac{1}{s}$$

$$\boxed{L[1] = \frac{1}{s}}$$

(2) $F(t) = e^{at}$

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt$$

$$L[e^{at}] = \int_0^\infty e^{-st} \cdot e^{at}\,dt$$

$$L[e^{at}] = \int_0^\infty e^{-(s-a)t}\,dt$$

$$L[e^{at}] = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{-1}{s-a}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{-1}{s-a}[0-1]$$

$$\boxed{L[e^{at}] = \frac{1}{s-a}}$$

(3) $F(t) = e^{-at}$

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt = \int_0^\infty e^{-st} e^{-at} dt$$

$$L[e^{-at}] = \int_0^\infty e^{-(s+a)t} dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{-1}{s+a}\left[e^{-\infty} - e^0\right]$$

$$\boxed{L[e^{-at}] = \frac{1}{s+a}}$$

(4) Let $F(t) = e^{iat}$

$$F(t) = \cos(at) + i\sin(at)$$

$$L[F(t)] = L[\cos(at)] + i\,L[\sin(at)] \longrightarrow (1)$$

$$\therefore L[e^{iat}] = \frac{1}{s-ai}$$

$$= \frac{s+ai}{(s-ai)(s+ai)}$$

$$= \frac{s+ai}{s^2-a^2i^2}$$

$$L[e^{iat}] = \frac{s+ai}{s^2+a^2}$$

$$L[\cos(at)] + i\,L[\sin(at)] = \frac{s}{s^2+a^2} + i\,\frac{a}{s^2+a^2}$$

$$\therefore L[\cos(at)] = \frac{s}{s^2+a^2}$$

$$L[\sin(at)] = \frac{a}{s^2+a^2}$$

(5) Let $F(t) = \cosh(at)$

$$\Rightarrow F(t) = \frac{e^{at} + e^{-at}}{2}$$

$$\Rightarrow L[F(t)] = \frac{1}{2} L\left[e^{at} + e^{-at}\right]$$

$$= \frac{1}{2}\left[L[e^{at}] + L[e^{-at}]\right]$$

$$\Rightarrow f(s) = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+\cancel{a}+s-\cancel{a}}{(s-a)(s+a)}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{s^2-a^2}\right]$$

$$\therefore \boxed{L\cosh(at) = \frac{s}{s^2-a^2}}$$

(6) Let $F(t) = \sinh(at)$

$$F(t) = \frac{e^{at} - e^{-at}}{2}$$

$$= \frac{1}{2}\left[L[e^{at} - e^{-at}]\right]$$

$$= \frac{1}{2}\left[L[e^{at}] - L[e^{-at}]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right] = \frac{2a}{s^2-a^2}$$

$$= \frac{1}{\cancel{2}} \times \frac{\cancel{2}a}{s^2-a^2}$$

$$\Rightarrow \boxed{L[\sinh[at]] = \frac{a}{s^2-a^2}}$$

7] $L(t^n) = \dfrac{n!}{s^{n+1}}$, $\forall\, n \in Z^+$

$= \dfrac{\sqrt{n+1}}{s^n+1}$

8] Changing of Scale Property.

If $L[f(t)] = f(s)$, $L[G(t)] = g(s)$, Then for any constant, c_1 and c_2, then

$$L[c_1 F(t) \pm c_2 G(t)] = c_1 L[F(t)] \pm c_2 L[G(t)]$$

9] If $L[F(t)] = f(s)$, then $L[e^{at} F(t)] = f(s-a)$ and

$$L[e^{-at} F(t)] = f(s+a)$$

10] If $L[F(t)] = f(s)$, then $L[t^n F(t)] = (-1)^n \dfrac{d^n f(s)}{ds^n}$.

11] If $L[F(t)] = f(s)$, then $L\left[\dfrac{F(t)}{t}\right] = \int_s^\infty f(s)\,ds$.

Find the Laplace transform of the following function.

i] Sin3t

Let $F(t) = \sin 3t$

$L[F(t)] = L[\sin 3t]$

$\Rightarrow \boxed{f(s) = \dfrac{3}{s^2+9}}$

ii] $\cos^2 2t$

Let $F(t) = \cos^2 2t$

$\Rightarrow F(t) = \dfrac{1+\cos 2(2t)}{2}$

$\Rightarrow F(t) = \dfrac{1+\cos 4t}{2}$

$\Rightarrow L[F(t)] = \dfrac{1}{2} L[1+\cos 4t]$

$= \dfrac{1}{2}\left[L[1] + L[\cos 4t]\right]$

$\Rightarrow \boxed{f(s) = \dfrac{1}{2}\left[\dfrac{1}{s} + \dfrac{s}{s^2+16}\right]}$

iii] Sin t

Let $F(t) = \text{Sinh } t$

$L[F(t)] = L[\text{Sinh}3t]$

$\Rightarrow f(s) = \frac{3}{s^2-9}$

iv] Cosh4t

Let $F(t) = \text{Cosh}4t$

$\Rightarrow L[F(t)] = L[\text{Cosh}4t]$

$\Rightarrow f(s) = \frac{s}{s^2-16}$

v] Sin5t . Cos2t

$F(t) = \text{Sin}5t . \text{Cos}2t$

$F(t) = \frac{1}{2}\left[2\text{Sin}5t . \text{Cos}2t\right]$

$\Rightarrow F(t) = \frac{1}{2}\left[\text{Sin}(5t+2t) + \text{Sin}(5t-2t)\right]$

$= \frac{1}{2}\left[\text{Sin}7t + \text{Sin}3t\right]$

$= \frac{1}{2}\left[L(\text{Sin}7t) + L(\text{Sin}3t)\right]$

$\Rightarrow f(s) = \frac{1}{2}\left[\frac{7}{s^2+49} + \frac{3}{s^2+9}\right]$

vi] Sint . Sin2t . Sin3t

$F(t) = \underbrace{\text{Sin}t . \text{Sin}2t}\ \text{Sin}3t$

$= \frac{1}{2}\left[2\text{Sin}t . \text{Sin}2t\right]\text{Sin}3t$

$= \frac{1}{2}\left[\text{Cos}t - \text{Cos}3t\right]\text{Sin}3t$

$= \frac{1}{2}\left[\text{Cos}t\,\text{Sin}3t - \text{Cos}3t\,\text{Sin}3t\right]$

$= \frac{1}{4}\left[2\text{Sin}3t\,\text{Cos}t - 2\text{Sin}3t\,\text{Cos}3t\right]$

$= \frac{1}{4}\left[\text{Sin}4t + \text{Sin}2t - \text{Sin}6t\right]$

$L[f(t)] = \frac{1}{4}\left\{L[\text{Sin}4t] + L[\text{Sin}2t] - L[\text{Sin}6t]\right\}$

$= \frac{1}{4}\left[\frac{4}{s^2+16} + \frac{2}{s^2+4} - \frac{6}{s^2+36}\right]$ //

vii] Cost. Cos2t. Cos3t

$$F(t) = \underbrace{\cos t \cdot \cos 2t} \cdot \cos 3t$$

$$= \frac{1}{2}\left[2\cos t \cdot \cos 2t\right] \cdot \cos 3t$$

$$= \frac{1}{2}\left[2\cos 2t \cdot \cos t\right] \cos 3t$$

$$= \frac{1}{2}\left[\cos 3t + \cos t\right] \cos 3t$$

$$= \frac{1}{2}\left[\cos 3t \cos 3t + \cos t \cos 3t\right]$$

$$= \frac{1}{4}\left[2\cos 3t \cos 3t + 2\cos 3t \cos t\right]$$

$$= \frac{1}{4}\left[\cos 6t + 1 + \cos 4t + \cos 2t\right]$$

$$\Rightarrow L[f(t)] = \frac{1}{4}\left\{L[\cos 6t] + L[1] + L[\cos 4t] + L[\cos 2t]\right\}$$

$$\Rightarrow f(s) = \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{1}{s} + \frac{s}{s^2+16} + \frac{s}{s^2+4}\right]$$

viii] $(3t+4)^3 + 5^t$

$$= (3t)^3 + 4^3 + 3(3t)(4)^2 + 3(3t)^2(4) + 5^t$$

$$F(t) = 27t^3 + 64 + 108t^2 + 144t + e^{(\log_e 5)t}$$

$$L[F(t)] = 27L[t^3] + 64L[1] + 108L[t^2] + 144L[t] + L[e^{\log 5t}]$$

$$\Rightarrow f(s) = 27\left[\frac{3!}{s^4}\right] + \frac{1 \times 64}{s} + 108\left[\frac{2!}{s^3}\right] + 144\left[\frac{1!}{s^2}\right] + \frac{1}{s-\log 5}$$

$$\Rightarrow f(s) = \frac{162}{s^4} + \frac{216}{s^3} + \frac{144}{s^2} + \frac{64}{s} + \frac{1}{s-\log 5}$$

(ix) $e^{-2t}\sinh 4t$

Let $F(t) = e^{-2t}\sinh 4t$

$$= e^{-2t}\left[\frac{e^{4t}-e^{-4t}}{2}\right]$$

$$= \frac{1}{2}\left[e^{2t}-e^{-6t}\right]$$

$$\Rightarrow L[F(t)] = \frac{1}{2}\left[L[e^{2t}] - L[e^{-6t}]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-2} - \frac{1}{s+6}\right]$$

$$\Rightarrow f(s) = \frac{1}{2}\left[\frac{1}{s-2} - \frac{1}{s+6}\right]$$

Find the Laplace transform of the following functions.

i] $e^{-2t}[2\cos 5t - \sin 5t]$

$\rightarrow$ Let $F(t) = 2\cos 5t - \sin 5t$

$$L[F(t)] = 2L[\cos 5t] - L[\sin 5t]$$

$$\Rightarrow 2\frac{s}{s^2+25} - \frac{5}{s^2+25}$$

$$f(s) = \frac{2s-5}{s^2+25}$$

WKT $L[e^{at}F(t)] = f(s-a)$

$$L[e^{-2t}\, 2\cos 5t - \sin 5t] = f(s+2)$$

$$f(s+2) = \frac{2(s+2)-5}{(s+2)^2+25}$$

$$= \frac{2s+4-5}{s^2+4+25+2s} = \frac{2s-1}{s^2+2s+29}$$

$$\Rightarrow L\left[e^{-2t}\,2\cos 5t - \sin 5t\right] = \frac{2s-1}{s^2+2s+29}$$

ii] $e^{-t}\cos^2 3t$

$$F(t) = \cos^2 3t$$

$$\Rightarrow \frac{1-\cos 2(3t)}{2} = \frac{1+\cos 6t}{2}$$

$$L[F(t)] = \frac{1}{2}\left[L[1] + L[\cos 6t]\right]$$

$$f(s) = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

W.KT $L[e^{at}F(t)] = f(s-a)$

$$L[e^{-t}F(t)] = f(s+1)$$

$$\Rightarrow f(s+1) = \frac{1}{2}\left[\frac{1}{s+1} + \frac{s+1}{(s+1)^2+36}\right]$$

iii] $e^{3t}\sin 5t \sin 3t.$

$$F(t) = \sin 5t \sin 3t$$

$$= \frac{1}{2}\left[2\sin 5t \sin 3t\right]$$

$$F(t) = \frac{1}{2}\left[\cos 2t - \cos 8t\right]$$

$$L[F(t)] = \frac{1}{2}\left[L[\cos 2t] - L[\cos 8t]\right]$$

$$f(s) = \frac{1}{2}\left[\frac{s}{s^2+4} - \frac{s}{s^2+64}\right]$$

WKT $L[e^{at}F(t)] = f(s-a)$

$$L[e^{3t}\sin 5t \sin 3t] = f(s-3)$$

$$\Rightarrow f(s-3) = \frac{1}{2}\left[\frac{s-3}{(s-3)^2+4} - \frac{(s-3)}{(s-3)^2+64}\right]$$

$$= \frac{1}{2}\left[\frac{s-3}{s^2+9-6s+4} - \frac{s-3}{s^2+9-6s+64}\right]$$

$$\Rightarrow f(s) = \frac{1}{2}\left[\frac{s-3}{s^2-6s+13} - \frac{s-3}{s^2-6s+73}\right]_{//}$$

iv] $e^{-3t} \sin 2t$

$$F(t) = \sin 2t$$

$$L[F(t)] = L[\sin 2t]$$

$$f(s) = \frac{2}{s^2+4}$$

WKT $L[e^{at} F(t)] = f(s-a)$

$$L[e^{-3t} \sin 2t] = f(s+3)$$

$$\Rightarrow f(s+3) = \frac{2}{(s+3)^2+4} = \frac{2}{s^2+9+6s+4}$$

$$= \frac{2}{s^2+6s+13}$$

v] $e^{-2t} \sin 3t \cos 5t$.

$$F(t) = \sin 3t \cos 5t$$

$$F(t) = \frac{1}{2}[2\sin 3t \cos 5t]$$

$$= \frac{1}{2}[\sin 8t + \sin(-2t)]$$

$$L[F(t)] = \frac{1}{2}[L(\sin 8t) - L(\sin 2t)]$$

$$f(s) = \frac{1}{2}\left[\frac{8}{s^2+64} - \frac{2}{s^2+4}\right]$$

WKT $L[e^{at} F(t)] = f(s-a)$

$$L[e^{-2t} \sin 3t \cos 5t] = f(s+2)$$

$$\Rightarrow f(s+2) = \frac{1}{2}\left[\frac{8}{(s+2)^2+64} - \frac{2}{(s+2)^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{8}{s^2+4+4s+64} - \frac{2}{s^2+4+4s+4}\right]$$

$$= \frac{1}{2}\left[\frac{8}{s^2+4s+68} - \frac{2}{s^2+4s+8}\right]$$

vi] $\left[\frac{4t+5}{e^{2t}}\right]^2$

$$\Rightarrow G(t) = \left[\frac{4t+5}{e^{2t}}\right]^2$$

$$= \frac{(4t+5)^2}{(e^{2t})^2} = \frac{(4t+5)^2}{e^{4t}}$$

$$= e^{-4t}(4t+5)^2 = G(t)$$

$$F(t) = (4t+5)^2$$

$$F(t) = 16t^2+25+40t$$

$$L[F(t)] = 16L[t^2] + 25L[1] + 40L[t]$$

$$f(s) = 16\left[\frac{2!}{s^3}\right] + 25\left[\frac{1}{s}\right] + 40\left[\frac{1}{s^2}\right]$$

$$f(s) = \frac{32}{s^3} + \frac{25}{s} + \frac{40}{s^2}$$

$$L[e^{at}F(t)] = f(s-a)$$

$$\Rightarrow L[e^{-4t}F(4t+5)^2] = f(s+4)$$

$$\Rightarrow f(s+4) = \frac{32}{(s+4)^3} + \frac{25}{(s+4)} + \frac{40}{(s+4)^2}$$

Find the Laplace transform of the following.

i] $t\sin t$

Let $F(t) = L[\sin t]$

$$f(s) = \frac{1}{s^2+1}$$

W.K.T $L[t^n F(t)] = (-1)^n \frac{d^n f(s)}{ds^n}$

$$\Rightarrow L[t\sin t] = (-1)\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$\Rightarrow f(s) = -\left[\frac{(s^2+1)(0) - 2s}{(s^2+1)^2}\right]$$

$$\therefore \boxed{L[t\sin t] = \frac{2s}{(s^2+1)^2}}$$

ii] $t\cos at$

$\Rightarrow F(t) = \cos at$

$$L[F(t)] = L[\cos at]$$

$$f(s) = \frac{s}{s^2+a^2}$$

WKT $L[t^n F(t)] = (-1)^n \frac{d^n f(s)}{ds^n}$

$$\Rightarrow L[t\cos at] = (-1)\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$\Rightarrow f(s) = -\left[\frac{(s^2+a^2)(1) - (2s)(s)}{(s^2+a^2)^2}\right]$$

$$\Rightarrow f(s) = -\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$\Rightarrow f(s) = -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$\Rightarrow \boxed{L[t\cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}}$$

iii] $te^{-at}\sin 4t$

$\rightarrow$ $F(t) = \sin 4t$

$L[F(t)] = L[\sin 4t]$

$\Rightarrow f(s) = \dfrac{4}{s^2+16}$

$\therefore$ WKT $L[t^n F(t)] = (-1)^n \dfrac{d^n f(s)}{ds^n}$

$\Rightarrow L[t F(t)] = (-1) \dfrac{d}{ds}\left[\dfrac{4}{s^2+16}\right]$

$\Rightarrow L[t\sin 4t] = -\left[\dfrac{(s^2+16)(0) - 4(2s)}{(s^2+16)^2}\right]$

$\Rightarrow L[t\sin 4t] = \dfrac{8s}{(s^2+16)^2}$

$L[e^{-at} F(t)] = f(s-a)$

$= f(s+2)$

$\Rightarrow L[e^{-2t} t\sin 4t] = \dfrac{8(s+2)}{[(s+2)^2+16]^2}$

iv] $t^2 e^{-3t}\sin 2t$

$\rightarrow$ Let $F(t) = \sin 2t$

$L[F(t)] = L[\sin 2t]$

$f(s) = \dfrac{2}{s^2+4}$

$\therefore$ WKT $L[t^n F(t)] = (-1)^n \dfrac{d^n f(s)}{ds^n}$

$\Rightarrow L[t^2 F(t)] = (-1)^2 \dfrac{d^2}{ds^2}\left[\dfrac{2}{s^2+4}\right]$

$= \dfrac{d}{ds}\left[\dfrac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2}\right]$

$= \dfrac{d}{ds}\left[\dfrac{-4s}{(s^2+4)^2}\right]$

$$\Rightarrow f(s) = -4\frac{d}{ds}\left[\frac{s}{(s^2+4)^2}\right]$$

$$\Rightarrow f(s) = -4\left[\frac{(s^2+4)^2(1) - s.2(s^2+4)2s}{(s^2+4)^4}\right]$$

$$\Rightarrow f(s) = -4\left[\frac{(s^2+4) - 4s^2}{(s^2+4)^3}\right]$$

$$\Rightarrow f(s) = -4\left[\frac{-3s^2+4}{(s^2+4)^3}\right]$$

$$\Rightarrow L[t^2 \sin 2t] = \frac{4(3s^2-4)}{(s^2+4)^3}$$

$$\therefore L[e^{-3t} t^2 \sin 2t] = \left[\frac{4(3s^2-4)}{(s^2+4)^3}\right]_{s \to s+3}$$

$$\Rightarrow L[e^{-3t} t^2 \sin 2t] = \frac{4[3(s+3)^2-4]}{[(s+3)^2+4]^3}$$

v] $t e^{-3t} \sin 3t$

$\rightarrow$ Let $F(t) = \sin 3t$

$$L[F(t)] = L[\sin 3t]$$

$$f(s) = \frac{3}{s^2+9}$$

$\therefore$ WKT $L[t^n F(t)] = (-1)^n \frac{d^n f(s)}{ds^n}$

$$\Rightarrow L[t F(t)] = (-1)\frac{d f(s)}{ds}$$

$$\Rightarrow L[t \sin 3t] = -\frac{d}{ds}\left[\frac{3}{s^2+9}\right]$$

$$= -\left[\frac{(s^2+9)(0) - 3(2s)}{(s^2+9)^2}\right]$$

$$\Rightarrow L[t\sin 3t] = \frac{6s}{(s^2+9)^2}$$

$$\Rightarrow L[e^{-3t}t\sin 3t] = \left[\frac{6s}{(s^2+9)^2}\right]_{s\to s+3}$$

$$\Rightarrow L[e^{-3t}t\sin 3t] = \frac{6(s+3)}{[(s+3)^2+9]^2}$$

vi] $te^{-4t}\sin 3t$.

$\rightarrow$ $F(t) = \sin 3t$

$$L[F(t)] = L[\sin 3t]$$

$$f(s) = \frac{3}{s^2+9}$$

$\therefore$ WKT $L[t^n F(t)] = (-1)^n \frac{d^n f(s)}{ds^n}$

$$\Rightarrow L[t\,F(t)] = -1\frac{d}{ds}\left[\frac{3}{s^2+9}\right]$$

$$\Rightarrow L[t\sin 3t] = -1\left[\frac{(s^2+9)(0)-3(2s)}{(s^2+9)^2}\right]$$

$$\Rightarrow L[t\sin 3t] = \frac{6s}{(s^2+9)^2}$$

$$\therefore L[e^{-4t}t\sin 3t] = \left[\frac{6s}{(s^2+9)^2}\right]_{s\to s+4}$$

$$L[e^{-4t}t\sin 3t] = \frac{6(s+4)}{((s+4)^2+9)^2}$$

vii] $te^{-2t}\cos 2t$.

→ Let $F(t) = \cos 2t$

$$L[F(t)] = L[\cos 2t]$$

$$f(s) = \frac{s}{s^2+4}$$

∴ WKT $L[t^n F(t)] = (-1)^n \frac{d^n f(s)}{ds^n}$

$$L[t F(t)] = (-1)\frac{d}{ds}\left[\frac{s}{s^2+4}\right]$$

$$\Rightarrow L[t\cos 2t] = -\left[\frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right]$$

$$= -\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right]$$

$$= \frac{s^2-4}{(s^2+4)^2}$$

$$\therefore L[e^{-2t} t\cos 2t] = \left[\frac{s^2-4}{(s^2+4)^2}\right]_{s\to s+2}$$

$$\Rightarrow L[e^{-2t} t\cos 2t] = \frac{(s+2)^2-4}{[(s+2)^2+4]^2}.$$

Find the Laplace transform of the following.

i] $\dfrac{\sin t}{t}$

$\Rightarrow$ Let $F(t) = \sin t$

$$L[F(t)] = L[\sin t]$$

$$f(s) = \frac{1}{s^2+1}$$

WKT $L\left[\dfrac{F(t)}{t}\right] = \displaystyle\int_s^\infty f(s)\,ds$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \left[\tan^{-1}(s)\right]_s^\infty = \tan^{-1}(\infty) - \tan^{-1} s$$

$$L\left[\frac{\sin t}{t}\right] = \frac{\pi}{2} - \tan^{-1}(s)$$

ii] $\dfrac{1-e^t}{t}$

$\Rightarrow$ $F(t) = 1 - e^t$

$$L[F(t)] = L[1] - L[e^t]$$

$$f(s) = \frac{1}{s} - \frac{1}{s-1}$$

$\therefore$ WKT $L\left[\dfrac{F(t)}{t}\right] = \displaystyle\int_s^\infty f(s)\,ds$

$$= L\left[\frac{1-e^t}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds$$

$$= \int_s^\infty \frac{1}{s}\,ds - \int_s^\infty \frac{1}{s-1}\,ds$$

$$= \left[\log s - \log(s-1)\right]_s^\infty$$

$$= \log\left[\frac{s}{s-1}\right]_s^\infty - \log\left[\frac{s}{s(1-1/s)}\right]_s^\infty$$

$$= \log\left[\frac{1}{1-1/s}\right]_s^\infty - \log\left[\frac{1}{1-1/\infty}\right] - \log\left[\frac{1}{1-1/s}\right]$$

$$= \log 1 - \log\left[\frac{s}{s-1}\right]$$

$$= 0 - \log\left[\frac{s}{s-1}\right]$$

$$= \log\left|\frac{s}{s-1}\right|^{-1}$$

$$L\left[\frac{1-e^t}{t}\right] = \log\left|\frac{s-1}{s}\right|_{//}$$

iii] $\dfrac{\cos at - \cos bt}{t}$

$\Rightarrow$ $F(t) = \cos at - \cos bt$

$\Rightarrow L[F(t)] = L[\cos at] - L[\cos bt]$

$$f(s) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$\therefore$ WKT $L\left[\dfrac{F(t)}{t}\right] = \displaystyle\int_s^\infty f(s)\,ds$

$$\Rightarrow L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right] ds$$

$$= \frac{1}{2}\left[\log|s^2+a^2| - \log|s^2+b^2|\right]_s^\infty$$

$$= \frac{1}{2}\log\left|\frac{s^2+a^2}{s^2+b^2}\right|_s^\infty$$

$$= 0 - \frac{1}{2}\log\left|\frac{s^2+a^2}{s^2+b^2}\right|$$

$$= \log \left| \frac{s^2+a^2}{s^2+b^2} \right|^{-1/2}$$

$$L\left[\frac{\cos at - \cos bt}{t} \right] = \log \sqrt{\frac{s^2+b^2}{s^2+a^2}}$$ //

iv) $\left[\frac{\sin 2t}{\sqrt{t}} \right]^2 = \frac{\sin^2 2t}{(\sqrt{t})^2}$

$$= \frac{\frac{1-\cos 2(2t)}{2}}{t}$$

$$\left(\frac{\sin 2t}{\sqrt{t}} \right)^2 = \frac{1}{2}\left(\frac{1-\cos 4t}{t} \right)$$

$$L\left[\left(\frac{\sin 2t}{\sqrt{t}} \right)^2 \right] = \frac{1}{2} L\left[\frac{1-\cos 4t}{t} \right] \longrightarrow (1)$$

$$\therefore L[1-\cos 4t] = L[1] - L[\cos 4t]$$

$$L[1-\cos 4t] = \frac{1}{s} - \frac{s}{s^2+16}$$

$$L\left[\frac{1-\cos 4t}{t} \right] = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+16} \right) ds$$

$$= \int_s^\infty \frac{1}{s} ds - \frac{1}{2} \int_s^\infty \frac{2s}{s^2+16} ds$$

$$= \left[\log s - \frac{1}{2} \log |s^2+16| \right]_s^\infty$$

$$= \left[\log s - \log \sqrt{s^2+16}. \right]_s^\infty$$

$$= \log \left| \frac{s}{\sqrt{s^2+16}} \right|_s^\infty - \log \left| \frac{s}{s\sqrt{1+16(1/s^2)}} \right|_s^\infty$$

$$= \log \left| \frac{1}{\sqrt{1+16(1/s)^2}} \right|_s^\infty$$

$$= \log \left| \frac{1}{\sqrt{1+0}} \right| - \log \left| \frac{s}{\sqrt{s^2+16}} \right|$$

$$= \log 1 - \log\left|\frac{s}{\sqrt{s^2+16}}\right|$$

$$= 0 - \log\left|\frac{s}{\sqrt{s^2+16}}\right|$$

$$L\left[\frac{1-\cos 4t}{t}\right] = \log\frac{\sqrt{s^2+16}}{s}$$

$$(1) \Rightarrow L\left[\left(\frac{\sin 2t}{\sqrt{t}}\right)^2\right] = \frac{1}{2}\log\left|\frac{\sqrt{s^2+16}}{s}\right|_{//}$$

v] $te^{-2t}\sin 2t + \frac{\cos 2t - \cos 3t}{t}$

$$F(t) = te^{-2t}\sin 2t + \frac{\cos 2t - \cos 3t}{t}$$

$$L[F(t)] = L[te^{-2t}\sin 2t] + L\left[\frac{\cos 2t - \cos 3t}{t}\right] \longrightarrow (1)$$

$$F(t) = \sin 2t$$

$$L[F(t)] = \frac{2}{s^2+4}$$

$$\Rightarrow L[t^n F(t)] = (-1)^n \frac{d}{ds}[f(s)]$$

$$= (-1)\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$\Rightarrow L[t\sin 2t] = -\left[\frac{(s^2+4)(0) - 2s(2)}{(s^2+4)^2}\right] = \frac{4s}{(s^2+4)^2}$$

$$L[e^{-2t}\sin 2t] = \frac{4(s+2)}{[(s+2)^2+4]^2}$$

$$\therefore L\left[\cos 2t - \cos 3t\right] = L[\cos 2t] - L[\cos 3t]$$

$$= \frac{s}{s^2+4} - \frac{s}{s^2+9}$$

$$\therefore L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \int_s^{\infty}\left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right] ds.$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{2s}{s^2+4} - \frac{2s}{s^2+9}\right] ds$$

$$= \frac{1}{2}\left[\log|s^2+4| - \log|s^2+9|\right]_s^\infty$$

$$= \frac{1}{2}\log\left|\frac{s^2+4}{s^2+9}\right|_s^\infty \rightarrow \log\sqrt{\frac{s^2+9}{s^2+4}}$$

$$(1) \Rightarrow \frac{4(s+2)}{[(s+2)^2+4]^2} + \log\left|\sqrt{\frac{s^2+9}{s^2+4}}\right|$$

vii] $\frac{2\sin t . \sin 5t}{t}$

$\rightarrow$ Let $F(t) = 2\sin t . \sin 5t$

$\Rightarrow F(t) = 2\sin 5t . \sin t$

$\Rightarrow F(t) = \cos(5t-t) - \cos(5t+t)$

$\Rightarrow F(t) = \cos 4t - \cos 6t$

$L[F(t)] = L[\cos 4t] - L[\cos 6t]$

$f(s) = \frac{s}{s^2+16} - \frac{s}{s^2+36}$

$\therefore$ WKT $L\left[\frac{F(t)}{t}\right] = \int^\infty \left[\frac{s}{s^2+16} - \frac{s}{s^2+36}\right] ds$

$$\Rightarrow L\left[\frac{2\sin t \sin 5t}{t}\right] = \frac{1}{2}\int_s^\infty \left[\frac{2s}{s^2+16} - \frac{2s}{s^2+36}\right] ds$$

$$= \frac{1}{2}\left[\log(s^2+16) - \log(s^2+36)\right]_s^\infty$$

$$= \frac{1}{2}\log\left|\frac{s^2+16}{s^2+36}\right|_s^\infty$$

$$= 0 - \frac{1}{2}\log\left|\frac{s^2+16}{s^2+36}\right|_s$$

$$L\left[\frac{2\sin t \sin 5t}{t}\right] = \log\sqrt{\frac{s^2+36}{s^2+16}}$$

vi] $2^t + \dfrac{\cos 2t - \cos 3t}{t} + t\sin t$

$\rightarrow$ $$F(t) = 2^t + \frac{\cos 2t - \cos 3t}{t} + t\sin t$$

$$L[F(t)] = L[2^t] + L\left[\frac{\cos 2t - \cos 3t}{t}\right] + L[t\sin t] \longrightarrow (1)$$

$$\therefore L[2^t] = L\left[e^{(\log_e 2)t}\right]$$

$$= L\left[e^{(\log 2)t}\right]$$

$$\Rightarrow L[2^t] = \frac{1}{s-\log_e 2}$$

$$L\left[\frac{\cos 2t - \cos 3t}{t}\right] = L[\cos 2t] - L[\cos 3t]$$

$$= \frac{s}{s^2+4} - \frac{s}{s^2+9}$$

$$\therefore L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \int_s^\infty \left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right] ds$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{2s}{s^2+4} - \frac{2s}{s^2+9}\right] ds$$

$$= \frac{1}{2}\Big[\log[s^2+4] - \log[s^2+9]\Big]_s^\infty$$

$$= \frac{1}{2}\log\left|\frac{s^2+4}{s^2+9}\right|_s^\infty$$

$$= 0 - \frac{1}{2}\log\left|\frac{s^2+4}{s^2+9}\right|$$

$$\therefore L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \log\left|\sqrt{\frac{s^2+9}{s^2+4}}\right|$$

$$\therefore L[\sin t] = \frac{1}{s^2+1}$$

$$\therefore L[t\sin t] = (-1)^1 \frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$\Rightarrow \quad -\left[\frac{(s^2+1)(0) - 1(2s)}{(s^2+1)^2}\right]$$

$$\Rightarrow L[t\sin t] = \frac{2s}{(s^2+1)^2}$$

$$(1) \Rightarrow \frac{1}{s-\log 2} + \log\left|\sqrt{\frac{s^2+9}{s^2+4}}\right| + \frac{2s}{(s^2+1)^2}$$

Laplace Transform of Periodic Functions:-

Let F(t) be a real valid function defined over the interval $I \subseteq R$, for any constant $T \geq 0$, such that $F(t+T) = F(t)$ is called the periodic function with the period T and it can be calculated as upper limit - lower limit.

The Laplace transform of a periodic function F(t) with the period T can be defined as.

$$L F(t) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\,dt$$

1] Find the Laplace transform of square wave function of a period a given by $f(t) = \begin{cases} 1 & 0 < t < a/2 \\ -1 & a/2 < t < a \end{cases}$

$\rightarrow \therefore$ WKT $L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$ when $T=a$

$$L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}} \left\{ \int_0^{a/2} e^{-st} f(t)\,dt + \int_{a/2}^a e^{-st} f(t)\,dt \right\}$$

$$F(s) = \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st}(1)\,dt + \int_{a/2}^a e^{-st}(-1)\,dt \right]$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st}\,dt - \int_{a/2}^a e^{-st}\,dt \right]$$

$$= \frac{1}{1-e^{-as}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^{a/2} - \left[\frac{e^{-st}}{-s}\right]_{a/2}^{a}\right\}$$

$$= \frac{-1}{s(1-e^{-as})}\left\{\left[e^{-st}\right]_0^{a/2} - \left[\frac{e^{-st}}{-s}\right]_{a/2}^{a}\right\}$$

$$= \frac{-1}{s(1-e^{-as})}\left[(e^{-as/2} - e^{0}) - (e^{-as} - e^{-as/2})\right]$$

$$= \frac{-1}{s(1-e^{-as})}\left[2e^{-as/2} - 1 - e^{-as}\right]$$

$$= \frac{1}{s(1-e^{-as})}\left[1 - 2e^{-as/2} + e^{-as}\right]$$

$$= \frac{1}{s[(1)^2 - (e^{-as/2})^2]}\left[(1)^2 - 2(1)(e^{-as/2}) + (e^{-as/2})^2\right]$$

$$= \frac{(1-e^{-as/2})^2}{s(1-e^{-as/2})(1+e^{-as/2})} = \frac{(1-e^{-as/2}) \times e^{as/4}}{s(1+e^{-as/2})\, e^{as/4}}$$

$$= \frac{1}{s}\left[\frac{e^{as/4} - e^{-as/4}}{e^{as/4} + e^{-as/4}}\right] = \frac{1}{s}\tanh\left(\frac{as}{4}\right)$$

$$\therefore \boxed{L[f(t)] = \frac{1}{s}\tanh\left(\frac{as}{4}\right)}$$

2] Find the laplace transform of a function for a period of a

$$f(t) = \begin{cases} E, & 0 < t < a/2 \\ -E, & a/2 < t < a \end{cases}$$

→ ∴ WKT $L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$ when $T=0$

$$L[f(t)] = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\left\{\int_0^{a/2} e^{-st} f(t)\,dt + \int_{a/2}^{a} e^{-st} f(t)\,dt\right.$$

$$F(s) = \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st}[E]\,dt + \int_{a/2}^{a} e^{-st}(-E)\,dt\right]$$

$$= \frac{E}{1-e^{-as}}\left[\int_a^{a/2} e^{-st}dt - \int_{a/2}^{a} e^{-st}dt\right]$$

$$= \frac{E}{1-e^{-as}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^{a/2} - \left[\frac{e^{-st}}{-s}\right]_{a/2}^{a}\right\}$$

$$= \frac{-E}{s(1-e^{-as})}\left\{\left[e^{-st}\right]_0^{a/2} - \left[e^{-st}\right]_{a/2}^{a}\right\}$$

$$= \frac{-E}{s(1-e^{-as})}\left[\left(e^{-as/2} - e^{0}\right) - \left(e^{-as} - e^{-as/2}\right)\right]$$

$$= \frac{-E}{s(1-e^{-as})}\left[2e^{-as/2} - 1 - e^{-as}\right]$$

$$= \frac{+E}{s(1-e^{-as})}\left[1 - 2e^{-as/2} + e^{-as}\right]$$

$$= \frac{E}{s\left((1)^2 - (e^{-as/2})^2\right)}\left[(1)^2 - 2(1)(e^{-as/2}) + (e^{-as/2})^2\right]$$

$$= \frac{E\left(1 - e^{-as/2}\right)^{\cancel{2}}}{s\,\cancel{(1-e^{-as/2})}\,(1+e^{-as/2})} = \frac{E\,(1-e^{-as/2})}{s\,(1+e^{-as/2})} \times \frac{e^{as/4}}{e^{as/4}}$$

$$= \frac{E}{s}\left[\frac{e^{as/4} - e^{-as/4}}{e^{as/4} + e^{-as/4}}\right] = \frac{E}{s}\tanh\left(\frac{as}{4}\right)$$

3] Find the Laplace transform of a function a period $2a$.

$$f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a. \end{cases}$$

$\Rightarrow$ $\therefore$ WKT $L[f(t)] = \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st} f(t)\,dt$ when $T = 2a$

$$= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st} f(t)dt + \int_0^{2a} e^{-st} f(t)dt\right]$$

$$L[f(t)] = \frac{1}{1-e^{-2as}}\left[\int_0^a t\, e^{-st}dt + \int_0^{2a} (2a-t)e^{-st}dt\right] \rightarrow (1)$$

$$\therefore \int_0^a t e^{-st}dt = t\int_0^a e^{-st}dt - \int_0^a \left[1 \int e^{-st}dt\right]dt$$

$$= -\frac{1}{s}\left[te^{-st}\right]_0^a + \frac{1}{s}\int_0^a e^{-st}dt$$

$$= -\frac{1}{s}\left[te^{-st}\right]_0^a - \frac{1}{s^2}\left[e^{-st}\right]_0^a$$

$$= -\frac{1}{s}\left[ae^{-as} - 0\right] - \frac{1}{s^2}\left[e^{-as} - e^0\right]$$

$$\int_0^a te^{-st}dt = -\frac{a}{s}e^{-as} - \frac{1}{s^2}e^{-as} + \frac{1}{s^2}$$

$$\therefore \int_a^{2a} (2a-t)e^{-st}dt = (2a-t)\int_a^{2a} e^{-st}dt - \int_0^{2a}\left[(-1)\int e^{-st}dt\right]dt$$

$$= -\frac{1}{s}\left[(2a-t)e^{-st}\right]_0^{2a} - \frac{1}{s}\int_a^{2a} e^{-st}dt$$

$$= -\frac{1}{s}\left[(2a-t)e^{-st}\right]_a^{2a} + \frac{1}{s^2}\left[e^{-st}\right]_a^{2a}$$

$$= -\frac{1}{s}\left[0 - ae^{-as}\right] + \frac{1}{s^2}\left[e^{-2as} - e^{-as}\right]$$

$$\therefore \int_0^{2a} (2a-t)e^{-st}dt = \frac{a}{s}e^{-as} + \frac{1}{s^2}e^{-2as} - \frac{1}{s^2}e^{-as}$$

$$\therefore (1) \Rightarrow f(s) = \frac{1}{1-e^{-2as}}\left[\cancel{-\frac{a}{s}e^{-as}} - \frac{1}{s^2}e^{-as} + \cancel{\frac{a}{s}e^{-as}} + \frac{1}{s^2}e^{-2as} - \frac{1}{s^2}e^{-as}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1}{s^2} - \frac{2}{s^2}e^{-as} + \frac{1}{s^2}e^{-2as}\right]$$

$$\Rightarrow F(s) = \frac{1}{\left[s^2\left[[1]^2 - (e^{-as})^2\right]\right]}\left[(1)^2 - 2(1)(e^{-as}) + (e^{-as})^2\right]$$

$$= \frac{(1-e^{-as})^{\cancel{2}}}{s^2\cancel{(1-e^{-as})}(1+e^{-as})}$$

$$= \frac{1}{s^2}\frac{(1-e^{-as})}{(1+e^{-as})} = \frac{1}{s^2}\frac{(1-e^{-as})}{(1+e^{-as})} \times \frac{e^{as/2}}{e^{as/2}}$$

$$= \frac{1}{s^2}\left(\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}\right)$$

$$F(s) = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)_{/\!/}$$

4] If $f(t) = \begin{cases} E\sin(\omega t), & 0 \le t < \pi/\omega \\ 0, & \pi/\omega \le t < 2\pi/\omega \end{cases}$ is a periodic function of the period $2\pi/\omega$, then show that $L[f(t)] = \frac{E\omega}{(s^2+\omega^2)(1-e^{-\pi/\omega})}$ for ω and E Constants.

→ given $f(t) = \begin{cases} E\sin(\omega t), & 0 \le t < \pi/\omega \\ 0, & \pi/\omega \le t < 2\pi/\omega. \end{cases}$

when $T = 2\pi/\omega$

$$\therefore L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{\frac{2\pi}{\omega}s}}\int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$\Rightarrow L[f(t)] = \frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\left[\int_0^{\pi/\omega} e^{-st} f(t)\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\pi/\omega} e^{-st}\,E\sin(\omega t)\,dt + 0\right]$$

$$= \frac{E}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{\pi/\omega} e^{-st}\sin(\omega t)\,dt$$

$$= \frac{E}{1-e^{-\frac{2\pi}{\omega}s}}\left\{\frac{e^{-st}}{(-s)^2+\omega^2}\left[-s\sin(\omega t) - \omega\cos(\omega t)\right]\right\}_0^{\pi/2}$$

$$= \frac{-E}{1-e^{-\frac{2\pi}{\omega}s}}\left\{\frac{e^{-st}}{s^2+\omega^2}\left[s\sin(\omega t) + \omega\cos(\omega t)\right]\right\}_0^{\pi/\omega}$$

$$= \frac{-E}{\left(1-e^{-\frac{2\pi}{\omega}s}\right)} \left\{ \frac{e^{-\pi\omega/s}}{s^2+\omega^2} \left[s\sin\left(\frac{\pi}{\omega}\cdot\omega\right) + \omega\cos\left(\frac{\pi}{\omega}\cdot\omega\right) \right] \right.$$

$$\left. - \frac{-e^0}{s^2+\omega^2} \left[s\sin(0) + \omega\cos(0) \right] \right\}$$

$$= \frac{-E}{1-e^{-\frac{2\pi}{\omega}s}} \left\{ \frac{-\omega e^{-\frac{\pi}{\omega}s}}{s^2+\omega^2} - \frac{\omega}{s^2+\omega^2} \right\}$$

$$= \frac{E\omega}{\left(1-e^{-\frac{2\pi}{\omega}s}\right)} \left[\frac{1+e^{-\frac{\pi}{\omega}s}}{(s^2+\omega^2)} \right]$$

$$= \frac{E\omega\left[1+e^{-\frac{\pi s}{\omega}}\right]}{(s^2+\omega^2)\left((1)^2-\left(e^{-\frac{\pi s}{\omega}}\right)^2\right)}$$

$$= \frac{E\omega\,\cancel{\left[1+e^{-\pi s/\omega}\right]}}{(s^2+\omega^2)\left(1-e^{-\pi s/\omega}\right)\cancel{\left(1+e^{-\pi s/\omega}\right)}}$$

$$\therefore \quad L[f(t)] = \frac{E\omega}{(s^2+\omega^2)\left(1-e^{-\pi s/\omega}\right)}$$

Unit Step (or) Heaviside function:-

For any $a \geq 0$, the unit step or heaviside function can be defined as $u(t-a)$ or $H(t-a) = \begin{cases} 0, & t \leq a \\ 1, & t > a \end{cases}$ and the Laplace transform can be defined as

$$L[u(t-a)] = \frac{e^{-as}}{s}$$

* Laplace transform of Unit step function:-

i] ~~For any~~ Suppose $f(t)$ be a real valid function, then $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$ where $\bar{f}(s) = L[f(t)]$.

ii] $f(t) = \begin{cases} f_1(t), & 0 < t < a \\ f_2(t), & t \geq a \end{cases}$ then the unit step form of $f(t)$ is $f_1(t) + [f_2(t) - f_1(t)]u(t-a)$

iii] $f(t) = \begin{cases} f_1(t), & t < a \\ f_2(t), & a \le t < b \\ f_3(t), & t \ge b \end{cases}$, then the unit step form.

of $f(t)$ is defined as

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) + f_2(t)]u(t-b).$$

Find the Laplace transform of the following.

i] $t^2 u(t-1)$

$\Rightarrow$ Let $f(t-1) = t^2$

take $t = t+1$

$\Rightarrow f(t+\cancel{1}-\cancel{1}) = (t+1)^2$

$f(t) = t^2 + 2t + 1$

$L[f(t)] = L[t^2] + [t] + L[1]$

$\bar{f}(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$

$\therefore$ WKT $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$

$\therefore L[f(t-1)u(t-1)] = e^{-s}\bar{f}(s)$

$\Rightarrow L[t^2 u(t-1)] = e^{-s}\left[\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right]$

ii] $(t^2-1)u(t-1)$

$f(t-1) = t^2 - 1$

Let $f(t-1) = t^2 - 1$

$t = t+1$

$f(t+\cancel{1}-\cancel{1}) = (t+1)^2 - 1$

$f(t) = t^2 + 2t$

$L[f(t)] = L[t^2] + 2L[t]$

$\bar{f}(s) = \frac{2}{s^3} + \frac{2}{s^2}$

WKT $L[f(t-a)\,u(t-a)] = e^{-as}\bar{f}(s)$

$\Rightarrow L[f(t-1)\,u(t-1)] = e^{-s}\bar{f}(s)$

$\Rightarrow L[t^2-1\,u(t-1)] = e^{-s}\left[\frac{2}{s^3} + \frac{2}{s^2}\right]$

iii] $(t-1)^2 u(t+1)$

Let $f(t+1) = (t-1)^2 \longrightarrow (1)$

$t = t-1$

$f(t-\cancel{1}+\cancel{1}) = ((t-1)-1)^2$

$f(t) = (t-2)^2$

$f(t) = t^2+4-4t$

$L[f(t)] = L[t^2] + 4[L(1)] - 4L[t]$

$\bar{f}(s) = \frac{2}{s^3} + \frac{4}{s} - \frac{4}{s^2}$

$\therefore$ WKT $L[f(t-a)\,u(t-a)] = e^{-as}\bar{f}(s)$

$\Rightarrow L[f(t+1)\,u(t-(-1))] = e^{s}\bar{f}(s)$

$\Rightarrow L[(t-1)^2\,u(t+1)] = e^{s}\left[\frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s}\right]$

iv] $\sin t \cdot u(t-\pi)$

Let $f(t-\pi) = \sin t \longrightarrow (1)$

$t \rightarrow t+\pi$

$(1) \Rightarrow f(t+\cancel{\pi}-\cancel{\pi}) = \sin(t+\pi)$

$\Rightarrow f(t) = -\sin t$

$\Rightarrow L[f(t)] = -L[\sin t]$

$\Rightarrow \bar{f}(s) = \frac{-1}{s^2+1}$

$\therefore$ WKT $L[f(t-a)\,u(t-a)] = e^{-as}\bar{f}(s)$

$\Rightarrow L[f(t-\pi)\,u(t-\pi)] = e^{-\pi s}\bar{f}(s)$

$\Rightarrow L[\sin t\; u(t-\pi)] = e^{-\pi s}\left[\frac{-1}{s^2+1}\right]$ //

Express the function $f(t) = \begin{cases} \pi - t, & 0 < t < \pi \\ \sin t, & t \geq \pi. \end{cases}$

in terms of unit step function & hence find its laplace transform.

$$f(t) = \begin{cases} \pi - t, & 0 < t < \pi \\ \sin t, & t \geq \pi \end{cases}$$

$$\Rightarrow \quad f(t) = (\pi - t) + [\sin t - (\pi - t)]\, u(t-\pi)] \longrightarrow ①$$

$$f(t) = (\pi - t) + [\sin t + t - \pi]\, u(t-\pi)$$

$$L[f(t)] = \pi L[1] - L[t] + L[\sin t + t - \pi]\, u(t-\pi) \longrightarrow ②$$

$$\therefore L[1] = \frac{1}{s} \qquad L[t] = \frac{1}{s^2}$$

and given $g(t-\pi) = \sin t + t - \pi$

$$t \longrightarrow t + \pi$$

$$g(t + \cancel{\pi} - \cancel{\pi}) = \sin(t+\pi) + t + \cancel{\pi} - \cancel{\pi}$$

$$g(t) = -\sin t + t$$

$$g(t) = t - \sin t$$

$$L[g(t)] = L[t] - L[\sin t]$$

$$\bar{g}(s) = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$\therefore$ WKT $L[g(t-a)\,u(t-a)] = e^{-as}\bar{g}(s)$

$$L[g(t-\pi)\,u(t-\pi)] = e^{-\pi s}\bar{g}(s)$$

$$L[\sin t + t - \pi\, u(t-\pi)] = e^{-\pi s}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$$

$$(2) \Rightarrow f(s) = \frac{\pi}{s} - \frac{1}{s^2} + e^{-\pi s}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]_{//}$$

2] Express the function $f(t) = \begin{cases} \cos t, & 0 < t \leq \pi \\ \cos 2t, & \pi < t \leq 2\pi \\ \cos 3t, & t > 2\pi \end{cases}$

$$\rightarrow \quad f(t) = \cos t + [\cos 2t - \cos t]\, u(t-\pi) + [\cos 3t - \cos 2t]\, u(t-2\pi)$$

$$L[f(t)] = L[\cos t] + L[\cos 2t - \cos t]\, u(t-\pi) + L[\cos 3t - \cos 2t \; u(t-2\pi)] \longrightarrow (1)$$

$$\therefore L[\cos t] = \frac{s}{s^2+1}$$

Let $g(t-\pi) = \cos 2t - \cos t$

$$t \to t+\pi$$

$$g(t) = \cos 2(t+\pi) - \cos(t+\pi)$$

$$g(t) = \cos(2t+2\pi) - \cos(t+\pi)$$

$$g(t) = \cos 2t + \cos t$$

$$L[g(t)] = L[\cos 2t] + L[\cos t]$$

$$= \frac{s}{s^2+4} + \frac{s}{s^2+1}$$

$$\therefore L[g(t-\pi)\, u(t-\pi)] = e^{-\pi s}\, \bar{g}(s)$$

$$L[\cos 2t - \cos t\, u(t-\pi)] = e^{-\pi s}\left[\frac{s}{s^2+4} + \frac{s}{s^2+1}\right]$$

$$h(t-2\pi) = \cos 3t - \cos 2t$$

$$t \to t+2\pi$$

$$h(t) = \cos 3(t+2\pi) - \cos 2(t+2\pi)$$

$$h(t) = \cos(3t+6\pi) - \cos(2t+4\pi)$$

$$h(t) = \cos 3t - \cos 2t$$

$$L[h(t)] = L[\cos 3t] - L[\cos 2t]$$

$$\bar{h}(s) = \frac{s}{s^2+9} - \frac{s}{s^2+4}$$

$$\therefore L[h(t-2\pi)\, u(t-2\pi)] = e^{-2\pi s}\, \bar{h}(s)$$

$$L[\cos 3t - \cos 2t\, u(t-2\pi)] = e^{-2\pi s}\left[\frac{s}{s^2+9} - \frac{s}{s^2+4}\right]$$

$$\bar{f}(s) = \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{s}{s^2+4} + \frac{s}{s^2+1}\right] + e^{-2\pi s}\left[\frac{s}{s^2+9} - \frac{s}{s^2+4}\right]$$

(3) Express the function $f(t) = \begin{cases} 1, & 0<t<1 \\ t, & 1<t<2 \\ t^2, & t>2 \end{cases}$ in terms of unit step function & hence find its laplace transform.

$\rightarrow$ $f(t) = 1 + (t-1)u(t-1) + (t^2-t)u(t-2)$

$L[f(t)] = L[1] + L[t-1]u[t-1] + L[t^2-t]u(t-2)] \rightarrow (1)$

$L[1] = \frac{1}{s}$

Let $g(t-1) = t-1$

$t \rightarrow t+1$

$L[g(t)] = L[t]$

$\bar{g}(s) = \frac{1}{s^2}$

$\therefore L[g(t-1)u(t-1)] = e^{-s}\bar{g}(s)$

$L[(t-1)u(t-1)] = \frac{e^{-s}}{s^2}$

Let $h(t-2) = t^2 - t$

$t \rightarrow t+2$

$h(t) = (t+2)^2 - (t+2)$

$h(t) = t^2 + 4t + 4 - t - 2$

$h(t) = t^2 + 3t + 2$

$L[h(t)] = L[t^2] + 3L[t] + 2L[1]$

$\bar{h}(s) = \frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}$

$\therefore L[h(t-2)u(t-2)] = e^{-2s}\bar{h}(s)$

$L[(t^2-t)u(t-2)] = e^{-2s}\left[\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right]$

$(1) \Rightarrow \bar{f}(s) = \frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-2s}\left[\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right]$ //

(4) $f(t) = \begin{cases} \sin t, & 0<t \le \pi/2 \\ \cos t, & \pi/2 < t \le \pi \\ 1, & t > \pi \end{cases}$ find its Laplace transform.

$\rightarrow\ f(t) = \sin t + [\cos t - \sin t]\, u(t - \pi/2) + (1 - \cos t)\, u(t - \pi)$

$L[f(t)] = L[\sin t] + L[\cos t - \sin t]\, u(t - \pi/2) + L[1 - \cos t]\, u(t - \pi)] \rightarrow (1)$

$\therefore L[\sin t] = \dfrac{1}{s^2+1}$

Let $g(t - \pi/2) = \cos t - \sin t$

$t \rightarrow t + \pi/2$

$g(t) = \cos(t + \pi/2) - \sin(t + \pi/2)$

$g(t) = -\sin t - \cos t$

$L[g(t)] = -L[\sin t] - L[\cos t]$

$\bar{g}(s) = \dfrac{-1}{s^2+1} - \dfrac{s}{s^2+1} = -\left[\dfrac{(s+1)}{s^2+1}\right]$

$\therefore L[g(t - \pi/2)\, u(t - \pi/2)] = e^{-\pi/2s}\, \bar{g}(s)$

$L[(\cos t - \sin t)\, u(t - \pi/2)] = e^{-\pi/2s}\left(\dfrac{s+1}{s^2+1}\right)$

Let $h(t - \pi) = 1 - \cos t$

$t \rightarrow t + \pi$

$h(t) = 1 - \cos(t + \pi)$

$h(t) = 1 + \cos t$

$L[h(t)] = L[1] + L[\cos t]$

$= \dfrac{1}{s} + \dfrac{s}{s^2+1}$

$\therefore L[h(t - \pi)\, u(t - \pi)] = e^{-\pi s}\, \bar{h}(s)$

$L[1 - \cos t]\, u(t - \pi)] = e^{-\pi s}\left[\dfrac{1}{s} + \dfrac{s}{s^2+1}\right]$

$(1) \Rightarrow \bar{f}(s) = \dfrac{1}{s^2+1} - e^{-\pi/2s}\, \dfrac{s+1}{s^2+1} + e^{-\pi s}\left[\dfrac{1}{s} + \dfrac{s}{s^2+1}\right]$

(5) $f(t) = \begin{cases} 1, & 0 < t < 1 \\ 2t, & 1 < t < 2 \\ 3t, & t > 2 \end{cases}$

$f(t) = 1 + (2t - 1)\, u(t-1) + (3t - 2t)\, u(t-2)$

$f(t) = 1 + (2t - 1)\, u(t-1) + t\, u(t-2)$

$$L[f(t)] = L[1] + L[(2t-1)\,u(t-1)] + L[t]\,u[t-2] \rightarrow (1)$$

$$\therefore \; L[1] = \frac{1}{s}$$

Let $g(t-1) = 2t-1$

$$t \rightarrow t+1$$

$$g(t) = 2(t+1) - 1$$

$$g(t) = 2t + 2 - 1$$

$$g(t) = 2t + 1$$

$$L[g(t)] = 2L[t] + L[1]$$

$$\bar{g}(s) = \frac{2}{s^2} + \frac{1}{s}$$

$$\therefore \; L[g(t-1)\,u(t-1)] = e^{-s}\bar{g}(s)$$

$$L[(2t-1)\,u(t-1)] = e^{-s}\left[\frac{2}{s^2} + \frac{1}{s}\right]$$

and let $h(t-2) = t$

$$t \rightarrow t+2$$

$$h(t) = t + 2$$

$$L[h(t)] = L[t] + 2L[1]$$

$$\bar{h}(s) = \frac{1}{s^2} + \frac{2}{s}$$

$$\therefore \; L[h(t-2)\,u(t-2)] = e^{-2s}\bar{h}(s)$$

$$L[t\,u(t-2)] = e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$$

$$(1) \Rightarrow \bar{f}(s) = \frac{1}{s} + e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right] + e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$$

- $f(t) = \begin{cases} t^2, & 0 < t \le 2 \\ 4t, & 2 < t \le 4 \\ 8, & t > 4 \end{cases}$ find the laplace transform.

$$\Rightarrow f(t) = t^2 + (4t - t^2)\,u(t-2) + (8 - 4t)\,u(t-4)$$

$$L[f(t)] = L[t^2] + L[4t - t^2]\,u(t-2) + L[(8-4t)]\,u(t-4) \rightarrow (1)$$

$$\therefore \; L[t^2] = \frac{2}{s^3}$$

Let $g(t-2) = 4t - t^2$

$$t \rightarrow t+2$$

$$g(t) = 4(t+2) - (t+2)^2$$

$$g(t) = 4t + 8 - (t^2 + 4t + 4)$$
$$g(t) = \cancel{4t} + 8 - t^2 - \cancel{4t} - 4$$
$$g(t) = 4 - t^2$$
$$L[g(t)] = 4L[1] - L[t^2]$$
$$\bar{g}(s) = \frac{4}{s} - \frac{2}{s^3}$$

and Let $h(t-4) = 8 - 4t$
$$t \to t+4$$
$$h(t) = 8 - 4(t+4)$$
$$h(t) = 8 - 4t - 16 \Rightarrow -4t - 8$$
$$h(t) = -(4t + 8)$$
$$L[h(t)] = -\{4L[t] + 8L[1]\}$$
$$\bar{h}(s) = -\left[\frac{4}{s^2} + \frac{8}{s}\right]$$
$$\therefore L[h(t-4)u(t-4)] = e^{-4s}\bar{h}(s)$$
$$L[(8-4t)u(t-4)] = -e^{-4s}\left[\frac{4}{s^2} + \frac{8}{s}\right]$$
$$(1) \Rightarrow \bar{f}(s) = \frac{2}{s^3} + e^{-2s}\left[\frac{4}{s} - \frac{2}{s^3}\right] - e^{-4s}\left[\frac{4}{s^2} + \frac{8}{s}\right]_{//}$$

$$f(t) = \begin{cases} \sin t, & 0 < t \le \pi \\ \sin 2t, & \pi < t \le 2\pi \\ \sin 3t, & t > 2\pi \end{cases}$$

$$\Rightarrow f(t) = \sin t + [\sin 2t - \sin t]\, u(t-\pi) + (\sin 3t - \sin 2t)\, u(t-2\pi)$$
$$L[f(t)] = L[\sin t] + L[\sin 2t - \sin t]\, u(t-\pi) + L[(\sin 3t - \sin 2t)$$
$$u(t-2\pi)] \to (2)$$
$$\therefore L[\sin t] = \frac{1}{s^2+1}$$

Let $g(t-\pi) = \sin 2t - \sin t$
$$t \to t + \pi$$
$$g(t) = \sin 2(t+\pi) - \sin(t+\pi)$$
$$g(t) = \sin(2t + 2\pi) - \sin(t+\pi) = \sin 2t + \sin t$$

$$L[g(t)] = L[\sin 2t] + L[\sin t]$$

$$\bar{g}(s) = \frac{2}{s^2+4} + \frac{1}{s^2+1}$$

$$\therefore L[g(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{g}(s)$$

$$L[(\sin 2t - \sin t)\,u(t-\pi)] = e^{-\pi s}\left[\frac{2}{s^2+4} + \frac{1}{s^2+1}\right]$$

and let $h(t-2\pi) = \sin 3t - \sin 2t$

$$t \to t + 2\pi$$

$$h(t) = \sin 3(t+2\pi) - \sin 2(t+2\pi)$$

$$h(t) = \sin 3t - \sin 2t$$

$$L[h(t)] = L[\sin 3t] - L[\sin 2t]$$

$$\bar{h}(s) = \frac{3}{s^2+9} - \frac{2}{s^2+4}$$

$$\therefore L[h(t-2\pi)u(t-2\pi)] = e^{-2\pi s}\bar{h}(s)$$

$$L[(\sin 3t - \sin 2t)\,u(t-2\pi)] = e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]$$

$$(1) \Rightarrow \bar{f}(s) = \frac{1}{s^2+1} + e^{-\pi s}\left[\frac{2}{s^2+4} + \frac{1}{s^2+1}\right] + e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]$$

⇒ Inverse Laplace Transform.

Suppose $f(s)$ be the laplace transform of $F(t)$, then its inverse laplace transform can be defined as $L^{-1}[f(s)] = F(t)$ and it follows the standard results, as below.

$$L^{-1}\left[\frac{1}{s}\right] = 1 \qquad L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

$$L^{-1}\left[\frac{1}{s-a}\right] = e^{at} \qquad L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$$

$$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at} \qquad L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$$

$$L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$$

$$L^{-1}\left[f(s-a)\right] = e^{at} F(t)$$

$$L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$$

$$L^{-1}\left[f(s+a)\right] = e^{-at} F(t)$$

$$L^{-1}\left[f^{(n)}(s)\right] = (-1)^n t^n F(t).$$

1] Find the Laplace transform of the following:-

i] $\dfrac{(s^2-1)^2}{s^5}$

$\Rightarrow$ given $f(s) = \dfrac{(s^2-1)^2}{s^5}$

$$f(s) = \frac{s^4-2s^2+1}{s^5}$$

$$f(s) = \frac{s^4}{s^5} - \frac{2s^2}{s^5} + \frac{1}{s^5}$$

$$f(s) = \frac{1}{s} - 2\frac{1}{s^3} + \frac{1}{s^5}$$

$$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s}\right] - 2L^{-1}\left[\frac{1}{s^{2+1}}\right] + L^{-1}\left[\frac{1}{s^{4+1}}\right]$$

$$F(t) = 1 - \frac{2t^2}{2!} + \frac{t^4}{4!}$$

$$F(t) = 1 - t^2 + \frac{t^4}{24}$$

ii] $\dfrac{s^3+2s^2-s+1}{s^4}$

$\rightarrow$ $f(s) = \dfrac{s^3+2s^2-s+1}{s^4}$

$$f(s) = \frac{s^3}{s^4} + \frac{2s^2}{s^4} - \frac{s}{s^4} + \frac{1}{s^4}$$

$$f(s) = \frac{1}{s} + \frac{2}{s^2} - \frac{1}{s^3} + \frac{1}{s^4}$$

$$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s}\right] + 2L^{-1}\left[\frac{1}{s^{1+1}}\right] - L^{-1}\left[\frac{1}{s^{2+1}}\right] + L^{-1}\left[\frac{1}{s^{3+1}}\right]$$

$$F(t) = 1 + 2t - \frac{t^2}{2} + \frac{t^3}{6}$$

iii] $\dfrac{1}{s(s+1)(s+2)}$

$$f(s) = \frac{1}{s(s+1)(s+2)} \longrightarrow (1)$$

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$1 = A(s+1)(s+2) + B(s)(s+2) + C(s)(s+1) \longrightarrow (2)$$

when $s=0$

$1 = A(1)(2)$

$\boxed{A = 1/2}$

when $s=-2$

$1 = C(-2)(-1)$

$1 = 2C$

$\boxed{C = 1/2}$

when $s=-1$

$1 = B(-1)(1)$

$\boxed{B = -1}$

$$(1) \Rightarrow f(s) = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{(s+1)} + \frac{1}{2}\,\frac{1}{s+2}$$

$$L^{-1}[f(s)] = \frac{1}{2}L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right]$$

$$F(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$

iv] $\dfrac{s}{(s+2)(s+3)}$

$\rightarrow$ given $f(s) = \dfrac{s}{(s+2)(s+3)} \longrightarrow (1)$

$$\frac{s}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$s = A(s+3) + B(s+2)$$

when $s=-2$

$-2 = A(1)$

$\boxed{A = -2}$

when $s=-3$

$-3 = B(-1)$

$\boxed{B = +3}$

$$f(s) = \left(\frac{1}{s+2} - \frac{1}{s+3}\right)s = \frac{-2}{s+2} + \frac{3}{s+3}$$

$$L^{-1}[f(s)] = L^{-1}\left[\frac{-2}{s+2} + \frac{3}{s+3}\right]$$

$$F(t) = -2e^{-2t} + 3e^{-3t}$$

v] $\frac{s+3}{s^2-4s+13}$

$\rightarrow$ $$f(s) = \frac{s+3}{s^2-4s+13}$$

$$= \frac{s+3}{(s^2-2s(2)+2^2)-2^2+13}$$

$$= \frac{s+3}{(s-2)^2-4+13}$$

$$= \frac{(s-2)+2+3}{(s-2)^2+9}.$$

$$f(s) = \frac{(s-2)+5}{(s-2)^2+9}$$

$$L^{-1}[f(s)] = L^{-1}\left[\frac{(s-2)+5}{(s-2)^2+9}\right]$$

$$= e^{2t} L^{-1}\left[\frac{s+5}{s^2+9}\right]$$

$$= e^{2t} L^{-1}\left[\frac{s}{s^2+9} + \frac{5}{s^2+9}\right]$$

$$F(t) = e^{2t}\left[\cos 3t + \frac{5}{3}\sin 3t\right]$$

vi] $\frac{4s+5}{(s+1)^2(s+2)}$

$\rightarrow$ $$f(s) = \frac{4s+5}{(s+1)^2(s+2)} \longrightarrow (1)$$

$$\therefore \frac{4s+5}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$$

$$4s+5 = A(s+1)(s+2) + B(s+2) + C(s+1)^2 \longrightarrow (2)$$

when $s=-1$

$$-4+5 = B(1)$$

$$\boxed{B=1}$$

when $s=1$

$$9 = A(2)(3)+3-3(2)^2$$

$$9 = 6A+3-12$$

$$9+9 = 6A$$

$$18 = 6A$$

$$\boxed{A=3}$$

Equating s^2-coefficient we get.

$$A+C=0$$

$$A=-C$$

$$\boxed{A=3}$$

$$(1) \Rightarrow \quad f(s) = \frac{3}{(s+1)} + \frac{1}{(s+1)^2} - \frac{3}{(s+2)}$$

$$\Rightarrow L^{-1}[f(s)] = 3L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2}\right] - 3L^{-1}\left[\frac{1}{s+2}\right]$$

$$\Rightarrow L^{-1}[f(s)] = 3e^{-t} + e^{-t}L^{-1}\left[\frac{1}{s^2}\right] - 3e^{-2t}$$

$$F(t) = 3e^{-t} + te^{-t} - 3e^{-2t}$$

vii] $\dfrac{2s^2+5s-4}{s^3+s^2-2s}$

$$\rightarrow \text{let } f(s) = \frac{2s^2+5s-4}{s^3+s^2-2s}$$

$$f(s) = \frac{2s^2+5s-4}{s(s^2+s-2)}$$

$$f(s) = \frac{2s^2+5s-4}{s(s^2-s+2s-2)} = \frac{2s^2+5s-4}{s(s-1)(s+2)} \longrightarrow (1)$$

$$\therefore \frac{2s^2+5s-4}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2}$$

$$2s^2+5s-4 = A(s-1)(s+2) + B(s)(s+2) + C(s+1)s \longrightarrow (2)$$

when $s=0$

$(2) \Rightarrow -4 = A(-1)(2)$

$-2A = -4$

$\boxed{A = 2}$

When $s = 1$

$(2) \Rightarrow 3 = B(3)$

$\boxed{B = 1}$

When $s = -2$

$(2) \Rightarrow -6 = C(-2)(-2-1)$

$-6 = 6C$

$\boxed{C = -1}$

$(1) \Rightarrow f(s) = \frac{2}{s} + \frac{1}{s-1} - \frac{1}{s+2}$

$\Rightarrow L^{-1}[f(s)] = 2L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+2}\right]$

$F(t) = 2 + e^{t} - e^{-2t}$

x] $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$

$s=1$	1	-6	11	-6
	0	1	-5	6
$s=2$	1	-5	6	0
	0	2	-6	
$s=3$	1	-3	0	
	0	3		
	1	0		

$\rightarrow f(s) = \dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)} \longrightarrow (1)$

$\therefore \dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \dfrac{A}{s-1} + \dfrac{B}{s-2} + \dfrac{C}{s-3}$

$2s^2-6s+5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2) \longrightarrow (2)$

When $s = 1$

$(2) \Rightarrow 1 = A(-1)(-2)$

$\boxed{A = 1/2}$

When $s = 2$

$(2) \Rightarrow 1 = B(1)(-1)$

$\boxed{B = -1}$

When $s = 3$

$(2) \Rightarrow 5 = C(2)(1)$

$\boxed{C = 5/2}$

$(2) \Rightarrow f(s) = \frac{1}{2}\,\frac{1}{s-1} - \frac{1}{s-2} + \frac{5}{2}\,\frac{1}{s-3}$

$$\Rightarrow L^{-1}[f(s)] = \frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s-2}\right] + \frac{5}{2}L^{-1}\left[\frac{1}{s-3}\right]$$

$$\Rightarrow F(t) = \frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}$$

$$\frac{s}{s^2+6s+13}$$

$$\Rightarrow f(s) = \frac{s}{(s^2+2(s)(3)+3^2)-3^2+13}$$

$$= \frac{s}{(s+3)^2+4}$$

$$f(s) = \frac{(s+3)-3}{(s+3)^2+4}$$

$$L^{-1}[f(s)] = L^{-1}\left[\frac{(s+3)-3}{(s+3)^2+4}\right]$$

$$= e^{-3t}L^{-1}\left[\frac{s-3}{s^2+4}\right]$$

$$= e^{-3t}L^{-1}\left[\frac{s}{s^2+4} - \frac{3}{s^2+4}\right]$$

$$F(t) = e^{-3t}\left[\cos 2t - \frac{3}{2}\sin 2t\right]$$

$$\frac{1}{(s-1)(s^2+1)}$$

$$\Rightarrow f(s) = \frac{1}{(s-1)(s^2+1)} \longrightarrow (1)$$

$$\frac{1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)(s-1)$$

$$1 = As^2 + A + Bs^2 - Bs + Cs - C$$

$$1 = s^2(A+B) + s(C-B) + (A-C) \longrightarrow (2)$$

when $s=1$ $(2) \Rightarrow 1 = A + \cancel{B} + \cancel{C} - \cancel{B} + A - \cancel{C}$

$$1 = 2A$$

$$\boxed{A = 1/2}$$

Equating co-efficient s^2

$$A + B = 0$$
$$A = -B$$
$$\boxed{B = -1/2}$$

Equating co-efficient of s

$$C - B = 0$$
$$C = B$$
$$\boxed{C = -1/2}$$

$$(1) \Rightarrow f(s) = \frac{1}{2}\frac{1}{s-1} + \left(-\frac{1}{2}\right)\frac{s-1}{s^2+1}$$

$$f(s) = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\left[\frac{s}{s^2+1}\right] - \frac{1}{2}\left[\frac{1}{s^2+1}\right]$$

$$\Rightarrow L^{-1}[f(s)] = \frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{2}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$F(t) = \frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\sin t$$

Find the inverse laplace transform of the following:-

i] $\log\left|\frac{s+a}{s+b}\right|$

Let $f(s) = \log[s+a] - \log[s+b] \rightarrow (1)$

diff eq^n (1) w.r.t s

$$\Rightarrow L^{-1}[f'(s)] = L^{-1}\left[\frac{1}{s+a}\right] - L^{-1}\left[\frac{1}{s+b}\right]$$

$$\Rightarrow (-1)^1 t^1 F(t) = e^{-at} - e^{-bt}$$

$$\Rightarrow F(t) = \frac{e^{-bt} - e^{-at}}{t}$$

ii] $\log\left|\frac{s^2+1}{s(s+1)}\right|$

$$\rightarrow f(s) = \log(s^2+1) - \log[s(s+1)]$$

$$\Rightarrow f(s) = \log(s^2+1) - \log s - \log(s+1) \rightarrow (1)$$

diff eq^n(1) w.r.t s.

$$\Rightarrow f'(s) = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$$

$$L^{-1}[f'(s)] = 2L^{-1}\left[\frac{s}{s^2+1}\right] - L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$\Rightarrow (-1)^1 + F(t) = 2\cos t - 1 - e^{-t}$$

$$F(t) = \frac{1+e^{-t}-2\cos t}{t}$$

iii] $f(s) = \cot^{-1}\left(\frac{s}{2}\right)$

Let $f(s) = \cot^{-1}\left(\frac{s}{2}\right) \longrightarrow (1)$

Diff (1) w.r.t 's'

$$\Rightarrow f'(s) = \frac{1}{1+(s/2)^2} \cdot \left(\frac{1}{2}\right)$$

$$\Rightarrow f'(s) = -\frac{1}{2} \cdot \frac{1}{1+s^2/4}$$

$$\Rightarrow f'(s) = -\frac{1}{2} \cdot \frac{4}{s^2+4}$$

$$\Rightarrow f'(s) = \frac{-2}{s^2+4}$$

$$\Rightarrow L^{-1}[f'(s)] = L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$\Rightarrow (-1)^1 t^1 \; F(t) = -\sin 2t$$

$$\Rightarrow F(t) = \frac{\sin 2t}{t}$$

iv] $\log\left|\frac{s(s+5)}{(s^2+25)(s-7)}\right|$

$$\rightarrow f(s) = \log|s(s+5)| - \log|(s^2+25)(s-7)|$$

$$\rightarrow f(s) = \log s + \log(s+5) - \log(s^2+25) - \log(s-7)$$

$$\Rightarrow f(s) = \frac{1}{s} + \frac{1}{s+5} - \frac{2s}{s^2+25} - \frac{1}{s-7}$$

$$\Rightarrow L^{-1}[f'(s)] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s+5}\right] - 2L^{-1}\left[\frac{s}{s^2+25}\right] - L^{-1}\left[\frac{1}{s-7}\right]$$

$$(-1)^1 \frac{d}{dt} F(t) = 1 + e^{-5t} - 2\cos 5t - e^{7t}$$

$$\Rightarrow F(t) = \frac{2\cos 5t + e^{7t} - e^{-5t} - 1}{t}$$

⇒ Laplace transform for Convolution theorem:-

Suppose $f(t)$, $g(t)$ be the two functions and let $\bar{f}(s)$, $\bar{g}(s)$ be the laplace transform of $f(t)$ & $g(t)$ be the ~~laplace transform of~~ can be defined as.

$$f(t) * g(t) = \int_{u=0}^{t} f(u)\, g(t-u)\, du = \int_{u=0}^{s} f(t-u)\, g(u)\, du.$$

where the symbol '*' is called convolution operator and the laplace transform of the convolution function can be defined as.

$$L[f(t) * g(t)] = L\left[\int_{u=0}^{t} f(u)\, g(t-u)\, du\right] = L\left[\int_{u=0}^{t} f(t-u)\, g(u)\, du\right]$$

$$= \bar{f}(s) \cdot \bar{g}(s).$$

and its inverse laplace transform can be defined as.

$$L^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t) = \int_{u=0}^{t} f(u)\, g(t-u)\, du =$$

$$\int_{u=0}^{t} f(t-u)\, g(u)\, du.$$

Find the inverse laplace transform for the following with Convolution theorem.

1] $\dfrac{1}{s(s+1)}$

→ Let $\bar{f}(s) \cdot \bar{g}(s) = \dfrac{1}{s(s+1)} = \dfrac{1}{s} \cdot \dfrac{1}{s+1}$

$$\bar{f}(s) = \frac{1}{s} \Rightarrow L^{-1}\left[\bar{f}(s)\right] = L^{-1}\left[\frac{1}{s}\right] = 1 = f(t)$$

$$\bar{g}(s) = \frac{1}{s+1} \Rightarrow L^{-1}\left[\bar{g}(s)\right] = L^{-1}\left[\frac{1}{s+1}\right] = e^{-t} = g(t)$$

$\therefore$ WKT $L^{-1}\left[\bar{f}(s).\bar{g}(s)\right] = \int_{u=0}^{t} f(t-u)g(u)du$

$$= \int_{u=0}^{t} 1.\, e^{-u}du = \int_{u=0}^{t} e^{-u}.du$$

$$= \left[\frac{e^{-u}}{-1}\right]_0^t$$

$$= -\left[e^{-u}\right]_0^t = -\left[e^{-t} - e^0\right]$$

$$= -\left[e^{-t} - 1\right]$$

$$\Rightarrow L^{-1}\left[\frac{1}{s(s+1)}\right] = 1 - e^{t}$$

2] $\frac{1}{s(s^2+1)}$

Let $\bar{f}(s).\bar{g}(s) = \frac{1}{s(s^2+1)} = \frac{1}{s}.\frac{1}{s^2+1}$

$$\bar{f}(s) = \frac{1}{s} = L^{-1}\left[\bar{f}(s)\right] = L^{-1}\left[\frac{1}{s}\right] = 1 = f(t)$$

$$\bar{g}(s) = \frac{1}{s^2+1} = L^{-1}\left[\bar{g}(s)\right] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t = g(t)$$

W.K.T $L^{-1}\left[\bar{f}(s).\bar{g}(s)\right] = \int_{u=0}^{t} f(t-u)g(u)du$

$$= \int_{u=0}^{t} 1.\sin u du \Rightarrow \left[-\cos u\right]_0^t$$

$$= -\left[\cos t - \cos 0\right] = -\left[\cos t - 1\right]$$

$$\Rightarrow L^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - \cos t$$

3) $\dfrac{1}{(s-1)(s^2+4)}$

$\Rightarrow$ Let $\bar{f}(s).\bar{g}(s) = \dfrac{1}{(s-1)(s^2+4)} = \dfrac{1}{s-1} \cdot \dfrac{1}{s^2+4}$

$\bar{f}(s) = \dfrac{1}{s-1} = L^{-1}[\bar{f}(s)] = L^{-1}\left[\dfrac{1}{s-1}\right] = e^t = f(t)$

$\bar{g}(s) = \dfrac{1}{s^2+4} = L^{-1}[\bar{g}(s)] = L^{-1}\left[\dfrac{1}{s^2+4}\right] = \dfrac{1}{2}\sin 2t = g(t)$

$\therefore$ WKT $L^{-1}[\bar{f}(s).\bar{g}(s)] = \displaystyle\int_{u=0}^{t} f(t-u)\,g(u)\,du.$

$$= \frac{1}{2}\int_{u=0}^{t} e^{t-u}\sin 2u\,du.$$

$$= \frac{1}{2}\int_{u=0}^{t} e^{t}e^{-u}\sin 2u\,du$$

$$= \frac{1}{2}e^{t}\int_{u=0}^{t} e^{-u}\sin 2u\,du$$

$$= \frac{1}{2}e^{t}\left\{\frac{e^{-u}}{(-1)^2+2^2}\left[-\sin 2u - 2\cos 2u\right]\right\}_0^t$$

$$= -\frac{e^t}{2}\left\{\frac{e^{-u}}{5}\left[\sin 2u + 2\cos 2u\right]\right\}_0^t$$

$$= -\frac{e^t}{2}\left\{\frac{e^{-t}}{5}(\sin 2t - 2\cos 2t) - \frac{1}{5}(0+2)\right\}$$

$$= -\frac{e^t}{2}\left\{\frac{e^{-t}}{5}(\sin 2t - 2\cos 2t) - \frac{2}{5}\right\}$$

$$= -\frac{1}{10}(\sin 2t + 2\cos 2t) + \frac{e^t}{5}$$

$\therefore L^{-1}\left[\dfrac{1}{(s-1)(s^2+4)}\right] = \dfrac{e^t}{5} - \dfrac{1}{10}(\sin 2t + 2\cos 2t)$ //

4] $\dfrac{1}{s(s^2+a^2)}$

$\Rightarrow \bar{f}(s).\bar{g}(s) = \dfrac{1}{s(s^2+a^2)} = \dfrac{1}{s}\cdot\dfrac{1}{(s^2+a^2)}$

$\bar{f}(s) = \dfrac{1}{s} = L^{-1}[\bar{f}(s)] = L^{-1}\left[\dfrac{1}{s}\right] = 1 = f(t)$

$\bar{g}(s) = \dfrac{1}{s^2+a^2} = L^{-1}[\bar{g}(s)] = L^{-1}\left[\dfrac{1}{s^2+a^2}\right] = \dfrac{1}{a}\sin at = g(t)$

$\therefore$ WKT $L^{-1}[\bar{f}(s).\bar{g}(s)] = \displaystyle\int_{u=0}^{t} f(t-u)\,g(u)\,du.$

$$= \int_{u=0}^{t} 1.\frac{1}{a}\sin au\,du$$

$$= \frac{1}{a}\int_{u=0}^{t}\sin au\,du$$

$$= -\frac{1}{a}\left[\frac{\cos au}{u}\right]_0^t$$

$$= \frac{-1}{a^2}\left[\cos at - \cancel{\cos 0}^{\,1}\right]$$

$$= \frac{-1}{a^2}\left[\cos at - 1\right]$$

$$= \frac{1}{a^2} - \frac{1}{a^2}\cos at$$

$$\therefore L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1}{a^2}\left[1-\cos at\right]_{//}$$

5] $\dfrac{s}{(s^2+a^2)^2}$

$\Rightarrow$ Let $\bar{g}(s).\bar{f}(s) = \dfrac{s}{s^2+a^2} = \dfrac{1}{(s^2+a^2)}\cdot\dfrac{s}{(s^2+a^2)}$

$\therefore \bar{f}(s) = \dfrac{1}{s^2+a^2} = L^{-1}[\bar{f}(s)] = L^{-1}\left[\dfrac{1}{s^2+a^2}\right] = \dfrac{1}{a}\sin at = f(t)$

$\bar{g}(s) = \dfrac{s}{s^2+a^2} = L^{-1}[\bar{g}(s)] = L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at = g(t)$

$$= \frac{1}{2}\int_{u=0}^{t} 2\cos au \, \cos(at-au)\,du.$$

$$= \frac{1}{2}\int_{u=0}^{t} \left[\cos[au+at-au] + \cos[au-at+au]\right]du$$

$$= \frac{1}{2}\int_{u=0}^{t} \cos at + \cos[2au-at]\,du$$

$$= \frac{1}{2}\left\{\cos at \int_{u=0}^{t} 1\,du + \int_{u=0}^{t} \cos(2au-at)\,du\right.$$

$$= \frac{1}{2}\left\{\cos at\,[u]_0^t + \left[\frac{\sin(2au-at)}{2a}\right]_0^t\right\}$$

$$= \frac{1}{2}\left\{\cos at \cdot t + \frac{1}{2a}\left[\sin(2at-at) + \sin at\right]\right\}$$

$$= \frac{1}{2}\left\{t\cos at + \frac{1}{2a}\left(\cancel{\sin at - \sin at}\right)\right\}$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)}\right] = \frac{1}{2}t\cos at_{//}$$

6] $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

Let $\bar{f}(s).\,\bar{g}(s) = \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}$

$\bar{f}(s) = \frac{s}{s^2+a^2} = L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at = f(t)$

$\bar{g}(s) = \frac{s}{s^2+b^2} = L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos bt = g(t)$

W.K.T $L^{-1}[\bar{f}(s).\,\bar{g}(s)] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$

$$= \int_{u=0}^{t} \cos au \, \cos b(t-u)\,du$$

$$= \frac{1}{2}\int_{u=0}^{t} 2\cos au \, \cos(bt-bu)\,du$$

$$= \frac{1}{2}\int_{u=0}^{t}\left\{\cos[(a-b)u+bt] + \cos[(a+b)u-bt]\right\}du$$

$$= \frac{1}{2}\left\{\left[\frac{\sin[a-b]u+bt}{a-b}\right]_0^t + \left[\frac{\sin[(a+b]u-bt}{a+b}\right]_0^t\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{a-b}\left[\sin at - \sin bt\right] + \frac{1}{a+b}\left[\sin at + \sin bt\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{1}{a-b} + \frac{1}{a+b}\right]\sin at - \left[\frac{1}{a-b} - \frac{1}{a+b}\right]\sin bt\right\}$$

$$= \frac{1}{2}\left\{\frac{2a}{a^2-b^2}\sin at - \frac{2b}{a^2-b^2}\sin bt\right\}$$

$$= \frac{a}{a^2-b^2}\sin at - \frac{b}{a^2-b^2}\sin bt$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{1}{a^2-b^2}\left[a\sin at - b\sin bt\right]$$

7) $\frac{1}{(s^2+a^2)(s^2+b^2)}$

→ Let $\bar{f}(s).\bar{g}(s) = \frac{1}{s^2+a^2}\cdot\frac{1}{s^2+b^2}$

$$\therefore \bar{f}(s) = \frac{1}{s^2+a^2} = L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at = f(t).$$

$$\bar{g}(s) = \frac{1}{s^2+b^2} = L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{1}{s^2+b^2}\right] = \frac{1}{b}\sin bt = g(t).$$

$$\therefore \text{WKT } L^{-1}[\bar{f}(s).\bar{g}(s)] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$$

$$= \int_{u=0}^{t}\frac{1}{a}\sin(au) - \frac{1}{b}\sin(b(t-u))\,du.$$

$$= \frac{1}{2ab}\int_{u=0}^{t} 2\sin(au)\sin(bt-bu)\,du.$$

$$= \frac{1}{2ab}\int_{u=0}^{t}\left\{\cos[(au-bt+bu)] - \cos[au+bt-bu]\right\}du.$$

$$= \frac{1}{2ab}\int_{u=0}^{t}\left\{\cos\left[(a+b)u-bt\right]-\cos\left[(a-b)u+bt\right]\right\}du.$$

$$= \frac{1}{2ab}\left\{\left[\frac{\sin(a+b)u-bt}{a+b}\right]_{u=0}^{t}-\left[\frac{\sin[(a-b)u+bt]}{a-b}\right]_{0}^{t}\right\}$$

$$= \frac{1}{2ab}\left\{\frac{1}{a+b}\left[\sin at+\sin bt\right]-\frac{1}{a-b}\left[\sin at-\sin bt\right]\right\}$$

$$= \frac{1}{2ab}\left\{\left[\frac{1}{a+b}-\frac{1}{a-b}\right]\sin at+\left[\frac{1}{a+b}+\frac{1}{a-b}\right]\sin bt\right\}$$

$$= \frac{1}{2ab}\left\{\frac{-2b}{a^2-b^2}\sin at+\frac{2a}{a^2-b^2}\sin bt\right\}$$

$$= \frac{1}{a^2-b^2}\left[-\frac{1}{a}\sin at+\frac{1}{b}\sin bt\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]=\frac{1}{a^2-b^2}\left[-\frac{1}{a}\sin at+\frac{1}{b}\sin bt\right]$$

8] $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$

$\therefore$ Let $\bar{f}(s).\bar{g}(s) = \dfrac{1}{(s^2+a^2)}\cdot\dfrac{s}{(s^2+b^2)}$

$\therefore$ $\bar{f}(s) = \dfrac{1}{s^2+a^2} = L^{-1}\left[\dfrac{1}{s^2+a^2}\right] = \dfrac{1}{a}\sin at = f(t)$

$\bar{g}(s) = \dfrac{s}{s^2+b^2} = L^{-1}\left[\dfrac{s}{s^2+b^2}\right] = \cos bt = g(t)$

W.K.T $L^{-1}\left[\bar{f}(s).\bar{g}(s)\right] = \displaystyle\int_{u=0}^{t} f(t-u)\, g(t)\,dt$

$$= \int_{u=0}^{t}\frac{1}{a}\sin a(t-u)\cos(bu)\,du$$

$$= \frac{1}{a}\int_0^{u}\left[\sin(at-au).\cos(bu)\right]du$$

$$= \frac{1}{2a}\int_0^{t}\left[\sin(at-au+bu)+\sin(at-au-bu)\right]du$$

$$= \frac{1}{2a}\left[\frac{-\cos(at-au+bu)}{(-a+b)} - \frac{\cos(at-au-bu)}{(-a-b)}\right]_{u=0}^{t}$$

$$= \frac{1}{2a}\left[\frac{-\cos(at-at+bt)}{(b-a)} - \frac{\cos(at-at-bt)}{-(a+b)}\right] + \frac{\cos bt}{b-a} + \left.\frac{\cos at}{-(a+b)}\right]$$

$$= \frac{1}{2a}\left[\frac{-\cos bt}{b-a} + \frac{\cos(-bt)}{a+b} + \frac{\cos at}{b-a} + \frac{\cos at}{a+b}\right]$$

$$= \frac{1}{2a}\left[\frac{-\cos bt}{b-a} - \frac{\cos bt}{a+b} + \frac{\cos at}{b-a} + \frac{\cos at}{a+b}\right]$$

$$= \frac{1}{2a}\left[\left(\frac{1}{a+b} - \frac{1}{b-a}\right)\cos bt + \left(\frac{1}{b-a} + \frac{1}{a+b}\right)\cos at\right]$$

$$= \frac{1}{2a}\left[\left(\frac{b-a-a-b}{b^2-a^2}\right)\cos bt + \left(\frac{a+b+b-a}{b^2-a^2}\right)\cos at\right]$$

$$= \frac{1}{2a}\left[\left(\frac{-2a}{b^2-a^2}\right)\cos bt + \left(\frac{2b}{b^2-a^2}\right)\cos at\right]$$

$$= \frac{1}{a(b^2-a^2)}\left[b\cos at - a\cos at\right]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+b^2)}\right] = \frac{1}{a(b^2-a^2)}\left[b\cos at - a\cos bt\right]$$

9] $\frac{1}{(s^2+a^2)^2}$

$\Rightarrow$ Let $\bar{f}(s).\bar{g}(s) = \frac{1}{(s^2+a^2)^2} = \frac{1}{(s^2+a^2)} \cdot \frac{1}{(s^2+a^2)}$

$\therefore \bar{f}(s) = \frac{1}{s^2+a^2} = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at = f(t)$

$\bar{g}(s) = \frac{1}{s^2+a^2} = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at = g(t)$

W.K.T = $L^{-1}\left[\bar{f}(s).\,\bar{g}(s)\right] = \int_{u=0}^{t} f(t-u)\,g(u)\,du$

$$= \int_{u=0}^{t} \frac{1}{a}\sin a(t-u)\,\frac{1}{a}\sin au\,du$$

$$= \frac{1}{2a^2}\int_{u=0}^{t} 2\sin(at-au)\sin au\,du$$

$$= \frac{1}{2a^2}\int_{u=0}^{t}\left[\cos(at-au-au) - \cos(at-au+au)\right]du.$$

$$= \frac{1}{2a^2}\int_{u=0}^{t}\left[\cos(at-2au) - \cos at\right]du$$

$$= \frac{1}{2a^2}\left\{\left[\frac{\sin(at-2au)}{-2a}\right]_0^t - \cos at\,(u)_0^t\right\}$$

$$= \frac{1}{2a^2}\left\{-\frac{1}{2a}\left(-\sin at - \sin at\right) - \cos at.\,t\right\}$$

$$= \frac{1}{2a^2}\left\{-\frac{1}{2a}\left(-2\sin at\right) - \cos at\right\}$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at\right]_{\prime\prime}$$

10) $\dfrac{1}{(s^2+1)(s^2+9)}$

$\Rightarrow$ Let $\bar{f}(s).\,\bar{g}(s) = \dfrac{1}{(s^2+1)(s^2+9)} = \dfrac{1}{s^2+1}\cdot\dfrac{1}{s^2+9}$

$\therefore\ \bar{f}(s) = \dfrac{1}{s^2+1} = L^{-1}\left[\dfrac{1}{s^2+1}\right] = \sin t = f(t)$

$\bar{g}(s) = \dfrac{1}{s^2+9} = L^{-1}\left[\dfrac{1}{s^2+9}\right] = \dfrac{1}{3}\sin 3t = g(t)$

WKT $L^{-1}\left[\bar{f}(s).\bar{g}(s)\right] = \int_{u=0}^{t} f(t-u)\,g(u)\,du$

$$= \int_{u=0}^{t} \sin(t-u)\,\frac{1}{3}\sin 3u\,du.$$

$$= \frac{1}{2\times 3}\int_{u=0}^{t} 2\sin(t-u)\cdot \sin 3u\,du.$$

$$= \frac{1}{6}\int_{u=0}^{t}\left[\cos(t-u-3u) - \cos(t-u+3u)\right]du$$

$$= \frac{1}{6}\int_{u=0}^{t}\left[\cos(t-4u) - \cos(t+2u)\right]du.$$

$$= \frac{1}{6}\left\{\left[\frac{\sin t-4u}{-4}\right]_0^t - \left[\frac{\sin(t+2u)}{2}\right]_0^t\right\}$$

$$= \frac{1}{6}\left\{\left[\frac{\sin(t-4t)}{-4} + \frac{\sin(t-0)}{4}\right] - \left[\frac{\sin 3t}{2} - \frac{\sin t}{2}\right]\right\}$$

$$= \frac{1}{24}\sin 3t + \frac{\sin t}{24} - \frac{\sin 3t}{12} + \frac{\sin t}{12}$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+1)(s^2+9)}\right] = -\frac{1}{24}\sin 3t + \frac{1}{8}\sin t_{//}$$

Applications of Laplace transforms:-

Solutions of differential equation using laplace transform.

Step 1:- Consider the differential Equation as $f(t, y, y', y'' \ldots y^n) = 0$ with the initial Conditions $y(0), y'(0), y''(0)$ ------.

Step 2:- Apply laplace transform on both sides & Hence Substitute following.

$$L\left[y^{(n)}(t)\right] = s^n \bar{y}(s) - s^{n-1}y(0) - s^{n-2}y'(0) - s^{n-3}y''(0) - \ldots$$

and $L[y(t)] = \bar{y}(s)$

$$L[y'(t)] = s\,\bar{y}(s) - y(0)$$

$$L[y''(t)] = s^2\,\bar{y}(s) - s\,y(0) - y'(0).$$

$$L[y'''(t)] = s^3\bar{y}(s) - s^2 y(0) - s y'(0) - y''(0)$$

Step 3:- Solve the Algebraic Expression and Hence write $\bar{y}(s)$ in terms of s.

Step 4:- Apply inverse laplace transform on b.s and get the solution 'y' in terms of 't'.

1] Solve by using laplace transform techniques.

$y'' - 3y' + 2y = e^{3t}$. $y(0)=1$, $y'(0)=-1$.

$$\Rightarrow \quad y'' - 3y' + 2y = e^{3t} \longrightarrow (1)$$

and $y(0)=1$, $y'(0)=-1$

$$\therefore (1) \Rightarrow L[y''] - 3L[y'] + 2L[y] = L[e^{3t}]$$

$$\Rightarrow [s^2\bar{y}(s)] - s y(0) - y'(0)] - 3[s\bar{y}(s) - y(0)] + 2\bar{y}(s) = \frac{1}{s-3}$$

$$\Rightarrow [s^2\bar{y}(s) - s + 1] - 3[s\bar{y}(s) - 1] + 2\bar{y}(s) = \frac{1}{s-3}$$

$$\Rightarrow s^2\bar{y}(s) - s + 1 - 3s\bar{y}(s) + 3 + 2\bar{y}(s) = \frac{1}{s-3}$$

$$\Rightarrow (s^2 - 3s + 2)\bar{y}(s) - (s-4) = \frac{1}{s-3}$$

$$\Rightarrow (s^2 - 3s + 2)\bar{y}(s) = \frac{1}{s-3} + (s-4)$$

$$\Rightarrow (s^2 - s - 2s + 2)\bar{y}(s) = \frac{1 + (s-3)(s-4)}{s-3}$$

$$\Rightarrow [s(s-1) - 2(s-1)]\bar{y}(s) = \frac{s^2 - 7s + 13}{s-3}$$

$$\Rightarrow (s-1)(s-2)\bar{y}(s) = \frac{s^2 - 7s + 13}{s-3}$$

$$\Rightarrow \bar{y}(s) = \frac{s^2 - 7s + 13}{(s-1)(s-2)(s-3)} \longrightarrow (2)$$

$$\therefore \frac{s^2 - 7s + 13}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$\Rightarrow S^2 - 7S + 13 = A(S-2)(S-3) + B(S-1)(S-3) + C(S-1)(S-2)$$

when $S=1$

$\therefore (1) \Rightarrow 7 = A(-1)(-2)$

$$\boxed{A = 7/2}$$

when $S=2$

$(2) \Rightarrow 3 = B(1)(-1)$

$$\boxed{B = -3}$$

when $S=3$

$(2) \Rightarrow 1 = C(2)(1)$

$$\boxed{C = 1/2}$$

$$(2) = \bar{y}(S) = \frac{7}{2}\left(\frac{1}{S-1}\right) - 3\left(\frac{1}{S-2}\right) + \frac{1}{2}\left(\frac{1}{S-3}\right)$$

$$L^{-1}[\bar{y}(S)] = \frac{7}{2}e^{t} - 3e^{2t} + \frac{1}{2}e^{3t}$$

(2) $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$ where $x(0) = 2$, $x'(0) = -1$.

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$$

$$\Rightarrow x''(t) - 2x'(t) + x(t) = e^t \longrightarrow (1)$$

and $x(0) = 2$, $x'(0) = -1$

$$\therefore (1) \Rightarrow L[x''(t)] - 2L[x'(t)] + L[x(t)] = L[e^t]$$

$$\Rightarrow \left[S^2\bar{x}(S) - S(x)(0) - x'(0)\right] - 2\left[S\bar{x}(S) - x(0)\right] + \bar{x}(S) = \frac{1}{S-1}$$

$$\Rightarrow \left[S^2\bar{x}(S) - 2S + 1\right] - 2\left[S\bar{x}(S) - 2\right] + \bar{x}(S) = \frac{1}{S-1}$$

$$\Rightarrow S^2\bar{x}(S) - 2S + 1 - 2S\bar{x}(S) + 4 + \bar{x}(S) = \frac{1}{S-1}$$

$$\Rightarrow (S^2 - 2S + 1)\bar{x}(S) - (2S-5) = \frac{1}{S-1}$$

$$\Rightarrow (S-1)^2\bar{x}(S) = \frac{1}{S-1} + (2S-5)$$

$$\Rightarrow (S-1)^2\bar{x}(S) = \frac{1 + (S-1)(2S-5)}{S-1}$$

$$\Rightarrow \bar{x}(S) = \frac{1 + 2S^2 - 5S - 2S + 5}{(S-1)^3}$$

$$\Rightarrow \bar{x}(S) = \frac{2S^2 - 7S + 6}{(S-1)^3} \longrightarrow (2)$$

$$\frac{2S^2 - 7S + 6}{(S-1)^3} = \frac{A}{S-1} + \frac{B}{(S-1)^2} + \frac{C}{(S-1)^3}$$

$$2S^2 - 7S + 6 = A(S-1)^2 + B(S-1) + C \longrightarrow (3)$$

when $s=1$

$(3) \Rightarrow 1 = C$

$\boxed{C=1}$

s^2 Co-efficient

$\boxed{A=2}$

Constant Equaling.

$A+B+C=6$

$2-B+1=6$

$-B=6-3$

$\boxed{B=-3}$

$$(2) \Rightarrow \bar{x}(s) = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$\Rightarrow L^{-1}\left[\bar{x}(s)\right] = 2L^{-1}\left[\frac{1}{s-1}\right] - 3L^{-1}\left[\frac{1}{(s-1)^2}\right] + L^{-1}\left[\frac{1}{(s-1)^3}\right]$$

$$\Rightarrow \quad 2e^t - 3e^t \cdot L^{-1}\left[\frac{1}{s^2}\right] + e^t L^{-1}\left[\frac{1}{s^3}\right]$$

$$\Rightarrow \quad x(t) = 2e^t - 3te^t + \frac{t^2}{2}e^t$$

(3) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} = e^{-t}$, when $y(0)=0$, $y''(0)=0$.

given $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}$

$$\Rightarrow y''(t) + 4y'(t) + 4y(t) = e^{-t} \longrightarrow (1)$$

and $y(0)=0$, $y'(0)=0$.

$$\therefore (1) \Rightarrow L\left[y''(t)\right] + 4L\left[y'(t)\right] + 4L\left[y(t)\right] = L\left[e^{-t}\right]$$

$$\Rightarrow \left[s^2\bar{y}(s) - sy(0) - y'(0)\right] + 4\left[s\bar{y}(s) - y(0)\right] + 4\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow s^2\bar{y}(s) + 4s\bar{y}(s) + 4\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow (s^2+4s+4)\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow (s+2)^2\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow \bar{y}(s) = \frac{1}{(s+1)(s+2)^2} \longrightarrow (2)$$

$$\therefore \frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$\Rightarrow 1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1) \longrightarrow (3)$$

when $s=-1$

$(1) \quad 1 = A \quad \boxed{A=1}$

when $s=-2$

$(2) \Rightarrow 1 = -C$

$\boxed{C=-1}$

s^2-Co-efficient

$A+B=0$

$B=-A$

$\boxed{B=-1}$

$\therefore$ (2) $\Rightarrow \bar{y}(s) = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$

$\Rightarrow L^{-1}[\bar{y}(s)] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{(s+2)^2}\right]$

$\Rightarrow e^{-t} L^{-1}\left[\frac{1}{s}\right] - e^{-2t} L^{-1}\left[\frac{1}{s}\right] - e^{-2t} L^{-1}\left[\frac{1}{s^2}\right]$

$\Rightarrow y(t) = e^{-t} - e^{-2t} - te^{-2t}.$

(4) $y'' + 3y' + 2y = e^{-t}$ when $y(0)=0$, $y'(0)=1$.

Given :- $y''(t) + 3y'(t) + 2y(t) = e^{-t} \longrightarrow (1)$

and $y(0)=0$, $y'(0)=1$

$\Rightarrow L[y''(t)] + 3L[y'(t)] + 2L[y(t)] = L[e^{-t}]$

$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] + 3[s\bar{y}(s) - y(0)] + 2[\bar{y}(s)] = \frac{1}{s+1}$

$\Rightarrow (s^2\bar{y}(s) - 1) + 3(s\bar{y}(s)) + 2\bar{y}(s) = \frac{1}{s+1}$

$\Rightarrow s^2\bar{y}(s) - 1 + 3s\bar{y}(s) + 2\bar{y}(s) = \frac{1}{s+1}$

$\Rightarrow (s^2 + 3s + 2)\bar{y}(s) - 1 = \frac{1}{s+1}$

$\Rightarrow (s^2 + s + 2s + 2)\bar{y}(s) - 1 = \frac{1}{s+1}$

$\Rightarrow [s(s+1) + 2(s+1)]\bar{y}(s) - 1 = \frac{1}{s+1}$

$\Rightarrow (s+1)(s+2)\bar{y}(s) = \frac{1}{s+1} + 1$

$\Rightarrow \bar{y}(s) = \frac{1 + s + 1}{(s+1)(s+1)(s+2)}$

$\Rightarrow \bar{y}(s) = \frac{1(s+1)}{(s+1)^2(s+2)} \longrightarrow (2)$

$\frac{s+2}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$

$s+2 = A(s+1)(s+2) + B(s+2) + C(s+1)^2 \longrightarrow (3)$

when $s=-1$

$(3) \Rightarrow 1 = B(1)$

$\boxed{B=1}$

when $s=-2$

$(3) \Rightarrow \boxed{C=0}$

s^2- Co-efficient

$A+C=0$

$A=-C$

$\boxed{A=0}$

$\therefore$ Eq^n (2) $= \bar{y}(s) = \frac{1}{(s+1)^2} \Rightarrow L^{-1}[\bar{y}(s)] = L^{-1}\left[\frac{1}{(s+1)^2}\right]$

$$y(t) = \underline{\underline{te^{-t}}}$$

(5) $y'' + 2y' - 3y = \sin t \longrightarrow ①$ when $y(0)=0$, $y'(0)=0$.

Given :- $y'' + 2y' - 3y = \sin t \longrightarrow (1)$

$\therefore (1) \rightarrow L[y''(t)] + 2L[y'(t)] - 3L[y(t)] = L[\sin t]$

$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] + 2[s\bar{y}(s) - y(0)] - 3\bar{y}(s) = \frac{1}{s^2+1}$

$\Rightarrow (s^2+2s-3)\,\bar{y}(s) = \frac{1}{s^2+1}$

$\Rightarrow (s-1)(s+3)\,\bar{y}(s) = \frac{1}{s^2+1}$

$\Rightarrow \bar{y}(s) = \frac{1}{(s-1)(s+3)(s^2+1)} \longrightarrow (2).$

$\therefore \frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$

$\Rightarrow 1 = A(s+3)(s^2+1) + B(s-1)(s^2+1) + Cs + D(s-1)(s+3) \longrightarrow (2)$

$$\Rightarrow \quad 1 = A(s^3+s+3s^2+3) + B(s^3+s-s^2-1) + Cs+D(s^2+2s-3)$$

$$\Rightarrow \quad 1 = A(s^3+3s^2+s+3) + B(s^3-s^2+s-1) + Cs^3+2Cs^2-3Cs+$$
$$Ds^2+2Ds-3D$$

$$\Rightarrow \quad 1 = (A+B+C)s^3 + (3A-B+2C+D)s^2 + (A+B-3C+2D)s$$
$$+ (3A-B-3D) \longrightarrow (3)$$

when $s=1$

$(2) \Rightarrow 1 = 8A$

$$\boxed{A = \frac{1}{8}}$$

when $s=-3$

$(2) \Rightarrow 1 = -40B$

$$\boxed{B = -\frac{1}{40}}$$

when $A+B+C=0$

$C = -A-B$

$$\boxed{C = -\frac{1}{10}}$$

$3A-B-3D=1$

$3D-3A-B=1$

$3D = \frac{3}{8} + \frac{1}{40} - 1$

$$\boxed{D = -\frac{1}{5}}$$

$$\therefore \quad \bar{y}(s) = \frac{1}{8}\,\frac{1}{s-1} - \frac{1}{40}\,\frac{1}{s+3} - \frac{1}{10}\,\frac{s}{s^2+1} - \frac{1}{5}\,\frac{1}{s^2+1}$$

$$y(t) = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t$$

6] $y'' - 3y' + 2y = e^{3t}$ with $y(0)=1$; $y'(0)=0$.

given :- $y''(t) - 3y'(t) + 2y(t) = e^{3t} \longrightarrow (1)$

$$\Rightarrow L[y''(t)] - 3[Ly'(t)] + 2L[y(t)] = L[e^{3t}]$$

$$\Rightarrow [s^2\bar{y}(s) - s\,y(0)] - 3s\bar{y}(s) + 3y(0) + 2\bar{y}(s) = \frac{1}{s-3}$$

$$\Rightarrow \quad s^2\bar{y}(s) - s - 3s\bar{y}(s) + 3 + 2\bar{y}(s) = \frac{1}{s-3}$$

$$\Rightarrow (s^2-3s+2)\,\bar{y}(s) - s+3 = \frac{1}{s-3}$$

$$\Rightarrow (s^2-3s+2)\,\bar{y}(s) - (s-3) = \frac{1}{s-3}$$

$$\Rightarrow (s-1)(s-2)\,\bar{y}(s) = \frac{1+(s-3)^2}{(s-3)}$$

$$\Rightarrow \bar{y}(s) = \frac{1+s^2+9-3\times 2s}{(s-1)(s-2)(s-3)}$$

$$\Rightarrow \bar{y}(s) = \frac{s^2-6s+10}{(s-1)(s-2)(s-3)} \longrightarrow (2)$$

$$\therefore \frac{s^2-6s+10}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$\Rightarrow s^2-6s+10 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2) \longrightarrow (2)$$

when $s=1$

$(2) \Rightarrow 1-6+10 = A(-1)(-2)$

$2A = 5$

$\boxed{A = 5/2}$

when $s=2$

$(2) \Rightarrow 4-12+10 = B(2-1)(2-3)$

$2 = B(-1)$

$\boxed{B = -2}$

when $s=3$

$(2) \Rightarrow 9-18+10 = C(3-1)(3-2)$

$1 = C(2)$

$\boxed{C = 1/2}$

$$(2) \Rightarrow \bar{y}(s) = \frac{5}{2}\,\frac{1}{s+1} + [-2]\frac{1}{s-2} + \frac{1}{2}\left[\frac{1}{s-3}\right]$$

$$L^{-1}[\bar{y}(s)] = \frac{5}{2}L^{-1}\left[\frac{1}{s+1}\right] + [-2]L^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s-3}\right]$$

$$y(t) = \frac{5}{2}e^{t} \cdot 2e^{2t} + \frac{1}{2}e^{3t} \;//$$

(7) $x'' - 2x' + x = e^{2t}$ with $x(0)=1$, $x'(0)=-1$.

Given:- $x''(t) - 2x'(t) + x(t) = e^{2t}$

$$\Rightarrow L[x''(t)] - 2L[x'(t)] + L[x(t)] = L[e^{2t}]$$

$$\Rightarrow [s^2\bar{x}(s) - s\,x'(0) - x'(0)] - 2[s\bar{x}(s) - x(0)] + \bar{x}(s) = \frac{1}{s-2}$$

$$\Rightarrow s^2\bar{x}(s) - s + 1 - 2s\bar{x}(s) + 2 + \bar{x}(s) = \frac{1}{s-2}$$

$$\Rightarrow (s^2 - 2s + 1)\bar{x}(s) - s + 3 = \frac{1}{s-2}$$

$$\Rightarrow (s^2 - 2s + 1)\bar{x}(s) - (s-3) = \frac{1}{s-2}$$

$$\Rightarrow (s-1)^2(\bar{x}(s)) = \frac{1}{s-2} + (s-3)$$

$$\Rightarrow (s-1)^2\bar{x}(s) = \frac{1 + (s-3)(s-2)}{(s-2)}$$

$$\Rightarrow (s-1)^2\bar{x}(s) = \frac{s^2 - 5s + 7}{(s-2)}$$

$$\bar{x}(s) = \frac{s^2 - 5s + 7}{(s-2)(s-1)^2} \longrightarrow (1)$$

$$s^2 - 5s + 7 = A(s-1)^2 + B(s-1)(s-2) + C(s-2) \longrightarrow (2)$$

when $s = 2$

$4 - 10 + 7 = A(1)^2$

$\boxed{1 = A}$

when $s = 1$

$1 - 5 + 7 = C(-1)$

$3 = -C$

$\boxed{C = -3}$

$1 = A - B.$

$B = 1 - A$

$B = 1 - 1$

$\boxed{B = 0}$

$$\Rightarrow \bar{x}(s) = \frac{1}{(s\cancel{+}2)} - \frac{3}{(s-1)^2}$$

$$\Rightarrow L^{-1}[\bar{x}(s)] = L^{-1}\left[\frac{1}{s-2}\right] + 3L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$x(t) = e^{2t} - 3e^t t //$$

(8) $y'' + 4y' + 3y = e^{-t}$ with $y(0) = 1$ and $y'(0) = 1$

Given:- $y''(t) + 4y'(t) + 3y(t) = e^{-t}$

$$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] + 4[s\bar{y}(s) - y(0)] + 3\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow s^2\bar{y}(s) - s - 1 + 4s\bar{y}(s) - 4 + 3\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow [s^2 + 4s + 3]\bar{y}(s) - (s+5) = \frac{1}{s+1}$$

$$\Rightarrow (s^2+4s+3)\,\bar{y}(s) = \frac{1}{s+1} + (s+5)$$

$$\Rightarrow (s+1)(s+3)\,\bar{y}(s) = \frac{1+(s+1)(s+5)}{(s+1)}$$

$$\therefore \bar{y}(s) = \frac{s^2+6s+6}{(s+1)^2(s+3)} \longrightarrow ②$$

$$\therefore \frac{s^2+6s+6}{(s+1)^2(s+3)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3} \longrightarrow ②$$

$$s^2+6s+6 = A(s+1)(s+3) + B(s+3) + C(s+1)^2 \longrightarrow ③$$

when $s=-1$

(2) $1 = 2B$

$\boxed{B = 1/2}$

when $s=-3$

$-3 = 4C$

$\boxed{C = \frac{-3}{4}}$

$A + C = 1$

$A = 1 - C$

$\boxed{A = 7/4}$

$$(2) \Rightarrow \bar{y}(s) = \frac{7}{4}\cdot\frac{1}{s+1} + \frac{1}{2}\,\frac{1}{(s+1)^2} - \frac{3}{4}\,\frac{1}{s+3}$$

$$L^{-1}[\bar{y}(s)] = \frac{7}{4}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}\right] - \frac{3}{4}L^{-1}\left[\frac{1}{s+3}\right]$$

$$y(t) = \frac{7}{4}e^{-t} + \frac{1}{2}e^{t} - \frac{3}{4}e^{-3t}$$