

21MAT31 MODULE-2

FOURIER SERIES

Introduction :-

Expressing the function $f(x)$ in the combinations of constants and trigonometric ratios in any other interval $(0, \pi)$ $(0, 2\pi)$ is called the Fourier Series.

$$\therefore f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots + a_n \cos nx + \dots + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots + b_n \sin nx + \dots$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a_0 is called the constant.

a_n, b_n are called Fourier Co-efficients.

Fourier series Expansion of $f(x)$ over the period 2π :-

The Fourier series expansion of $f(x)$ over the interval $(c, c+2\pi)$ can be denoted as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

$$\text{where } a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx dx$$

This is also called Dirichlet's Property.

If $c=0$, then the constant and the Fourier coefficients of $f(x)$ can be evaluated as $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx$$

If $c=-\pi$, then the Fourier series coefficients of $f(x)$ in the interval $(-\pi, \pi)$ can be evaluated as

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Note:-

① Suppose $f(x)$ is the continuous function in the interval $[0, 2\pi]$ then $f(x)$ is an even function when $f(2\pi-x) = f(x)$ and $f(x)$ is an odd function when $f(2\pi-x) = -f(x)$

② If $f(x)$ is a discontinuous function over the interval $[0, 2\pi]$ $f(x) = \begin{cases} \phi(x), & 0 \leq x \leq \pi \\ \psi(x), & \pi \leq x < 2\pi \end{cases}$, then

$f(x)$ is an even function when $\phi(2\pi-x) = \psi(x)$ and $f(x)$ is an odd function when $\phi(2\pi-x) = -\psi(x)$

③ Suppose $f(x)$ is the continuous function in the interval $[-\pi, \pi]$, then

$f(x)$ is an even function when $f(-x) = f(x)$ and

$f(x)$ is an odd function when $f(-x) = -f(x)$.

④ Suppose $f(x)$ is a discontinuous function in the interval $[-\pi, \pi]$, that is $f(x) = \begin{cases} \phi(x), & -\pi \leq x < 0 \\ \psi(x), & 0 \leq x < \pi \end{cases}$, then

$f(x)$ is an even function when $\phi(-x) = \psi(x)$ and

$f(x)$ is an function when $\phi(-x) = \psi(x)$ and

$f(x)$ is an odd function when $\phi(-x) = -\psi(x)$.

Even and odd functions:-

WKT the Fourier series expansion of $f(x)$ over the period 2π is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$, then

1. If $f(x)$ is an even function, then $b_n = 0$, hence the Fourier series of $f(x)$ can be expanded as

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ is also called as half-range

cosine series in the $[0, \pi]$ for which

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

2. If $f(x)$ is an odd function, then a_0 and a_n values will be zero, hence the Fourier series expansion of

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ is also called the Fourier half-range

sin series in the $[0, \pi]$ for which $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$

Important results:-

1. $\sin(n\pi) = 0$, $\sin(2n\pi) = 0$, $\forall n \in \mathbb{Z}$
2. $\cos(n\pi) = (-1)^n$, $\cos(2n\pi) = 1$, $\forall n \in \mathbb{Z}$
3. $\sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$, $\forall n \in \mathbb{Z}$
4. $\cos\left(n + \frac{1}{2}\right)\pi = 0$, $\forall n \in \mathbb{Z}$

$$5. \sin\left(\frac{n\pi}{2}\right) = \begin{cases} (-1)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$6. \cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{\frac{n}{2}}, & \text{if } n \text{ is even} \end{cases}$$

① Obtain the Fourier Series of $f(x) = \frac{\pi-x}{2}$ in $[0, 2\pi]$ and hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

$$\text{Given } f(x) = \frac{\pi-x}{2}, \quad x \in [0, 2\pi]$$

$$\begin{aligned} f(2\pi-x) &= \frac{\pi - (2\pi-x)}{2} \\ &= \frac{\pi - 2\pi + x}{2} \\ &= \frac{-\pi + x}{2} \end{aligned}$$

$$f(2\pi-x) = -f(x)$$

$\therefore f(x)$ is an odd function in $[0, 2\pi]$

$$a_0 = 0, \quad a_n = 0$$

The Fourier Series of

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$\begin{aligned} \text{when } b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi-x}{2}\right) \sin nx \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} (\pi-x) \sin nx \, dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\pi} \left\{ (\pi-x) \int_0^{\pi} \sin nx \, dx - \int_0^{\pi} (-1) \int \sin nx \, dx \right\} \\ &= \frac{1}{\pi} \left\{ -\frac{1}{n} (\pi-x) \cos nx \right\}_0^{\pi} - \frac{1}{\pi} \int_0^{\pi} \cos nx \, dx \Bigg\} \\ &= \frac{1}{\pi} \left\{ -\frac{1}{n} (\pi-x) \cos nx \right\}_0^{\pi} - \frac{1}{n^2} [\sin nx]_0^{\pi} \Bigg\} \end{aligned}$$

$$= \frac{1}{\pi} \left\{ -\frac{1}{n} [0 - \pi] - \frac{1}{n^2} [0 - 0] \right\}$$

$$= \frac{1}{\pi} \left[\frac{\pi}{n} \right]$$

$$b_n = \frac{1}{n}$$

$$\textcircled{1} \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

$$\frac{\pi-x}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx \quad \text{--- (2)}$$

$$\textcircled{2} \Rightarrow \frac{\pi - \pi/2}{2} = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{1} \sin\left(\frac{\pi}{2}\right) + \frac{1}{2} \sin(\pi) + \frac{1}{3} \sin\left(\frac{3\pi}{2}\right) + \frac{1}{4} \sin(2\pi) + \frac{1}{5} \sin\left(\frac{5\pi}{2}\right) + \dots$$

$$\Rightarrow \frac{\pi}{4} = 1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} - \frac{1}{7} + \dots$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} //$$

$\textcircled{2}$ Obtain a series of $f(x) = \left(\frac{\pi-x}{2}\right)^2$ in $[0, 2\pi]$, Hence deduce that 1. $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

$$2. \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$$

$$\Rightarrow f(x) = \left(\frac{\pi-x}{2}\right)^2, \quad x \in [0, 2\pi]$$

$$f(x) = \left(\frac{x-\pi}{2}\right)^2 = \frac{1}{4} (x-\pi)^2$$

$$f(2\pi-x) = \frac{1}{4} [2\pi-x-\pi]^2$$

$$= \frac{1}{4} [\pi-x]^2$$

$$= \frac{1}{4} [x-\pi]^2$$

$$f(2\pi-x) = f(x)$$

$f(x)$ is an even function.

$$\Rightarrow b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \frac{1}{4} (x-\pi)^2 dx$$

$$= \frac{1}{2\pi} \int_0^{\pi} (x^2 - 2\pi x + \pi^2) dx$$

$$= \frac{1}{2\pi} \left[\frac{x^3}{3} - \frac{2\pi x^2}{2} + \pi^2 x \right]_0^{\pi}$$

$$= \frac{1}{2\pi} \left[\frac{\pi^3}{3} - \pi^3 + \pi^3 \right]$$

$$= \frac{1}{2\pi} \times \frac{\pi^3}{3}$$

$$a_0 = \frac{\pi^2}{6}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{1}{4} (x-\pi)^2 \cos nx dx$$

$$= \frac{1}{2\pi} \int_0^{\pi} (x-\pi)^2 \cos nx dx$$

$$= \frac{1}{2\pi} \left\{ \frac{1}{n} [(x-\pi)^2 \sin nx]_0^{\pi} - \frac{2}{n} \int_0^{\pi} (x-\pi) \sin nx dx \right\}$$

$$= \frac{1}{2\pi} \left[0 - \frac{2}{n} \int_0^{\pi} (x-\pi) \sin nx dx \right]$$

$$= -\frac{1}{n\pi} \int_0^{\pi} (x-\pi) \sin nx dx$$

$$= -\frac{1}{n\pi} \left\{ (x-\pi) \int_0^{\pi} \sin nx dx - \int_0^{\pi} [1 \cdot \int \sin nx dx] dx \right\}$$

$$= -\frac{1}{n\pi} \left\{ -\frac{1}{n} [(x-\pi) \cos nx]_0^{\pi} + \left[\frac{1}{n^2} \sin nx \right]_0^{\pi} \right\}$$

$$= -\frac{1}{n\pi} \left\{ -\frac{1}{n} [0 + \pi] + \frac{1}{n^2} (0) \right\}$$

$$= -\frac{1}{n\pi} \left[-\frac{\pi}{n} \right]$$

$$a_n = \frac{1}{n^2}$$

$$\textcircled{1} \Rightarrow \frac{1}{4} (x-\pi)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

$$\Rightarrow \frac{1}{4} (x-\pi)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx \quad \text{--- (2)}$$

when $x=0$

$$\textcircled{2} \Rightarrow \frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{--- (1)}$$

$$\textcircled{2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{3\pi^2 - \pi^2}{12} = \frac{\pi^2}{6}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

(ii) When $x=\pi$

$$\textcircled{2} \Rightarrow 0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

$$\Rightarrow -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \dots = -\frac{\pi^2}{12}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12} //$$

$\textcircled{3}$ Find the Fourier series expansion of $f(x) = (\pi-x)^2$ in $[0, 2\pi]$.

Given $f(x) = (\pi-x)^2, x \in [0, 2\pi]$

$$f(x) = (x-\pi)^2$$

$$f(2\pi-x) = [2\pi-x-\pi]^2 \\ = (\pi-x)^2$$

$$f(2\pi-x) = (x-\pi)^2$$

$f(x)$ is an even function.

$$\Rightarrow b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

$$\begin{aligned}
 a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) dx \\
 &= \frac{2}{\pi} \int_0^{\pi} (x-\pi)^2 dx \\
 &= \frac{2}{\pi} \int_0^{\pi} (x^2 + \pi^2 - 2x\pi) dx \\
 &= \frac{2}{\pi} \left[\frac{x^3}{3} + \pi^2 x - \frac{2x^2\pi}{2} \right]_0^{\pi} \\
 &= \frac{2}{\pi} \left[\frac{\pi^3}{3} \right] \\
 &= \frac{2\pi}{3} \cdot 2
 \end{aligned}$$

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx \\
 &= \frac{2}{\pi} \int_0^{\pi} (x-\pi)^2 \cos nx dx \\
 &= \frac{2}{\pi} \left\{ (x-\pi)^2 \int_0^{\pi} \cos nx dx - \int_0^{\pi} [2(x-\pi)] \cos nx dx \right\} \\
 &= \frac{2}{\pi} \left\{ (x-\pi)^2 \sin nx \Big|_0^{\pi} - \frac{2}{n} (x-\pi) \sin nx dx \right\} \\
 &= \frac{2}{\pi} \left\{ -0 + \frac{2}{n^2} (x-\pi) \cos nx \Big|_0^{\pi} \right\} \\
 &= \frac{2}{\pi} \left\{ -\frac{2}{n^2} (-\pi) \right\}
 \end{aligned}$$

$$a_n = \frac{4}{n^2}$$

$$\textcircled{1} \Rightarrow (x-\pi)^2 = \frac{2\pi^2}{6} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

$$(x-\pi)^2 = \frac{2\pi^2}{6} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx \quad \textcircled{2}$$

when $x=0$

$$\textcircled{2} \Rightarrow \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{4}{n^2} &= \pi^2 - \frac{\pi^2}{3} \\
 \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{2\pi^2}{12}
 \end{aligned}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{\pi^2}{6}$$

When $x = \pi$

$$(2) \Rightarrow 0 + \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos n\pi$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4}{n^2} = -\frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \dots - \frac{\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12} //$$

(4) Expand the function $f(x) = x(2\pi - x)$ in Fourier series over the limits $[0, 2\pi]$, hence deduce $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Given $f(x) = x(2\pi - x)$, $x \in [0, 2\pi]$

$$\Rightarrow f(2\pi - x) = (2\pi - x)(2\pi - 2\pi + x) \\ = x(2\pi - x)$$

$f(x)$ is even function.

$$b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x(2\pi - x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} [2\pi x - x^2] dx$$

$$= \frac{2}{\pi} \left[\pi x^2 - \frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left(\frac{2\pi^3}{3} \right)$$

$$a_0 = \frac{4\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (2\pi x - x^2) \cos nx \, dx$$

$$= \frac{2}{\pi} \left[(2\pi x - x^2) \int_0^{\pi} \cos nx \, dx - \int_0^{\pi} (2\pi - 2x) \int \cos nx \, dx \right] dx \}$$

$$= \frac{2}{\pi} \left[\frac{1}{n} (2\pi x - x^2) \sin nx \right]_0^{\pi} - \frac{2}{n^2} (\cos nx) \Big|_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{1}{n} \int_0^{\pi} (2\pi - 2x) \sin nx \, dx \right]$$

$$= \frac{2}{\pi} \left[\frac{2}{n} \int_0^{\pi} (\pi - x) \sin nx \, dx \right]$$

$$= \frac{2}{\pi} \left[\frac{2}{n} (\pi - x) \cos nx \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\frac{2}{n^2} \pi \right]$$

$$a_n = -\frac{4}{n^2}$$

$$\textcircled{1} \Rightarrow x(2\pi - x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left(-\frac{4}{n^2}\right) \cos nx$$

$$x(2\pi - x) = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx \quad \textcircled{2}$$

when $x = 0$

$$\textcircled{2} \Rightarrow \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx = \frac{2\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{12}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

when $x = \pi$

$$\textcircled{2} \Rightarrow \pi^2 = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos n\pi$$

$$\sum_{n=1}^{\infty} \frac{4}{n^2} \cos n\pi = \frac{2\pi^2}{3} - \pi^2$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots - \frac{\pi^2}{12}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \dots - \frac{\pi^2}{12} //$$

⑤ obtain the fourier series of $f(x) = \begin{cases} x, & 0 \leq x < \pi \\ 2\pi - x, & \pi \leq x < 2\pi \end{cases}$ in $[0, 2\pi]$.

$$\text{Given : } f(x) = \begin{cases} x, & 0 \leq x < \pi \\ 2\pi - x, & \pi \leq x < 2\pi, \end{cases} \quad x \in [0, 2\pi]$$

$$\text{hence } \phi(x) = x, \quad \psi(x) = 2\pi - x$$

$$\text{Let } \phi(2\pi - x) = 2\pi - x = \psi(x)$$

$\therefore f(x)$ is an even function.

$$\Rightarrow b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{1}{\pi} [\pi^2 - 0]$$

$$a_0 = \pi$$

$$\therefore a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx$$

$$a_n = \frac{2}{\pi} \left\{ x \int_0^{\pi} \cos nx dx - \int_0^{\pi} 1 \cdot \int_0^{\pi} \cos nx dx \right\}$$

$$a_n = \frac{2}{\pi} \left\{ \frac{1}{n} (x \sin nx) \Big|_0^\pi + \frac{1}{n^2} (\cos nx) \Big|_0^\pi \right\}$$

$$a_n = \frac{2}{\pi} \left\{ \frac{1}{n} (0 - 0) + \frac{1}{n^2} (\cos n\pi - 1) \right\}$$

$$a_n = \frac{2}{\pi n^2} [(-1)^n - 1]$$

$$a_n = \begin{cases} \frac{-4}{\pi n^2}, & \forall n = 1, 3, 5, \dots \\ 0, & \forall n = 2, 4, 6, \dots \end{cases}$$

$$\textcircled{1} \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2 [(-1)^n - 1]}{\pi n^2} \cos nx //$$

- ⑥ Obtain the Fourier series of $f(x) = x^2$ in the interval $[-\pi, \pi]$ or find the Fourier half range cosin series of $f(x) = x^2$ in the interval $[0, \pi]$.

$$\text{Given } f(x) = x^2, \quad x \in [-\pi, \pi]$$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$f(x)$ is an even function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- ①}$$

$$a_0 = \frac{2}{\pi} \int_0^\pi f(x) dx$$

$$= \frac{2}{\pi} \int_0^\pi x^2 dx$$

$$= \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^\pi$$

$$a_0 = \frac{2}{3} \pi^2$$

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^\pi x^2 \cos nx dx$$

$$= \frac{2}{\pi} \left\{ x^2 \int_0^\pi \cos nx dx - \int_0^\pi [2x \int \cos nx dx] dx \right\}$$

$$= \frac{2}{\pi} \left\{ \frac{1}{\pi} [x^2 \sin nx]_0^{\pi} - \frac{2}{\pi} \int_0^{\pi} [x \sin nx] dx \right\}$$

$$= -\frac{4}{n\pi} \int_0^{\pi} x \sin nx dx$$

$$= -\frac{4}{n\pi} \left\{ x \int_0^{\pi} \sin nx dx - \int_0^{\pi} [1 \cdot \int \sin nx dx] dx \right\}$$

$$= -\frac{4}{n\pi} \left\{ -\frac{1}{n} [x \cos nx]_0^{\pi} + \frac{1}{n^2} (\sin nx)_0^{\pi} \right\}$$

$$= -\frac{4}{n\pi} \left\{ -\frac{1}{n} [\pi \cos n\pi - 0] \right\}$$

$$= \frac{4}{n^2 \pi} [(-1)^n]$$

$$a_n = \frac{4}{n^2} (-1)^n$$

$$\textcircled{D} \Rightarrow x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx //$$

$\textcircled{F}$ Obtain the fourier series $f(x) = |x|$ in the interval $[-\pi, \pi]$ and hence deduce the series $\frac{1}{1^2} + \frac{1}{3^2} + \dots + \frac{\pi^2}{8}$

$$\text{Given } f(x) = |x| = \begin{cases} -x, & -\pi \leq x \leq 0 \\ x, & 0 \leq x \leq \pi \end{cases}$$

$$\phi(x) = -x, \quad \psi(x) = x$$

$$\phi(-x) = -(-x) = x = \psi(x)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^2}{2} \right]$$

$$a_0 = \pi$$

$$\therefore a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx$$

$$= \frac{2}{\pi} \left\{ x \int_0^{\pi} \cos nx \, dx - \int_0^{\pi} 1 \cdot \int \cos nx \, dx \right\}$$

$$= \frac{2}{\pi} \left\{ \frac{1}{n} [x \sin nx]_0^{\pi} + \frac{1}{n^2} (\cos nx)_0^{\pi} \right\}$$

$$= \frac{2}{\pi n^2} [\cos n\pi - 1]$$

$$= \frac{2}{\pi n^2} [(-1)^n - 1]$$

$$\textcircled{1} \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} [(-1)^n - 1] \cos nx$$

$$\frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{(-1)^n - 1}{n^2} \right) \cos nx = \begin{cases} -x, & -\pi \leq x \leq 0 \\ x, & 0 \leq x \leq \pi \end{cases} \quad \text{--- (2)}$$

When $x = 0$

$$\textcircled{2} \Rightarrow \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} = 0$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} = -\frac{\pi^2}{4}$$

$$-\frac{2}{1^2} + 0 - \frac{2}{3^2} + 0 - \frac{2}{5^2} + \dots - \frac{\pi^2}{4}$$

$$-2 \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right] = -\frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8} //$$

$\textcircled{3}$ obtain the fourier series of $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \leq x \leq 0 \\ 1 - \frac{2x}{\pi}, & 0 \leq x \leq \pi \end{cases}$
hence deduce the series $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

$$\text{Given } f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \leq x \leq 0 \\ 1 - \frac{2x}{\pi}, & 0 \leq x \leq \pi \end{cases}$$

$$\phi(x) = 1 + \frac{2x}{\pi}, \quad \psi(x) = 1 - \frac{2x}{\pi}$$

$$\phi(x) = 1 - \frac{2x}{\pi} = \psi(x)$$

$\therefore f(x)$ is an even function

$$\Rightarrow b_n = 0$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx$$

$$= \frac{2}{\pi} \left[x - \frac{x^2}{\pi} \right]_0^{\pi}$$

$$= \frac{2}{\pi} (0 - 0)$$

$$a_0 = 0$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos nx dx$$

$$= \frac{2}{\pi} \left\{ \left(1 - \frac{2x}{\pi}\right) \int_0^{\pi} \cos nx dx - \int_0^{\pi} \left(-\frac{2}{\pi}\right) \cos nx dx \right\}$$

$$= \frac{2}{\pi} \left\{ \frac{1}{n} \left(1 - \frac{2x}{\pi}\right) \sin nx - \int_0^{\pi} \frac{2}{n^2 \pi} (\cos nx) \right\}$$

$$= \frac{2}{n} \left\{ \frac{1}{n} (0 - 0) - \frac{2}{n^2 \pi} (\cos n\pi - 1) \right\}$$

$$= \frac{4}{n^2 \pi^2} \{ (-1)^n - 1 \}$$

$$a_n = \frac{4}{n^2 \pi^2} [1 - (-1)^n]$$

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} [1 - (-1)^n] \cos nx$$

$$\frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^n}{n^2} \right] \cos nx = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \leq x \leq 0 \\ 1 - \frac{2x}{\pi}, & 0 \leq x \leq \pi \end{cases} \quad \text{--- (2)}$$

when $x = 0$

$$\textcircled{2} \Rightarrow \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} = 1$$

$$\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} = \frac{\pi^2}{4}$$

$$\frac{0}{1^2} + 0 + \frac{2}{3^2} + 0 + \frac{2}{5^2} + 0 + \dots \frac{\pi^2}{4}$$

$$2 \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \right] + \dots \frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \frac{\pi^2}{8} //$$

⑨ Obtain the fourier series for the function $f(x) = \begin{cases} -\pi, & -\pi \leq x < 0 \\ x, & 0 \leq x < \pi \end{cases}$

$$\text{Given } f(x) = \begin{cases} -\pi, & -\pi \leq x < 0 \\ x, & 0 \leq x < \pi \end{cases}$$

$$\phi(x) = -\pi \quad \psi(x) = x$$

$$\phi(x) \neq \psi(x) \text{ \& } \phi(x) \neq \psi(x)$$

The given function is neither even nor odd

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad \text{--- (1)}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^0 f(x) dx + \int_0^{\pi} f(x) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 (-\pi) dx + \int_0^{\pi} x dx$$

$$= \frac{1}{\pi} \left\{ -\pi [x]_{-\pi}^0 + \left[\frac{x^2}{2} \right]_0^{\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ -\pi (0 + \pi) + \frac{\pi^2}{2} - 0 \right\}$$

$$= \frac{1}{\pi} \left[-\pi^2 + \frac{\pi^2}{2} \right]$$

$$= \frac{1}{\pi} \left[-\frac{\pi^2}{2} \right]$$

$$a_0 = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 f(x) \cos nx \, dx + \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \left[-\pi \int_{-\pi}^0 \cos nx \, dx + \int_0^{\pi} x \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left\{ -\frac{\pi}{n} (\sin nx) \Big|_{-\pi}^0 + \int_0^{\pi} x \cos nx \, dx \right\}$$

$$= \frac{1}{\pi} \int_0^{\pi} x \cos nx \, dx$$

$$a_n = \frac{1}{\pi} \left\{ \frac{1}{n} (x \sin x) \Big|_0^{\pi} + \frac{1}{n^2} \cos nx \Big|_0^{\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{1}{n} (0 - 0) + \frac{1}{n^2} [\cos n\pi - 1] \right\}$$

$$= \frac{1}{n^2 \pi} [(-1)^n - 1]$$

$$a_n = \frac{(-1)^n - 1}{n^2 \pi}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 f(x) \sin nx \, dx + \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \left[-\pi \int_{-\pi}^0 \sin nx \, dx + \int_0^{\pi} x \sin nx \, dx \right]$$

$$= \frac{1}{\pi} \left\{ \frac{\pi}{n} [\cos nx]_{-\pi}^0 - \frac{1}{n} (x \cos nx) \Big|_0^{\pi} + \frac{1}{n^2} (\sin nx) \Big|_0^{\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{\pi}{n} (1 - \cos n\pi) - \frac{1}{n} [\pi \cos n\pi - 0] + \frac{1}{n^2} (0 - 0) \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{\pi}{n} - \frac{\pi}{n} \cos n\pi - \frac{\pi}{n} \cos n\pi \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{\pi}{n} - \frac{2\pi}{n} \cos n\pi \right\}$$

$$= \frac{1}{n} - \frac{2}{n} \cos n\pi$$

$$b_n = \frac{1 - 2(-1)^n}{n}$$

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 \pi} \cos nx + \left[\frac{1 - 2(-1)^n}{n} \right] \sin nx //$$

⑩ Find the Fourier Series of $f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ x, & 0 \leq x < \pi \end{cases}$

$$\text{Given } f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ x, & 0 \leq x < \pi \end{cases}$$

$\therefore f(x)$ & $y(x)$ either even nor odd

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad \text{--- ①}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^0 f(x) dx + \int_0^{\pi} f(x) dx \right\}$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 0 dx + \int_0^{\pi} x dx \right]$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} \right]$$

$$a_0 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) \cos nx dx + \int_0^{\pi} f(x) \cos nx dx \right]$$

$$= \frac{1}{\pi} \int_0^{\pi} x \cos nx dx$$

$$= \frac{1}{\pi} \left[x \int_0^{\pi} \cos nx dx - \int_0^{\pi} L_1 \cdot \cos nx dx \right] dx \}$$

$$= \frac{1}{\pi} \left\{ \frac{1}{n} (x \sin nx)_0^{\pi} + \frac{1}{n^2} [\cos nx]_0^{\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{1}{n} (0 - 0) + \frac{1}{n^2} (\cos n\pi - 1) \right\}$$

$$a_n = \frac{1}{\pi} \left\{ \frac{1}{n^2} (-1)^n - 1 \right\}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) \sin nx dx + \int_0^{\pi} f(x) \sin nx dx \right]$$

$$= \frac{1}{\pi} \int_0^{\pi} x \sin nx dx$$

$$= \frac{1}{\pi} \left\{ x \int_0^{\pi} \sin nx dx - \int_0^{\pi} L_1 \sin nx dx \right\}$$

$$= \frac{1}{\pi} \left\{ -\frac{1}{n} (x \cos nx)_0^{\pi} + \frac{1}{n^2} (\sin nx)_0^{\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ -\frac{1}{n} [\pi \cos n\pi - 0] + \frac{1}{n^2} (0 - 0) \right\}$$

$$b_n = -\frac{\pi \cos n\pi}{n\pi}$$

$$b_n = -\frac{(-1)^n}{n}$$

$$b_n = \frac{(-1)^{n+1}}{n}$$

$$\textcircled{1} \Rightarrow f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{1}{n^2} (-1)^n - 1 \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right]$$

② Find the Fourier series $f(x) = x - x^2, x \in [-\pi, \pi]$.

$$\text{Given } f(x) = x - x^2, x \in [-\pi, \pi]$$

$\therefore f(x)$ is neither even nor odd

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad \text{--- ①}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) dx$$

$$= \frac{1}{\pi} \left(\frac{x^2}{2} - \frac{x^3}{3} \right)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left\{ \left[\frac{\pi^2}{2} - \frac{\pi^3}{3} \right] - \left[\frac{\pi^2}{2} + \frac{\pi^3}{3} \right] \right\}$$

$$= \frac{1}{\pi} \left[-\frac{2\pi^3}{3} \right]$$

$$a_0 = -\frac{2}{3} \pi^2$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \cos nx dx$$

$$= \frac{1}{\pi} \left\{ (x - x^2) \int_{-\pi}^{\pi} \cos nx - \int_{-\pi}^{\pi} (1 - 2x) \int \cos nx dx \right\} dx$$

$$= \frac{1}{\pi} \left[(1 - 2x) \int_{-\pi}^{\pi} \sin nx dx (1 - 2x) \right]$$

$$= -\frac{1}{n\pi} \int_{-\pi}^{\pi} (1 - 2x) \sin x dx$$

$$= \frac{-1}{n\pi} \left\{ -\frac{1}{\pi} (1 - 2x) \cos nx \int_{-\pi}^{\pi} -\frac{2}{n^2} (\sin nx) \frac{\pi}{\pi} \right\}$$

$$= \frac{-1}{n\pi} \left\{ -\frac{1}{n} (1 - 2\pi) \cos n\pi - (1 + 2\pi) \cos n\pi \right\} - 0$$

$$= \frac{-1}{n\pi} \left\{ -\frac{1}{n} (\cos n\pi - 2\pi \cos n\pi - \cos n\pi - 2\pi \cos n\pi) \right\}$$

$$= \frac{-1}{n\pi} \left\{ -\frac{1}{n} (-4\pi \cos n\pi) \right\}$$

$$= \frac{-4\pi \cos n\pi}{n^2 \pi}$$

$$= \frac{-4}{n^2} \cos n\pi$$

$$= \frac{-4}{n^2} (-1)^n$$

$$= \frac{4}{n^2} (-1)^{n+1}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \sin nx dx$$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{\pi} (x - x^2) \sin nx dx - \int_{-\pi}^{\pi} (1 - 2x) \int \sin nx dx \right\}$$

$$= \frac{1}{\pi} \left\{ \left[-\frac{1}{n} (x - x^2) \cos nx \right]_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} (1 - 2x) \cos nx dx \right\}$$

$$= \frac{1}{\pi} \left\{ -\frac{1}{n} [\pi - \pi^2] \cos n\pi - (-\pi - \pi^2) \cos n\pi \right\} + \frac{1}{n} \int_{-\pi}^{\pi} (1 - 2x) \cos nx dx$$

$$= \frac{1}{\pi} \left\{ -\frac{1}{n} [2\pi \cos n\pi] + \frac{1}{n} I_1 \right\}$$

$$b_n = \frac{1}{\pi} \left[\frac{-2\pi \cos n\pi}{n} + \frac{1}{n} I_1 \right]$$

$$I_1 = \int_{-\pi}^{\pi} (1 - 2x) \cos nx dx$$

$$= \frac{1}{n} \left[(1 - 2x) \sin nx - \int_{-\pi}^{\pi} -\frac{2}{n} \cos nx \right]_{-\pi}^{\pi}$$

$$\Rightarrow I_1 = 0$$

$$b_n = \frac{-2\pi \cos n\pi}{n\pi}$$

$$= -\frac{2}{n} \cos n\pi$$

$$= -\frac{2}{n} (-1)^n$$

$$b_n = \frac{2}{n} (-1)^{n+1}$$

$$\textcircled{1} \Rightarrow (x - x^2) = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{4}{n^2} (-1)^{n+1} \cos nx + \frac{2}{n} (-1)^{n+1} \sin nx \right] \textcircled{2}$$

When $x = 0$

$$\textcircled{2} \Rightarrow 0 = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^{n+1}$$

$$4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{3}$$

$$4 \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right] = \frac{\pi^2}{3}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12} //$$

(12) Find the fourier series $f(x) = x^2 [0, \pi]$ or $f(x) = x^2 [-\pi, \pi]$

Given $f(x) = x^2, x \in [0, \pi]$

WKT

The fourier half-range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- ①}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 dx$$

$$= \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}$$

$$a_0 = \frac{2}{3} \pi^2$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$$

$$= \frac{2}{\pi} \left\{ x^2 \int_0^{\pi} \cos nx dx - \int_0^{\pi} [2x \int \cos nx dx] dx \right\}$$

$$= \frac{2}{\pi} \left\{ \frac{1}{\pi} [x^2 \sin nx]_0^{\pi} - \frac{2}{\pi} \int_0^{\pi} (x \sin nx) dx \right\}$$

$$= -\frac{4}{n\pi} \int_0^{\pi} x \sin nx dx$$

$$= -\frac{4}{n\pi} \left\{ x \int_0^{\pi} \sin nx dx - \int_0^{\pi} [1 \int \sin nx dx] dx \right\}$$

$$= -\frac{4}{n\pi} \left\{ -\frac{1}{n} [x \cos nx]_0^{\pi} + \frac{1}{n^2} [\sin nx]_0^{\pi} \right\}$$

$$= -\frac{4}{n\pi} \left\{ -\frac{1}{n} [\pi \cos n\pi - 0] \right\}$$

$$= \frac{4}{n^2\pi} [\pi (-1)^n]$$

$$a_n = \frac{4}{n^2} (-1)^n$$

$$\textcircled{1} \Rightarrow x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \dots = -\frac{\pi^2}{12}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

13) Find the fourier half range cosine & sin Series of $f(x) = x(\pi - x)$ in the $[0, \pi]$

$$\text{Given } f(x) = x(\pi - x) = \pi x - x^2, x \in [0, \pi]$$

WKT

The fourier half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \textcircled{1}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) dx$$

$$= \frac{2}{\pi} \left[\frac{\pi x^2}{2} - \frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^3}{6} \right]$$

$$= \frac{2\pi^2}{6}$$

$$a_0 = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \cos nx dx$$

$$= \frac{2}{\pi} \left\{ (\pi x - x^2) \int_0^{\pi} \cos nx dx - \int_0^{\pi} (\pi - 2x) \int \cos nx dx \right\}$$

$$= \frac{2}{\pi} \left\{ \frac{1}{n} (\pi x - x^2) \sin nx \right\}_0^{\pi} - \frac{1}{n} \int_0^{\pi} (\pi - 2x) \sin nx dx \}$$

$$= -\frac{2}{n\pi} \int_0^{\pi} (\pi - 2x) \sin nx dx$$

$$= -\frac{2}{n\pi} \left\{ (\pi - 2x) \int_0^{\pi} \sin nx dx - \int_0^{\pi} (-2) \int \sin nx dx \right\}$$

$$= -\frac{2}{n\pi} \left\{ -\frac{1}{n} [\pi - 2x] \cos nx \right\}_0^{\pi} - \frac{2}{n^2} [\sin nx]_0^{\pi} \}$$

$$= -\frac{2}{n\pi} \left\{ -\frac{1}{n} [-\pi \cos n\pi - \pi] - \frac{2}{n^2} [0 - 0] \right\}$$

$$= -\frac{2}{n\pi} \left\{ \frac{\pi}{n} (1 + \cos n\pi) \right\}$$

$$a_n = -\frac{2}{n\pi} [1 + (-1)^n]$$

$$\textcircled{1} \Rightarrow x(\pi - x) = \frac{\pi^2}{6} - 2 \sum_{n=1}^{\infty} \left[\frac{1 + (-1)^n}{n^2} \right] \cos nx$$

WKT

The Fourier half range sin series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (2)}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \sin nx dx$$

$$= \frac{2}{\pi} \left\{ (\pi x - x^2) \int_0^{\pi} \sin nx dx - \int_0^{\pi} [(\pi - 2x) \int \sin nx dx] dx \right\}$$

$$= \frac{2}{\pi} \left\{ -\frac{1}{n} [(\pi x - x^2) \cos nx]_0^{\pi} + \frac{1}{n} \int_0^{\pi} (\pi - 2x) \cos nx dx \right\}$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi - 2x) \cos nx dx$$

$$= \frac{2}{\pi} \left\{ (\pi - 2x) \int_0^{\pi} \cos nx dx - \int_0^{\pi} (-2) \int \cos nx dx \right\}$$

$$= \frac{2}{n\pi} \left\{ \frac{1}{n} [\pi - 2x] \sin nx \right\}_0^{\pi} - \frac{2}{n^2} [\cos nx]_0^{\pi} \}$$

$$= \frac{2}{n\pi} \left\{ \frac{1}{n} (0 - 0) - \frac{2}{n^2} [\cos n\pi - \cos 0] \right\}$$

$$= -\frac{4}{n^3\pi} [(-1)^n - 1]$$

$$b_n = \frac{4}{n^3\pi} [1 - (-1)^n] \sin nx //$$

(14) Expand the function $f(x) = x \sin x$ as a fourier series in the interval $0 \leq x \leq \pi$, deduce that $\frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \frac{1}{7 \cdot 9} + \dots = \frac{\pi}{4}$

$$f(x) = x \sin x, x \in [0, \pi]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- ①}$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x [2 \cos nx \cdot \sin x] dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x [\sin(n+1)x - \sin(n-1)x] dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \sin(n+1)x dx - \frac{1}{\pi} \int_0^{\pi} x \sin(n-1)x dx$$

$$\int_0^{\pi} x \sin(n+1)x dx = -\frac{1}{n+1} [x \cos(n+1)x]_0^{\pi} + \frac{1}{(n+1)^2} [\sin(n+1)x]_0^{\pi}$$

$$= -\frac{1}{n+1} [\pi \cos(n+1)\pi - 0]$$

$$= -\frac{\pi}{n+1} [\cos(n+1)\pi]$$

$$= -\frac{\pi (-1)^{n+1}}{n+1}$$

$$= \frac{\pi (-1)^n}{n+1}$$

$$a_n = \frac{1}{\pi} \frac{\pi (-1)^n}{n+1} - \frac{1}{\pi} \frac{\pi (-1)^n}{n-1}$$

$$= (-1)^n \left[\frac{1}{n+1} - \frac{1}{n-1} \right]$$

$$= (-1)^n \left[\frac{n-1-n+1}{n^2-1} \right]$$

$$a_n = \frac{-2(-1)^n}{n^2-1} \text{ for } n \geq 2$$

$$a_1 = \frac{2}{\pi} \int_0^{\pi} f(x) \cos x dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x [2 \sin x \cos x] dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \sin 2x dx$$

$$= \frac{1}{\pi} \left\{ -\frac{1}{2} [x \cos 2x]_0^{\pi} + \frac{1}{4} [\sin 2x]_0^{\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ -\frac{1}{2} [\pi - 0] \right\}$$

$$= \frac{1}{\pi} \left[-\frac{\pi}{2} \right]$$

$$= -\frac{1}{2}$$

$$\textcircled{1} \Rightarrow f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$x \sin x = 1 - \frac{1}{2} \cos x - 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} \cos nx \quad \textcircled{2}$$

$$\text{Let } x = \frac{\pi}{2}$$

$$\textcircled{2} \Rightarrow \frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) = 1 - \frac{1}{2} \cos\left(\frac{\pi}{2}\right) - 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} \cos\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{2} = 1 - 2 \left[\frac{1}{3} \cos \pi - \frac{1}{8} \cos \frac{3\pi}{2} + \frac{1}{15} \cos(2\pi) - \frac{1}{24} \cos\left(\frac{5\pi}{2}\right) + \frac{1}{35} \cos(3\pi) + \dots \right]$$

$$= -2 \left[-\frac{1}{3} - 0 + \frac{1}{15} - 0 - \frac{1}{35} + \dots \right] = \frac{\pi}{2} - 1$$

$$= -2 \left[-\frac{1}{3} + \frac{1}{15} - \frac{1}{35} + \dots \right] = \frac{\pi-2}{2}$$

$$= \frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \dots = \frac{\pi-2}{4}$$

$$= \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \dots = \frac{\pi-2}{4} \quad \parallel$$

⑤ Find the Fourier series of $f(x) = \sqrt{1 - \cos x}$ in the interval $[-\pi, \pi]$ or $[0, 2\pi]$ and hence deduce $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$

$$\text{Let } f(x) = \sqrt{1 - \cos x} = \sqrt{2 \sin^2\left(\frac{x}{2}\right)} = \sqrt{2} \sin\left(\frac{x}{2}\right), x \in [-\pi, \pi]$$

$$\begin{aligned} f(-x) &= \sqrt{1 - \cos(-x)} \\ &= \sqrt{1 - \cos x} \\ &= f(x) \end{aligned}$$

$f(x)$ is an even function

$$b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sqrt{2} \sin\left(\frac{x}{2}\right) dx \\ &= \frac{2\sqrt{2}}{\pi} \left[\frac{-\cos(x/2)}{1/2} \right]_0^{\pi} \end{aligned}$$

$$f(-x) = \sqrt{1 - \cos(-x)} = \sqrt{1 - \cos x} = f(x)$$

$f(x)$ is an even function.

$$b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sqrt{2} \sin\left(\frac{x}{2}\right) dx \\ &= \frac{2\sqrt{2}}{\pi} \left[\frac{-\cos(x/2)}{1/2} \right]_0^{\pi} \\ &= -\frac{4\sqrt{2}}{\pi} \left[\cos\left(\frac{x}{2}\right) \right]_0^{\pi} \\ &= -\frac{4\sqrt{2}}{\pi} \left[\cos \frac{\pi}{2} - \cos 0 \right] \end{aligned}$$

$$x = \frac{-4\sqrt{2}}{\pi} [0 - 1]$$

$$a_0 = \frac{4\sqrt{2}}{\pi}$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sqrt{2} \sin\left(\frac{x}{2}\right) \cos nx \, dx \\ &= \frac{\sqrt{2}}{\pi} \int_0^{\pi} 2 \cos nx \sin\left(\frac{x}{2}\right) \, dx \\ &= \frac{\sqrt{2}}{\pi} \int_0^{\pi} \left[\sin\left(n\frac{x}{2} + \frac{x}{2}\right) - \sin\left(n\frac{x}{2} - \frac{x}{2}\right) \right] \, dx \\ &= \frac{\sqrt{2}}{\pi} \int_0^{\pi} \left[\sin\left(n + \frac{1}{2}\right)x - \sin\left(n - \frac{1}{2}\right)x \right] \, dx \\ &= \frac{\sqrt{2}}{\pi} \left[\frac{-\cos\left(n + \frac{1}{2}\right)x}{n + \frac{1}{2}} + \frac{\cos\left(n - \frac{1}{2}\right)x}{n - \frac{1}{2}} \right]_0^{\pi} \\ &= \frac{\sqrt{2}}{\pi} \left\{ (0 - 0) - \left[-\frac{1}{n + \frac{1}{2}} + \frac{1}{n - \frac{1}{2}} \right] \right\} \\ &= -\frac{\sqrt{2}}{\pi} \left[\frac{1}{n - \frac{1}{2}} - \frac{1}{n + \frac{1}{2}} \right] \\ &= -\frac{\sqrt{2}}{\pi} \left[\frac{1}{\frac{2n-1}{2}} - \frac{1}{\frac{2n+1}{2}} \right] \\ &= -\frac{2\sqrt{2}}{\pi} \left[\frac{1}{2n-1} - \frac{1}{2n+1} \right] \\ &= -2\sqrt{2} \left[\frac{2n+1 - 2n-1}{4n^2-1} \right] \end{aligned}$$

$$a_n = \frac{-4\sqrt{2}}{\pi} \left[\frac{1}{4n^2-1} \right]$$

$$\textcircled{1} \Rightarrow \sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos nx$$

$$x=0$$

$$\textcircled{2} \Rightarrow 0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$$

$$\frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{2\sqrt{2}}{\pi}$$

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{2\sqrt{2}}{\pi} \cdot \frac{\pi}{4\sqrt{2}} = \frac{1}{2} //$$

Fourier Series having the period $2l$:-

The Fourier Series of $f(x) = [c, c+2l]$, over the period $2l$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]$$

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x) dx$$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Note:-

① If $c=0$, we can define $f(x)$ in the interval $[0, 2l]$ for

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]$$

$$a_0 = \frac{1}{l} \int_0^{2l} f(x) dx$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

② If $c=-\pi$, we can define $f(x)$ in the interval $[-\pi, \pi]$ for

$$a_0 = \frac{1}{l} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{l} \int_{-\pi}^{\pi} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_{-\pi}^{\pi} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

③ If $f(x)$ is an even function in the intervals $[0, 2l]$ or $[-l, l]$, then $b_n=0$, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \text{ and it also called the}$$

Fourier half-range cosine series in $[0, l]$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

④ If $f(x)$ is an odd function in $[0, 2l]$ or $[-l, l]$, then

$a_0 = 0$, $a_n = 0$, we have

$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ and it is also called the Fourier

half range sin series in $[0, l]$.

When $b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

⑤ If $f(x)$ is continuous in $[0, 2l]$, then $f(x)$ is even when $f(2l-x) = f(x)$, $f(x)$ is odd then, $f(2l-x) = -f(x)$

⑥ $f(x) = \begin{cases} \phi(x), & 0 \leq x \leq l \\ \psi(x), & l \leq x \leq 2l, \text{ then} \end{cases}$

$\phi(2l-x) = \psi(x)$ is even function

$\phi(2l-x) = -\psi(x)$ is odd function

⑦ If $f(x)$ is continuous $[-l, l]$, then $f(x)$ is even $f(-x) = f(x)$ and odd when $f(-x) = -f(x)$.

⑧ $f(x) = \begin{cases} \phi(x), & -l \leq x \leq 0 \\ \psi(x), & 0 \leq x \leq l \end{cases} \text{ then}$

$\phi(-x) = \psi(x)$ is even function

$\phi(-x) = -\psi(x)$ is odd function.

① Find the Fourier Series of $f(x) = x(2-x)$ in the interval $(0, 2)$, hence deduce $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

Given $f(x) = x(2-x)$, $x \in [0, 2]$

$$l = 1$$

$$\Rightarrow f(2-x) = (2-x)[2-(2-x)]$$

$$\Rightarrow f(2-x) = (2-x)[2-2+x]$$

$$\Rightarrow f(2-x) = x(2-x) = f(x)$$

$f(x)$ is an even function

$$\Rightarrow b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{--- ①}$$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$= \frac{2}{l} \int_0^l x(2-x) dx$$

$$= 2 \left[x^2 - \frac{x^3}{3} \right]_0^l$$

$$= 2 \left[1 - \frac{1}{3} \right]$$

$$= 2 \left[\frac{2}{3} \right]$$

$$a_0 = \frac{4}{3}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= 2 \int_0^1 f(x) \cos(n\pi x) dx$$

$$= 2 \int_0^1 (2x - x^2) \cos(n\pi x) dx$$

$$= 2 \left\{ (2x - x^2) \int_0^1 \cos(n\pi x) dx - \int_0^1 (2 - 2x) \int \cos(n\pi x) dx \right\}$$

$$= 2 \left\{ \frac{1}{n\pi} [(2x - x^2) \sin(n\pi x)]_0^1 - \frac{2}{n\pi} \int_0^1 (1-x) \sin(n\pi x) dx \right\}$$

$$\begin{aligned}
 &= -\frac{4}{n\pi} \int_0^1 (1-x) \sin(n\pi x) dx \\
 &= \frac{4}{n\pi} \int_0^1 (x-1) \sin(n\pi x) dx \\
 &= \frac{4}{n\pi} \left[(x-1) \int_0^1 \sin(n\pi x) dx - \int_0^1 1 \int_0^1 \sin(n\pi x) dx \right] dx \\
 &= \frac{4}{n\pi} \left[-\frac{1}{n\pi} [(x-1) \cos(n\pi x)]_0^1 + \frac{1}{n^2\pi^2} [\sin(n\pi x)]_0^1 \right] dx \\
 &= -\frac{4}{n^2\pi^2} [0+1] \\
 a_n &= -\frac{4}{n^2\pi^2}
 \end{aligned}$$

$$\therefore \textcircled{1} \Rightarrow x(2-x) = \frac{2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n\pi x) - \textcircled{2}$$

$$0 = \frac{2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2}{3} \times \frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

② Find the Fourier series of $f(x) = \begin{cases} \pi x, & 0 \leq x < 1 \\ \pi(2-x), & 1 \leq x < 2 \end{cases}$

$$\begin{aligned}
 \text{Given } f(x) &= \begin{cases} \pi x, & 0 \leq x < 1 \\ \pi(2-x), & 1 \leq x < 2 \end{cases} \\
 2l &= 2 \\
 l &= 1
 \end{aligned}$$

$$f(x) = \begin{cases} \pi x, & 0 \leq x < l \\ \pi(2-x), & l \leq x < 2l \end{cases}$$

$$\phi(x) = \pi x, \quad \psi(x) = \pi(2-x)$$

$$\phi(2-x) = \pi(2-x) = \psi(x)$$

$f(x)$ is an even function, $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) - \textcircled{1}$$

$$\begin{aligned}
 \therefore a_0 &= \frac{2}{l} \int_0^l f(x) dx \\
 &= \frac{2}{1} \int_0^1 \pi x dx \\
 &= 2\pi \int_0^1 x dx \\
 &= 2\pi \left[\frac{x^2}{2} \right]_0^1 \\
 &= 2\pi \left[\frac{1}{2} - 0 \right]
 \end{aligned}$$

$$\begin{aligned}
 a_n &= \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx \\
 &= \frac{2}{1} \int_0^1 f(x) \cos(n\pi x) dx \\
 &= 2 \int_0^1 \pi x \cos(n\pi x) dx \\
 &= 2\pi \int_0^1 x \cos(n\pi x) dx \\
 &= 2\pi \left\{ x \int_0^1 \cos(n\pi x) dx - \int_0^1 \left[1 \int \cos(n\pi x) dx \right] dx \right\} \\
 &= 2\pi \left\{ \frac{1}{2\pi} [x \sin(n\pi x)]_0^1 + \frac{1}{n^2\pi^2} \int_0^1 \cos(n\pi x) dx \right\} \\
 &= \frac{2\pi}{n^2\pi^2} [\cos n\pi - 1]
 \end{aligned}$$

$$\begin{aligned}
 a_n &= \frac{2}{n^2\pi} [(-1)^n - 1] \\
 \textcircled{1} &= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2} \right] \cos(n\pi x) = \begin{cases} \pi x, & 0 \leq x < 1 \\ \pi(2-x), & 1 \leq x < 2 \end{cases}
 \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} = -\frac{\pi^2}{4}$$

$$\Rightarrow -\frac{\pi^2}{4} = \frac{(-2)}{1^2} + \frac{(-2)}{3^2} + \frac{(-2)}{5^2} + \dots$$

$$-\frac{\pi^2}{4} = (-2) \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$$

$$\frac{\pi^2}{8} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \quad //$$

③ Find the Fourier Series of $f(x) = \begin{cases} 1+x, & -1 \leq x \leq 0 \\ 1-x, & 0 \leq x \leq 1 \end{cases}$

Given $f(x) = \begin{cases} 1+x, & -1 \leq x \leq 0 \\ 1-x, & 0 \leq x \leq 1 \end{cases}$

$\phi(x) = 1+x$, $\psi(x) = 1-x$

$\therefore \phi(-x) = 1-x = \psi(x)$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{--- ①}$

$a_0 = \frac{2}{l} \int_0^l f(x) dx$

$= \frac{2}{l} \int_0^1 (1-x) dx$

$= \frac{2}{l} \left[\frac{l^2 - \frac{l^2}{2}}{2} \right]$

$= \frac{2}{l} \left[\frac{l^2}{2} \right]$

$a_0 = l$

$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$

$= \frac{2}{l} \int_0^1 (1-x) \cos\left(\frac{n\pi x}{l}\right) dx$

$= \frac{2}{l} \left\{ (1-x) \int_0^l \cos\left(\frac{n\pi x}{l}\right) dx - \int_0^l [-1] \int_0^l \cos\left(\frac{n\pi x}{l}\right) dx \right\}$

$= \frac{2}{l} \left\{ \frac{1}{n\pi} (1-x) \sin\left(\frac{n\pi x}{l}\right) \Big|_0^l - \frac{l^2}{n^2\pi^2} \left[\cos\left(\frac{n\pi x}{l}\right) \right]_0^l \right\}$

$= \frac{2}{l} \left\{ 0 - \frac{l^2}{n^2\pi^2} (\cos n\pi - 1) \right\}$

$= \frac{2l^2}{ln^2\pi^2} ((-1)^n - 1)$

$= \frac{-2l}{n^2\pi^2} \{ (-1)^n - 1 \}$

$a_n = \frac{2l}{n^2\pi^2} (1 - (-1)^n)$

$$\textcircled{1} \Rightarrow \frac{l}{2} + \frac{2l}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n^2} \right) \cos \left(\frac{n\pi x}{l} \right) = \begin{cases} l+x, & -l \leq x \leq 0 \\ l-x, & 0 \leq x \leq l \end{cases} - \textcircled{2}$$

When $x=0$

$$\textcircled{2} \Rightarrow \frac{l}{2} + \frac{2l}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} = l$$

$$\frac{2l}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} = l - \frac{l}{2} = \frac{l}{2}$$

$$\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} = \frac{l}{2} \times \frac{\pi^2}{2l} = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \dots = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} = \dots = \frac{\pi^2}{8} //$$

$\textcircled{4}$ Find the Fourier Series of $f(x) = |x|$ in the interval $(-l, l)$

$$f(x) = |x| = \begin{cases} -x, & -l \leq x \leq 0 \\ x, & 0 \leq x \leq l \end{cases}$$

$$\phi(x) = -x, \quad \psi(x) = x$$

$$\phi(-x) = -(-x) = x = \psi(x) \Rightarrow b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left(\frac{n\pi x}{l} \right) - \textcircled{1}$$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$= \frac{2}{l} \int_0^l x dx$$

$$= \frac{2}{l} \left(\frac{x^2}{2} \right)_0^l$$

$$= \frac{2}{l} \left(\frac{l^2}{2} \right)$$

$$a_0 = l$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \left(\frac{n\pi x}{l} \right) dx$$

$$= \frac{2}{l} \int_0^l x \cos \left(\frac{n\pi x}{l} \right) dx$$

$$= \frac{2}{l} \left\{ x \int_0^l \cos\left(\frac{n\pi x}{l}\right) dx - \int_0^l x \cos\left(\frac{n\pi x}{l}\right) dx \right\}$$

$$= \frac{2}{l} \left\{ \frac{x}{n\pi} \left[x \sin\left(\frac{n\pi x}{l}\right) \right]_0^l + \frac{l^2}{n^2\pi^2} \left[\cos\left(\frac{n\pi x}{l}\right) \right]_0^l \right\}$$

$$= \frac{2l^2}{l n^2\pi^2} [\cos n\pi - 1]$$

$$= \frac{2l^2}{l n^2\pi^2} [(-1)^n - 1]$$

$$a_n = \frac{2l}{n^2\pi^2} [(-1)^n - 1]$$

$$\textcircled{1} \Rightarrow \frac{1}{2} + \frac{2l}{\pi^2} \sum_{n=1}^{\infty} [(-1)^n - 1]$$

$$\frac{1}{2} + \frac{2l}{n^2\pi^2} \sum_{n=1}^{\infty} [(-1)^n - 1] \cos\left(\frac{n\pi x}{l}\right) = \begin{cases} -x, & -l \leq x \leq 0 \\ x, & 0 \leq x \leq l \end{cases}$$

where $x = 0$

$$\frac{1}{2} + \frac{2l}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2} \right] = 0$$

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2} \right] = -\frac{1}{2} \times \frac{\pi^2}{2l}$$

$$\left[\frac{-2}{1^2} \right] + \left[\frac{-2}{3^2} \right] + \left[\frac{-2}{5^2} \right] + \dots = -\frac{\pi^2}{4}$$

$$-2 \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right] = -\frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8} //$$

- ⑤ Find the Fourier series of $f(x) = \begin{cases} 1 + \frac{4x}{3}, & -\frac{3}{2} \leq x \leq 0 \\ 1 - \frac{4x}{3}, & 0 \leq x \leq \frac{3}{2} \end{cases}$ and hence deduce that the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

$$f(x) = \begin{cases} 1 + \frac{4x}{3}, & -\frac{3}{2} \leq x \leq 0 \\ 1 - \frac{4x}{3}, & 0 \leq x \leq \frac{3}{2} \end{cases} = \begin{cases} \phi(x), & -1 \leq x \leq 0 \\ \psi(x), & 0 \leq x \leq 1 \end{cases}$$

$$\therefore \phi(x) = 1 + \frac{4x}{3} \quad \psi(x) = 1 - \frac{4x}{3} \quad , \quad l = \frac{3}{2}$$

$$\phi(-x) = 1 - \frac{4x}{3} = \psi(x)$$

$f(x)$ is an even function, $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{--- (1)}$$

$$\begin{aligned} a_0 &= \frac{2}{l} \int_0^l f(x) dx \\ &= \frac{2}{3} \int_0^{3/2} f(x) dx \\ &= \frac{2}{3} \int_0^{3/2} \left(1 - \frac{4x}{3}\right) dx \\ &= \frac{2}{3} \left[x - \frac{2x^2}{3} \right]_0^{3/2} \\ &= \frac{2}{3} \left[\frac{3}{2} - \frac{2}{3} \left(\frac{3}{2}\right)^2 \right] \\ &= \frac{2}{3} \left[\frac{3}{2} - \frac{3}{2} \right] \end{aligned}$$

$$a_0 = 0$$

$$\begin{aligned} a_n &= \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx \\ &= \frac{2}{3} \int_0^{3/2} \left(1 - \frac{4x}{3}\right) \cos\left(\frac{2n\pi x}{3}\right) dx \\ &= \frac{2}{3} \left\{ \left(1 - \frac{4x}{3}\right) \int_0^{3/2} \cos\left(\frac{2n\pi x}{3}\right) dx - \int_0^{3/2} \left(-\frac{4}{3}\right) \cos\left(\frac{2n\pi x}{3}\right) dx \right\} \\ &= \frac{2}{3} \left\{ \frac{3}{2n\pi} \left[\left(1 - \frac{4x}{3}\right) \sin\left(\frac{2n\pi x}{3}\right) \right]_0^{3/2} - \frac{4}{3} \frac{3^2}{4n^2\pi^2} \left[\cos\left(\frac{2n\pi x}{3}\right) \right]_0^{3/2} \right\} \\ &= \frac{2}{3} \left\{ 0 - \frac{3}{n^2\pi^2} [\cos n\pi - \cos 0] \right\} \\ &= \frac{-4}{n^2\pi^2} [\cos n\pi - 1] \\ &= \frac{-4}{n^2\pi^2} [(-1)^n - 1] \\ a_n &= \frac{4}{n^2\pi^2} (1 - (-1)^n) \end{aligned}$$

$$\textcircled{1} \Rightarrow \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{1 - (-1)^n}{n^2} \right) \cos\left(\frac{2n\pi x}{3}\right) = \begin{cases} 1 + \frac{4x}{3}, & -\frac{3}{2} \leq x \leq 0 \\ 1 - \frac{4x}{3}, & 0 \leq x \leq \frac{3}{2} \end{cases} \quad \text{--- (2)}$$

when $x = 0$

$$\textcircled{2} \Rightarrow \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} = 1$$

$$\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \dots = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8} //$$

$\textcircled{6}$ Find the half range cosine series of $f(x) = x(1-x)$ in the interval $[0, 1]$

Given $f(x) = x(1-x)$, $x \in [0, 1]$

WKT The fourier half range cosine series in $[0, 1]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{--- (1)}$$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$= \frac{2}{1} \int_0^1 (1x - x^2) dx$$

$$= \frac{2}{1} \left[\frac{1x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$= \frac{2}{1} \left[\frac{1^2}{2} - \frac{1^3}{3} \right]$$

$$a_0 = \frac{2}{1} \times \frac{1^3}{6}$$

$$a_0 = \frac{1^2}{3}$$

$$a_n = \frac{2}{l} \int_0^l a_n \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{1} \int_0^1 (1x - x^2) \cos\left(\frac{n\pi x}{1}\right) dx$$

$$= \frac{2}{1} \left\{ (1x - x^2) \int_0^1 \cos\left(\frac{n\pi x}{1}\right) dx - \int_0^1 [1 - 2x] \cos\left(\frac{n\pi x}{1}\right) dx \right\}$$

$$\begin{aligned}
 &= \frac{2}{l} \left\{ \frac{l}{n\pi} (0-0) - \frac{l}{n\pi} \int_0^l (1-2x) \sin\left(\frac{n\pi x}{l}\right) dx \right\} \\
 &= -\frac{2}{n\pi} \left\{ (1-2x) \int_0^l \sin\left(\frac{n\pi x}{l}\right) dx - \int_0^l (1-2x) \sin\left(\frac{n\pi x}{l}\right) dx \right\} \\
 &= -\frac{2}{n\pi} \left\{ -\frac{l}{n\pi} \left[(1-2x) \cos\left(\frac{n\pi x}{l}\right) \right]_0^l - \frac{2l^2}{n^2\pi^2} \left(\sin\left(\frac{n\pi x}{l}\right) \right)_0^l \right\} \\
 &= -\frac{2}{n\pi} \left\{ -\frac{l}{n\pi} [-1 \cos n\pi - 1] - 0 \right\} \\
 a_n &= -\frac{2l^2}{n^2\pi^2} [(-1)^{n+1}]
 \end{aligned}$$

$$\therefore f(x) = \frac{l^2}{6} + \left(-\frac{2l^2}{\pi^2}\right) \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{n^2}\right) \cos\left(\frac{n\pi x}{l}\right) //$$

⑦ Find the half range cosine series $f(x) = (x-1)^2$ in the interval $[0, 1]$.

Given $f(x) = (x-1)^2$, $x \in [0, 1]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{--- (1)}$$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$= \frac{2}{1} \int_0^1 (x-1)^2 dx$$

$$= 2 \left[\frac{(x-1)^3}{3} \right]_0^1$$

$$a_0 = \frac{2}{3}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= 2 \int_0^1 f(x) \cos(n\pi x) dx$$

$$= 2 \int_0^1 (x-1)^2 \cos(n\pi x) dx$$

$$= 2 \left\{ (x-1)^2 \int_0^1 \cos(n\pi x) dx - \int_0^1 [2(x-1) \cos(n\pi x)] dx \right\}$$

$$= 2 \left\{ \frac{1}{n\pi} (x-1)^2 [\sin(n\pi x)]_0^1 + \frac{2}{n^2\pi^2} [\cos(n\pi x)] 2(x-1) \right\}$$

$$a_n = \frac{4}{n^2 \pi^2} \{ (x-1) \cos(n\pi x) \} + 0 \}$$

$$= \frac{4}{n^2 \pi^2} \{ [0+1] \}$$

$$a_n = \frac{4}{n^2 \pi^2}$$

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n\pi x) //$$

⑧ Find the half range cosine series of $f(x) = (x+1)$ in the interval $0 \leq x \leq 1$

$$f(x) = (x+1), x \in [0, 1]$$

$$f(x) = (x+1), x \in [0, 1]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \text{--- ①}$$

$$a_0 = \frac{2}{1} \int_0^1 (x+1) dx$$

$$= 2 \left[\frac{x^2}{2} + x \right]_0^1$$

$$= 2 \left[\frac{1}{2} + 1 \right]$$

$$= 2 \left[\frac{1+2}{2} \right]$$

$$= 2 \left[\frac{3}{2} \right]$$

$$a_0 = 3$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{1} \int_0^1 (x+1) \cos\left(\frac{n\pi x}{1}\right) dx$$

$$= 2 \left\{ (x+1) \int_0^1 \cos\left(\frac{n\pi x}{1}\right) dx - \int_0^1 (1) \int \cos\left(\frac{n\pi x}{1}\right) dx \right\}$$

$$= 2 \left\{ \left[\frac{x+1}{n\pi} \sin(n\pi x) \right]_0^1 - \int_0^1 \sin\left(\frac{n\pi x}{1}\right) dx \right\}$$

$$= \frac{2}{n\pi} \{ (2 \cos(n\pi) - 1) - 0 \}$$

$$a_n = \frac{4}{n\pi} [(-1)^{n-1}]$$

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{4}{n\pi} [(-1)^{n-1}] \cos\left(\frac{n\pi x}{1}\right)$$

$$f(x) = \frac{3}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n-1}}{n}\right] \cos\left(\frac{n\pi x}{1}\right)$$

⑨ Find the half range sin series of e^x in the interval $0 \leq x \leq 1$

$$f(x) = e^x \quad x \in [0, 1]$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{1}\right)$$

$$b_n = \frac{2}{1} \int_0^1 f(x) \sin\left(\frac{n\pi x}{1}\right) dx$$

$$= \frac{2}{1} \int_0^1 e^x \sin\left(\frac{n\pi x}{1}\right) dx$$

$$= 2 \int_0^1 e^x \sin(n\pi x) dx$$

$$b_n = 2 \left[\frac{e^x}{1+n^2\pi^2} (\sin(n\pi x) - n\pi \cos(n\pi x)) \right]_0^1$$

$$= 2 \left[\frac{e}{1+n^2\pi^2} [\sin(n\pi) - n\pi \cos(n\pi)] - \frac{e^0}{1+n^2\pi^2} [\sin(n\pi \cdot 0) - n\pi \cos(n\pi \cdot 0)] \right]$$

$$= 2 \left[\frac{e}{1+n^2\pi^2} [-n\pi \cos(n\pi)] - \frac{1}{1+n^2\pi^2} (-n\pi) \right]$$

$$b_n = 2 \left[\frac{e}{1+n^2\pi^2} (-n\pi (-1)^n) + \frac{n\pi}{1+n^2\pi^2} \right]$$

$$= \frac{2n\pi}{1+n^2\pi^2} [-(-1)^n e + 1]$$

$$b_n = \frac{2n\pi}{1+n^2\pi^2} [1 - (-1)^n]$$

$$\textcircled{1} \Rightarrow f(x) = 2\pi \sum_{n=1}^{\infty} \frac{n}{1+n^2\pi^2} [1 - (-1)^n] \sin(n\pi x)$$

Harmonic Fourier Series:-

① Let $y = f(x)$ be a periodic function of the period 2π , then the Fourier series of $f(x)$ in the harmonics can be expressed as $f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots$

where $\frac{a_0}{2}$ is called the constant term. (a_1, b_1) are called the first harmonic coefficients, (a_2, b_2) are called the second harmonic coefficients and which will be evaluated as

$$a_0 = \frac{2}{N} \sum y$$

$$a_1 = \frac{2}{N} \sum y \cos x$$

$$b_1 = \frac{2}{N} \sum y \sin x$$

$$a_2 = \frac{2}{N} \sum y \cos 2x$$

$$b_2 = \frac{2}{N} \sum y \sin 2x$$

$$\text{Generally } y, a_n = \frac{2}{N} \sum y \cos nx, \forall n = 1, 2, 3, \dots$$

$$b_n = \frac{2}{N} \sum y \sin nx, \forall n = 1, 2, 3, \dots$$

where N is the number of terms given in the table which should always be even number.

② Suppose $y = f(x)$ be a periodic function over the period 2λ , then $f(x) = \frac{a_0}{2} + [a_1 \cos(\frac{\pi x}{\lambda}) + b_1 \sin(\frac{\pi x}{\lambda})] + [a_2 \cos(\frac{2\pi x}{\lambda}) + b_2 \sin(\frac{2\pi x}{\lambda})] + \dots$

where $a_0 = \frac{2}{N} \sum y$ $\therefore \theta = \frac{\pi x}{\lambda}$

$a_1 = \frac{2}{N} \sum y \cos \theta$

$b_1 = \frac{2}{N} \sum y \sin \theta$

$a_n = \frac{2}{N} \sum y \cos n\theta$

$b_n = \frac{2}{N} \sum y \sin n\theta$

① The following value of function y gives the displacement in inches of a certain machine part for rotation x of a flywheel. Expand y in terms of Fourier series upto the second harmonic.

rotations	x	0	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$	π
displacement	y	0	9.2	14.4	17.8	17.3	11.7	0

The Fourier series of $y=f(x)$ upto the second harmonic is

x	y	$y \cos x$	$y \sin x$	$y \cos 2x$	$y \sin 2x$
0°	0	0	0	0	0
30°	9.2	7.9674	4.6000	4.6	7.9674
60°	14.4	7.2	12.4707	-7.2	12.4707
90°	17.8	0	17.8	-17.8	0
120°	17.3	-8.65	14.9822	-8.65	-14.9822
150°	11.7	-10.1325	5.85	5.85	-10.1325
Σ	70.4	-3.6151	55.7029	-23.20	-4.6766

$$N = 6$$

$$\therefore a_0 = \frac{2}{N} \Sigma y = \frac{2}{6} (70.4) = 23.46$$

$$\therefore a_1 = \frac{2}{N} \Sigma y \cos x = \frac{2}{6} (-3.6151) = -1.2050$$

$$\therefore b_1 = \frac{2}{N} \Sigma y \sin x = \frac{2}{6} (55.7029) = 18.5676$$

$$a_2 = \frac{2}{N} \Sigma y \cos 2x = \frac{2}{6} (-23.20) = -7.7333$$

$$b_2 = \frac{2}{N} \Sigma y \sin 2x = \frac{2}{6} (-4.6766) = -1.5589$$

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)$$

$$= \frac{23.46}{2} + (-1.2050 \cos x + 18.5676 \sin x) + (-7.7333 \cos 2x -$$

$$1.5589 \sin 2x)$$

$$= 11.73 + (-1.2050 \cos x + 18.5676 \sin x) + (-7.7333 \cos 2x - 1.5589 \sin 2x) //$$

② Express y as a Fourier series upto first harmonic for the given data

x	0	30	60	90	120	150	180	210	240	270	300	330
y	1.8	1.1	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

x	y	$y \cos x$	$y \sin x$
0	1.8	1.8	0
30	1.1	0.9526	0.55
60	0.30	0.15	0.2598
90	0.16	0	0.16
120	1.50	-0.75	1.2990
150	1.30	-1.1258	0.65
180	2.16	-2.16	0
210	1.25	-1.0825	-0.625
240	1.30	-0.65	-1.1258
270	1.52	0	-1.52
300	1.76	0.88	-1.5242
330	2.00	1.7320	-1
Σ	16.15	-0.2537	-2.8762

$$\therefore a_0 = \frac{2}{N} \Sigma y = \frac{2}{12} (16.15) = 2.6916$$

$$a_1 = \frac{2}{N} \Sigma y \cos x = \frac{2}{12} (-0.2537) = -0.0422$$

$$b_1 = \frac{2}{N} \Sigma y \sin x = \frac{2}{12} (-2.8762) = -0.47936$$

$$f(x) = \frac{2.6916}{2} + (-0.0422 \cos x - 0.47936 \sin x)$$

$$= 1.3458 + (-0.0422 \cos x - 0.47936 \sin x) //$$

② Compute the first two harmonic of the Fourier series for the given data

x°	0	60	120	180	240	300
y	7.9	7.2	3.6	0.5	0.9	0.8

x	y	$y \cos x$	$y \sin x$	$y \cos 2x$	$y \sin 2x$
0	7.9	7.9	0	7.9	0
60	7.2	3.6	6.2353	-3.6	6.2353
120	3.6	-1.8	3.1176	-1.8	-3.1176
180	0.5	-0.5	0	0.5	0
240	0.9	-0.45	-0.7794	-0.45	0.7794
300	0.8	0.4	-0.6928	-0.4	-0.69282
Σ	20.9	9.15	7.8807	2.15	3.20428

$$N = 6$$

$$a_0 = \frac{2}{N} \Sigma y = \frac{2}{6} (20.9) = 6.96666$$

$$a_1 = \frac{2}{N} \Sigma y \cos x = \frac{2}{6} (9.15) = 3.05$$

$$b_1 = \frac{2}{N} \Sigma y \sin x = \frac{2}{6} (7.8807) = 2.6269$$

$$a_2 = \frac{2}{N} \Sigma y \cos 2x = \frac{2}{6} (2.15) = 0.716666$$

$$b_2 = \frac{2}{N} \Sigma y \sin 2x = \frac{2}{6} (3.20428) = 1.06809$$

$$\begin{aligned}
 f(x) &= \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + a_2 \cos 2x + b_2 \sin 2x \\
 &= \frac{6.96666}{2} + (3.05 \cos x + 2.6269 \sin x) + (0.71666 \cos 2x + 1.06809 \sin 2x) \\
 &= 3.48333 + (3.05 \cos x + 2.6269 \sin x) + (0.71666 \cos 2x + 1.06809 \sin 2x) //
 \end{aligned}$$

- ④ The displacement y (in cm) of a machine part occurs due to the rotation of x radians in given below:-

Rotation x (in radians)	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
Displacement y (in cms)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Expand y in terms of fourier series upto second harmonics.

x	y	$y \cos x$	$y \sin x$	$y \cos 2x$	$y \sin 2x$
0	1.0	1	0	1	0
60	1.4	0.7	1.2124	-0.7	1.2124
120	1.9	-0.95	1.6454	-0.95	-1.6454
180	1.7	-1.7	0	1.7	0
240	1.5	-0.75	-1.2990	-0.75	1.2990
300	1.2	0.6	-1.03923	-0.6	-1.03923
Σ	8.7	-1.1	0.51957	-0.3	-0.17323

$$N = 6$$

$$a_0 = \frac{2}{N} \Sigma y = \frac{2}{6} (8.7) = 2.9$$

$$a_1 = \frac{2}{N} \Sigma y \cos x = \frac{2}{6} (-1.1) = -0.36666$$

$$b_1 = \frac{2}{N} \Sigma y \sin x = \frac{2}{6} (0.51957) = 0.17319$$

$$a_2 = \frac{2}{N} \Sigma y \cos 2x = \frac{2}{6} (-0.3) = -0.1$$

$$b_2 = \frac{2}{N} \Sigma y \sin 2x = \frac{2}{6} (-0.17323) = -0.05774$$

$$\begin{aligned}
 f(x) &= \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) \\
 &= \frac{2.9}{2} + (-0.36666 \cos x + 0.17319 \sin x) + (-0.1 \cos 2x - 0.05774 \sin 2x) \\
 &= 1.45 + (-0.3666 \cos x + 0.17319 \sin x) + (-0.1 \cos 2x - 0.05774 \sin 2x)
 \end{aligned}$$

- ⑤ In a machine the displacement y of a given point in given for a certain angle x as follows.

x	0	30	60	90	120	150	180	210	240	270	300	330
y	7.9	8	7.2	5.6	3.6	1.7	0.5	0.2	0.9	2.5	4.7	6.8

Find the constant term and the first two harmonics in fourier series expansion of y .

x	y	$y \cos x$	$y \sin x$	$y \cos 2x$	$y \sin 2x$
0	7.9	7.9	0	7.9	0
30	8	6.9282	4	4	6.9282
60	7.2	3.6	6.2353	-3.6	6.2353
90	5.6	0	5.6	-5.6	0
120	3.6	-1.8	3.1176	-1.8	-3.1176
150	1.7	-1.4722	0.85	0.85	-1.4722
180	0.5	-0.5	0	0.5	0
210	0.2	-0.17320	-0.1	0.1	0.17320
240	0.9	-0.45	-0.779422	-0.45	0.77942
270	2.5	0	-2.5	-2.5	0
300	4.7	2.35	-4.0703	-2.34	-4.07031
330	6.8	5.8889	-3.4	3.4	-5.8889
Σ	49.6	22.2717	-7.45	0.45	-0.43289

$$N = 12$$

$$a_0 = \frac{2}{N} \Sigma y = \frac{2}{12} (49.6) = 8.2666$$

$$a_1 = \frac{2}{N} \Sigma y \cos x = \frac{2}{12} (22.2717) = 3.71195$$

$$b_1 = \frac{2}{N} \Sigma y \sin x = \frac{2}{12} (-7.45) = -1.24166$$

$$a_2 = \frac{2}{N} \Sigma y \cos 2x = \frac{2}{12} (0.45) = 0.075$$

$$b_2 = \frac{2}{N} \Sigma y \sin 2x = \frac{2}{12} (-0.43289) = -0.072148$$

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)$$

$$= \frac{8.22666}{2} + (3.71195 \cos x - 1.24166 \sin x) + (0.075 \cos 2x - 0.072148 \sin 2x)$$

$$= 4.13333 + (3.71195 \cos x - 1.24166 \sin x) + (0.075 \cos 2x - 0.072148 \sin 2x)$$

- ⑥ Obtain the constant term and first sin, cosine terms in the fourier expansion of y from the following table.

x	0	1	2	3	4	5
y	4	8	15	7	6	2

The given x varies as $0 \leq x < 6$

$$2l = 6$$

$$l = 3$$

The fourier series expansion upto first harmonic is defined as $f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{3}\right) + b_1 \sin\left(\frac{\pi x}{3}\right)$ — ①

x	y	$\theta = \frac{\pi x}{3}$	$y \cos \theta$	$y \sin \theta$
0	4	0	4	0
1	8	60°	4	6.9282
2	15	120°	-7.5	12.9903
3	7	180°	-7	0
4	6	240°	3	-5.1961
5	2	300°	1	-1.7320
Σ	42	—	-8.5	12.9901

$$\therefore a_0 = \frac{2}{N} \sum y = \frac{2}{6} (42) = 14$$

$$a_1 = \frac{2}{N} \sum y \cos \theta = \frac{2}{6} (-8.5) = -2.8333$$

$$b_1 = \frac{2}{N} \sum y \sin \theta = \frac{2}{6} (12.9901) = 4.3301$$

$$\therefore f(x) = 7 - (2.8333) \cos\left(\frac{\pi x}{3}\right) + (4.3301) \sin\left(\frac{\pi x}{3}\right) //$$

⑦ obtain the constant term and first sine & cosine terms in the fourier expansion of y from the following table.

x	0	1	2	3	4	5
y	9	18	24	28	26	20

The given x varies $0 \leq x < 6$

$$2l = 6$$

$$l = 3$$

The fourier series expansion upto the first harmonic is defined as

$$f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{3}\right) + b_1 \sin\left(\frac{\pi x}{3}\right) - 0$$

x	y	$\theta = \frac{\pi x}{3}$	$y \cos \theta$	$y \sin \theta$
0	9	0	9	0
1	18	60°	9	15.5884
2	24	120°	-12	20.7846
3	28	180°	-28	0
4	26	240°	-13	-22.5166
5	20	300°	10	-17.3205
Σ	125	-	-25	-3.4641

$$\therefore a_0 = \frac{2}{N} \Sigma y = \frac{2}{6} (125) = 41.667$$

$$a_1 = \frac{2}{N} \Sigma y \cos \theta = \frac{2}{6} (-25) = -8.3333$$

$$b_1 = \frac{2}{N} \Sigma y \sin \theta = \frac{2}{6} (-3.4641) = -1.547$$

$$f(x) = \frac{41.667}{2} - 8.3333 \cos\left(\frac{\pi x}{3}\right) - 1.547 \sin\left(\frac{\pi x}{3}\right)$$

$$= 20.8335 - 8.3333 \cos\left(\frac{\pi x}{3}\right) - 1.547 \sin\left(\frac{\pi x}{3}\right) //$$

- ⑨ The following table gives the variations of periodic current over a period T . show by numerical analysis that there is a direct current part 0.75 Amp. The Variable current and obtain the amplitude of first harmonic.

t (sec)	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
A	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Given the period to a circuit $2l = T$
 $l = \frac{T}{2}$

t	A	$\theta = \frac{\pi t}{l} = \frac{2\pi t}{T}$	$A \cos \theta$	$A \sin \theta$
0	1.98	0°	1.98	0
$T/6$	1.30	60°	0.65	1.1258
$T/3$	1.05	120°	-0.5250	0.9093
$T/2$	1.30	180°	-1.30	0
$2T/3$	-0.88	240°	0.44	0.7621
$5T/6$	-0.25	300°	-0.1250	0.2165

$$\therefore a_0 = \frac{2}{N} \sum A = \frac{2}{6} (4.5) = 1.5$$

$$a_1 = \frac{2}{N} \sum A \cos \theta = \frac{2}{6} (1.12) = 0.3733$$

$$b_1 = \frac{2}{N} \sum A \sin \theta = \frac{2}{6} (3.0137) = 1.0045$$

$$\text{Direct current} = \frac{a_0}{2} = \frac{1.5}{2} = 0.75 \text{ Amp}$$

$$\begin{aligned} \text{Amplitude} &= \sqrt{a_1^2 + b_1^2} = \sqrt{(0.3733)^2 + (1.0045)^2} \\ &= 1.0716 // \end{aligned}$$



