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LECTURE NOTES

TRANSFORM CALCULUS , FOURIER SERIES & NUMERICAL TECHNIQUES (21MATM31)

MODULE-4

NUMERICAL SOLUTION OF SECOND OREDER PDE

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Classification of Second Order PDE:-

* The general second order linear PDE in two independent variables 'x' & 'y' is of the form.

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU = 0 \longrightarrow (1)$$

where A, B, C, D, E, F are the general functions of x & y.

(1) If $B^2 - 4AC = 0$, then eqn (1) represent "parabola".

(2) If $B^2 - 4AC < 0$, then eqn (1) represent an "Ellipse"

(3) If $B^2 - 4AC > 0$, then eqn (1) represent "Hyperbola".

Few results of second order PDE.

PDE	A	B	C	$B^2 - 4AC$	Result
One dimensional wave eqn $c^2U_{xx} - U_{tt} = 0$	c^2	0	-1	$4c^2 > 0$	Hyperbolic

One dimensional heat Eqn $c^2u_{xx}+u_t=0$	c^2	0	0	0	Parabolic
Laplacian Equation $u_{xx}+u_{yy}=0$	1	0	1	$-4<0$	Elliptic

1. Classify the following P.D.E

a. $u_{xx}+4u_{xy}+4u_{yy}-u_x+2u_y=0$.

Let $u_{xx}+4u_{xy}+4u_{yy}-u_x+2u_y=0 \longrightarrow (1)$

Compare eqn (1) with.

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0$$

$\therefore A=1, \quad B=4, \quad C=4.$

$\therefore B^2-4AC = (4)^2-4(1)(4)$

$= 16-16 = 0$

$\therefore$ Eqn (1) can be represented as parabolic.

b. $x^2u_{xx}+(1-y^2)u_{yy}=0, \quad -1<y<1$

Let $x^2u_{xx}+(1-y^2)u_{yy}=0 \quad -1<y<1 \longrightarrow (1)$

Compare eqn (1) with.

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0$$

$\therefore A=x^2, \quad B=0, \quad C=(1-y^2)$

$\therefore B^2-4AC = 0-4(x^2)(1-y^2)<0$

if Eqn (1) can be Elliptic for Every $x, y \in Z$

$B^2-4AC=0$

$= -4x^2(1-y^2)=0$

$= x^2(1-y^2)=0$

$\Rightarrow x^2=0 \quad 1-y^2=0$

$\Rightarrow x=0, \quad y^2=1$

$\Rightarrow x=0 \quad y=\pm 1$

$\therefore$ Eqn (1) can represents parabolic for every $x=0$ & $y=\pm 1$

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c. $(1+x^2)U_{xx} + (5+2x^2)U_{xt} + (4+x^2)U_{tt} = 0$.

Let $(1+x^2)U_{xx} + (5+2x^2)U_{xt} + (4+x^2)U_{tt} \rightarrow (1)$

$\therefore A = (1+x^2)$, $B = (5+2x^2)$ $C = (4+x^2)$

$\therefore B^2 - 4AC = (5+2x^2)^2 - 4(1+x^2)(4+x^2)$

$= (2x^2+5)^2 - 4(x^2+1)(x^2+4)$

$= (4x^4 + 20x^2 + 25) - 4(x^4 + 4x^2 + x^2 + 4)$

$= 4x^4 + 20x^2 + 25 - 4x^4 - 20x^2 - 16$

$\therefore B^2 - 4AC = 9 > 0$.

$\therefore$ Eq^n (1) can be represented as hyperbolic.

d. $y^2U_{xx} - 2yU_{xy} + U_{yy} - U_y = 8y$.

Let $y^2U_{xx} - 2yU_{xy} + U_{yy} - U_y = 8y \rightarrow (1)$

compare Eq^n (1) with.

$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU = 0$

$A = y^2$, $B = -2y$, $C = 1$

$\therefore B^2 - 4AC = (-2y)^2 - 4y^2(1)$

$= 4y^2 - 4y^2$

$= 0$

Eq^n (1) can be represented as parabolic.

Numerical Solution of One dimensional Wave Equation.

Step 1:- Consider one dimensional wave Eq^n $c^2U_{xx} = U_{tt}$, subject to the boundry condition $u(0,t) = 0$, $u(l,t) = 0$, and intial conditions $u(x,0) = f(x)$, $u_t(x,0) = 0$ & it follows the procedure.

Step 2:- Based on given step size [k and h], we can form rectangular network of size 'h' and 'k'.

Step 3:- Given h only, we can get $k = \frac{h}{c}$, the points of division of x are

$x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, x_3 = x_0 + 3h, \ldots\ldots x_n = x_0 + nh$ and.

$y_0, y_1 = y_0 + k, y_2 = y_0 + 2k, y_3 = y_0 + 3k, \ldots\ldots y_n = y_0 + nk$.

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We can follow rectangular table for given set of input.

$t \downarrow$ \ $x \rightarrow$		x_0	x_1	x_2	x_3	----------	x_n
		0	1	2	3		n
t_0	0	$U_{0,0}$	$U_{1,0}$	$U_{2,0}$	$U_{3,0}$		$U_{n,0}$
t_1	1	$U_{0,1}$	$U_{1,1}$	$U_{2,1}$	$U_{3,1}$		$U_{n,1}$
t_2	2	$U_{0,2}$	$U_{1,2}$	$U_{2,2}$	$U_{3,2}$		$U_{n,2}$
t_3	3	$U_{0,3}$	$U_{1,3}$	$U_{2,3}$	$U_{3,3}$		$U_{n,3}$
⋮	⋮						
t_n	n	$U_{0,n}$	$U_{1,n}$	$U_{2,n}$	$U_{3,n}$		$U_{n,n}$

$\Rightarrow$ We instantly write the values of $U_{i,j}$ with the reference to the conditions $U(0,t)=0$ & $U(l,t)=0$ [Values along 1st & last column are zero].

$\Rightarrow$ The values along the 1st row are obtained by direct compitation using the condition $U(x,0)=f(x) \Rightarrow U_{x,0}=f(x)$.

$\Rightarrow$ The values of other 2nd row can be obtained by the relation

$$U_{i,1} = \frac{1}{2}\left[U_{i-1,0} + U_{i+1,0}\right]$$

$\Rightarrow$ Rest of the values are found from the relation

$$U_{i,j+1} = \left[U_{i-1,j} + U_{i+1,j} - U_{i,j-1}\right]$$

① Solve numerically $U_{xx} = 0.0625\, U_{tt}$, subject to the condition $U(0,t)=0=U(5,t)$, $U(x,0)=x^2(x-5)$ and $U_t(x,0)=0$ by taking $h=1$ for $0 \le t \le 1$.

Given:- one dimensional wave equation $U_{xx} = 0.0625\, U_{tt} \rightarrow ①$

$$\Rightarrow U_{tt} = \frac{1}{0.0625} U_{xx}$$

$$U_{tt} = 16 U_{xx}$$

$$c^2 = 16. \Rightarrow \boxed{c=4.}$$

And given the step size of $h=1$ & its is up to 5.

$$\therefore x = 0, 1, 2, 3, 4, 5.$$

$$\Rightarrow K = \frac{h}{c}$$

$$K = \frac{1}{4}$$

$$\Rightarrow K = 0.25$$

And 't' should be 0 to 1, therefore

$$\therefore t = 0, 0.25, 0.5, 0.75, 1.$$

Also given $u(0,t) = 0$, $u(5,t) = 0$

$t \downarrow$ \ $x \longrightarrow$		x_0	x_1	x_2	x_3	x_4	x_5
		0	1	2	3	4	5
t_0	0	$u_{0,0}=0$	$u_{1,0}$	$u_{2,0}$	$u_{3,0}$	$u_{4,0}$	$u_{5,0}=0$
t_1	0.25	$u_{0,1}=0$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}$	$u_{5,1}=0$
t_2	0.5	$u_{0,2}=0$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}$	$u_{5,2}=0$
t_3	0.75	$u_{0,3}=0$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3}$	$u_{5,3}=0$
t_4	1	$u_{0,4}=0$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4}$	$u_{5,4}=0$

To Evaluate the 1st row Values, given $u(x,0) = x^2(x-5)$

$$u_{x,0} = x^2(x-5)$$

$$u_{1,0} = 1(1-5) = -4$$

$$u_{2,0} = 4(2-5) = -12$$

$$u_{3,0} = 9(3-6) = -18$$

$$u_{4,0} = 16(4-5) = -16$$

To Evaluate the 2nd row, we have relation.

$$u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$$

$$= \frac{1}{2}\left[u_{0,0} + u_{2,0}\right]$$

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$$= \frac{1}{2}\left[0 + (-12)\right]$$

$$u_{1,1} = -6.$$

$$\Rightarrow u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right]$$

$$= \frac{1}{2}\left[(-4) + (-18)\right]$$

$$u_{2,1} = -11$$

$$\Rightarrow u_{3,1} = \frac{1}{2}\left[u_{2,0} + u_{4,0}\right]$$

$$= \frac{1}{2}\left[-12 - 16\right]$$

$$u_{3,1} = -14$$

$$\Rightarrow u_{4,1} = \frac{1}{2}\left[u_{3,0} + u_{5,0}\right]$$

$$= \frac{1}{2}\left[-18 + 0\right]$$

$$u_{4,1} = -9$$

To Evaluate other rows, we have the relation.

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

Let $i = 1$, $j = 1$

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0}.$$

$$= 0 - 11 + 4.$$

$$u_{1,2} = -7$$

Let $i = 2$, $j = 1$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0}$$

$$= -6 - 14 + 12$$

$$u_{2,2} = -8$$

Let $i = 3$, $j = 1$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0}$$

$$= -11 - 9 + 18$$

$$u_{3,2} = -2$$

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Let $i=4,\ j=1$

$U_{4,2} = U_{3,1} + U_{5,1} - U_{4,0}$

$= -14 + 0 + 16.$

$U_{4,2} = 2$

Let $i=1,\ j=2.$

$U_{1,3} = U_{0,2} + U_{2,2} - U_{1,1}$

$= 0 + (-8) + 6.$

$U_{1,3} = 2.$

Let $i=2,\ j=2.$

$U_{2,3} = U_{1,2} + U_{3,2} - U_{2,1}$

$= -7 - 2 + 11$

$U_{2,3} = 2$

Let $i=3,\ j=2.$

$U_{3,3} = U_{2,2} + U_{4,2} - U_{3,1}$

$= -8 + 2 + 14$

$U_{3,3} = 8$

Let $i=4,\ j=2$

$U_{4,3} = U_{3,2} + U_{5,2} - U_{4,1}$

$= -2 + 0 + 9$

$U_{4,3} = 7$

Let $i=1,\ j=3.$

$U_{1,4} = U_{0,3} + U_{2,3} - U_{1,2}$

$= 0 + 2 + 7$

$U_{1,4} = 9$

Let $i=2;\ j=3$

$U_{2,4} = U_{1,3} + U_{3,3} - U_{2,2}$

$= -2 + 8 + 8$

$U_{2,4} = 14$

Let $i=3,\ j=3$

$$U_{3,4} = U_{2,3} + U_{4,3} - U_{3,3}$$
$$= 2+7+2$$
$$U_{3,4} = 11$$

Let $i=4,\ j=3$.

$$U_{4,4} = U_{3,3} + U_{5,3} - U_{4,3}$$
$$= 8+0-2$$
$$U_{4,4} = 6.$$

$t \downarrow$ \ $x \rightarrow$		x_0	x_1	x_2	x_3	x_4	x_5
		0	1	2	3	4	5
t_0	0	0	-4	-12	-18	-16	0
t_1	0.25	0	-6	-11	-14	-9	0
t_2	0.5	0	-7	-8	-2	2	0
t_3	0.75	0	-2	2	8	7	0
t_4	1	0	9	14	11	6	0

02. The transverse displacement u of a point at a distance 'x' from one end at any time 't' vibrating string satisfies the equation $U_{tt} = 25U_{xx}$ with boundary conditions $u(0,t) = u(5,t) = 0$. and the intial Conditions $u(x,0) = \begin{cases} 20x, & 0 \le x \le 1 \\ 5(5-x), & 1 \le x \le 5 \end{cases}$ $U_t(x,0) = 0$. Solve the Equation numerically upto $t=5$. taking $h=1$, $K=0.2$.

$\Rightarrow$ Let $U_{tt} = 25\,U_{xx}$

$\Rightarrow C^2 = 25$

$\boxed{C=5}$

and $h=1$ & $K=0.2$

$\therefore x = 0,1,2,3,4,5.$

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$t = 0, 0.2, 0.4, 0.6, 0.8, 1.$

Also given:- $u(0,t) = 0$ $\quad u(5,t) = 0$

$t \downarrow$ / $x \rightarrow$		x_0	x_1	x_2	x_3	x_4	x_5
		0	1	2	3	4	5
t_0	0	$U_{0,0}$	$U_{1,0}$	$U_{2,0}$	$U_{3,0}$	$U_{4,0}$	$U_{5,0}$
t_1	0.2	$U_{0,1}$	$U_{1,1}$	$U_{2,1}$	$U_{3,1}$	$U_{4,1}$	$U_{5,1}$
t_2	0.4	$U_{0,2}$	$U_{1,2}$	$U_{2,2}$	$U_{3,2}$	$U_{4,2}$	$U_{5,2}$
t_3	0.5	$U_{0,3}$	$U_{1,3}$	$U_{2,3}$	$U_{3,3}$	$U_{4,3}$	$U_{5,3}$
t_4	0.8	$U_{0,4}$	$U_{1,4}$	$U_{2,4}$	$U_{3,4}$	$U_{4,4}$	$U_{5,4}$
t_5	1	$U_{0,5}$	$U_{1,5}$	$U_{2,5}$	$U_{3,5}$	$U_{4,5}$	$U_{5,5}$.

To find first row, given: $u(x,0) = \begin{cases} 20x & 0 \le x \le 1 \\ 5(5-x) & 1 \le x < 5. \end{cases}$

$U_{1,0} = 5(5-1) = 20.$

$U_{2,0} = 5(5-2) = 15$

$U_{3,0} = 5(5-3) = 10.$

$U_{4,0} = 5(5-4) = 5$

To find 2nd row, we have.

$$U_{i,1} = \frac{1}{2}\left[U_{i-1,0} + U_{i+1,0}\right]$$

Let $i = 1$

$\therefore U_{1,1} = \frac{1}{2}\left[U_{0,0} + U_{2,0}\right] = \frac{1}{2}\left[0 + 15\right] = 7.5$

$U_{2,1} = \frac{1}{2}\left[U_{1,0} + U_{3,0}\right] = \frac{1}{2}\left[20 + 10\right] = 15$

$U_{3,1} = \frac{1}{2}\left[U_{2,0} + U_{4,0}\right] = \frac{1}{2}\left[15 + 5\right] = 10.$

$U_{4,1} = \frac{1}{2}\left[U_{3,0} + U_{5,0}\right] = \frac{1}{2}\left[10 + 0\right] = 5.$

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To find 3rd row.

$$U_{i,j+1} = U_{i-1,j} + U_{i+1,j} - U_{i,j-1}$$

$i=1,\ j=1$

$$U_{1,2} = U_{0,1} + U_{2,1} - U_{1,0}$$
$$= 0 + 15 - 20$$
$$U_{1,2} = -5$$

$i=2,\ j=1$

$$U_{2,2} = U_{1,1} + U_{3,1} - U_{2,0}$$
$$= 7.5 + 10 - 15$$
$$U_{2,2} = 2.5$$

$i=3,\ j=1$

$$U_{3,2} = U_{2,1} + U_{4,1} - U_{3,0}$$
$$= 15 + 5 - 10.$$
$$U_{3,2} = 10.$$

$i=4,\ j=1$

$$U_{4,2} = U_{3,1} + U_{5,1} - U_{4,0}.$$
$$= 10 + 0 - 5$$
$$U_{4,2} = 5$$

To find 4th row,

$i=1,\ \ j=2.$

$$U_{1,3} = U_{0,2} + U_{2,2} - U_{1,1}$$
$$= 0 + 2.5 - 7.5$$
$$U_{1,3} = -5$$

$$U_{2,3} = U_{1,2} + U_{3,2} - U_{2,1}$$
$$= -5 + 10 - 15$$
$$U_{2,3} = -10.$$

$$U_{3,3} = U_{2,2} + U_{4,2} - U_{3,1}$$
$$= 2.5 + 5 - 10$$
$$U_{3,3} = -2.5$$

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$$U_{4,3} = U_{3,2} + U_{5,2} - U_{4,1}$$
$$= 10 + 0 - 5$$
$$U_{4,3} = 5$$

To find the 5th row:-

$i=1$, $j=3$.

$$U_{1,4} = U_{0,3} + U_{2,3} - U_{1,2} = 0 - 10 + 5 = -5$$

$$U_{2,4} = U_{1,3} + U_{3,3} - U_{2,2} = -5 - 2.5 - 2.5 = -10.$$

$$U_{3,4} = U_{2,3} + U_{4,3} - U_{3,2} = -10 + 5 - 10 = -15$$

$$U_{4,4} = U_{3,3} + U_{5,3} - U_{4,2} = -2.5 + 0 - 5 = -7.5$$

To find the 6th row:-

$i=1$, $j=4$.

$$U_{1,5} = U_{0,4} + U_{2,4} - U_{1,3} = 0 - 10 + 5 = -5$$

$$U_{2,5} = U_{1,4} + U_{3,4} - U_{2,3} = -5 - 15 + 10 = -10.$$

$$U_{3,5} = U_{2,4} + U_{4,4} - U_{3,3} = -10 - 7.5 + 2.5 = -15$$

$$U_{4,5} = U_{3,4} + U_{5,4} - U_{4,3} = -15 + 0 - 5 = -20.$$

$t \downarrow$ \ $x \rightarrow$		x_0	x_1	x_2	x_3	x_4	x_5
		0	1	2	3	4	5
t_0	0	0	20	15	10	5	0
t_1	0.2	0	7.5	15	10	5	0
t_2	0.4	0	-5	2.5	10	5	0
t_3	0.5	0	-5	-10	-2.5	5	0
t_4	0.8	0	-5	-10	-15	-7.5	0
t_5	1	0	-5	-10	-15	-20	0

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(Q2) Solve the Wave equation $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$ subject to $u(0,t)=0$, $u(4,t)=0$, $u_t(x,0)=0$ & $u(x,0) = x(4-x)$ by taking $h=1$, $k=0.5$ upto four steps.

→ Given :- $u_{tt} = 4u_{xx}$

$c^2 = 4$

$\boxed{c=2}$

$\therefore$ $x = 0, 1, 2, 3, 4$

$t = 0, 0.5, 1, 1.5, 2$

$h=1$, $k=0.5$

$t\downarrow$ \ $x\rightarrow$		x_0	x_1	x_2	x_3	x_4
		0	1	2	3	4
t_0	0	$u_{0,0}$	$u_{1,0}$	$u_{2,0}$	$u_{3,0}$	$u_{4,0}$
t_1	0.5	$u_{0,1}$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}$
t_2	1	$u_{0,2}$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}$
t_3	1.5	$u_{0,3}$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3}$
t_4	2	$u_{0,4}$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4}$

1st row:-

$u(x,0) = x(4-x)$

$u(1,0) = 1(4-1) = 3.$

$u(2,0) = 2(4-2) = 4$

$u(3,0) = 3(4-3) = 3.$

2nd row:- $u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$

$u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}]$

$= \frac{1}{2}[0+4]$

$u_{1,1} = 2$

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$$U_{2,1} = \frac{1}{2}\left[U_{1,0} + U_{3,0}\right] = \frac{1}{2}\left[3+3\right] = 3$$

$$U_{3,1} = \frac{1}{2}\left[U_{2,0} + U_{4,0}\right] = \frac{1}{2}\left[4+0\right] = 2.$$

3rd row:- $U_{i,j+1} = U_{i-1,j} + U_{i+1,j} - U_{i,j-1}$

$$U_{1,2} = U_{0,1} + U_{2,1} - U_{1,0} = 0+3-3 = 0.$$

$$U_{2,2} = U_{1,1} + U_{3,1} - U_{2,0} = 2+2-4 = 0.$$

$$U_{3,2} = U_{2,1} + U_{4,1} - U_{3,0} = 3+0-3 = 0.$$

4th row.

$$U_{1,3} = U_{0,2} + U_{2,2} - U_{1,1} = 0+0-2 = -2.$$

$$U_{2,3} = U_{1,2} + U_{3,2} - U_{2,1} = 0+0-3 = -3$$

$$U_{3,3} = U_{2,2} + U_{4,2} - U_{3,1} = 0+0-2 = -2.$$

5th row.

$$U_{1,4} = U_{0,3} + U_{2,3} - U_{1,2} = 0-3-0 = -3.$$

$$U_{2,4} = U_{1,3} + U_{3,3} - U_{2,2} = -2-2-0 = -4$$

$$U_{3,4} = U_{2,3} + U_{4,3} - U_{3,2} = -3+0+0 = -3.$$

$t \downarrow$ \ $x \rightarrow$		x_0	x_1	x_2	x_3	x_4
		0	1	2	3	4
t_0	0	0	3	4	3	0
t_1	0.5	0	2	3	2	0
t_2	1	0	0	0	0	0
t_3	1.5	0	-2	-3	-2	0
t_4	2	0	-3	-4	-3	0

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⇒ Numerical Solutions of One - dimensional heat Equation.

* Step 1:- observe the given heat Equation is in the standard form $u_t = c^2 u_{xx}$ and find the Value of c.

* Step 2:- Find the Step Size of t, $k = \frac{h^2}{2c^2}$

* Step 3:- Using given boundary Condition $u(0,t)=0$, $u(l,t)=0$ Prepare the table accordingly.

* Step 4:- Find the 1st row. by using the initial Condition $u(x,0) = f(x)$.

* Step 5:- Find the other rows of the Values Using the relation.

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right].$$

* Step 6:- In other Way Using Schmid ~~expression~~ Explicit formula we can find other rows.

$$u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha)u_{i,j} + \alpha u_{i+1,j}$$

where $\alpha = \frac{kc^2}{h^2}$

1] Find the numerical Solution of parabolic Equation $\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial t}$, $u(0,t)=0$, $u(4,t)=0$ & $u(x,0) = x(4-x)$, by taking $h=1$. Find the Values upto $t=5$.

⇒ given $u_{xx} = 2u_t$

$$\Rightarrow u_t = \frac{1}{2}u_{xx}$$

$$\boxed{c^2 = \frac{1}{2}}$$

$\therefore h=1$ $\quad k = \frac{h^2}{2c^2}$

$$= \frac{1}{2\left(\frac{1}{2}\right)}$$

$$\boxed{k = 1}$$

$x = 0,1,2,3,4.$ $\quad \therefore t = 0,1,2,3,4,5.$

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boundary condition:-

$u(0,t) = 0 = u(4,t)$.

$t \downarrow$ \ $x \rightarrow$		x_0	x_1	x_2	x_3	x_4
		0	1	2	3	4
t_0	0	$u_{0,0}$	$u_{1,0}$	$u_{2,0}$	$u_{3,0}$	$u_{4,0}$
t_1	1	$u_{0,1}$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}$
t_2	2	$u_{0,2}$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}$
t_3	3	$u_{0,3}$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3}$
t_4	4	$u_{0,4}$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4}$
t_5	5	$u_{0,5}$	$u_{1,5}$	$u_{2,5}$	$u_{3,5}$	$u_{5,5}$

To find 1st row:- $u(x,0) = x(4-x)$

$u_{1,0} = 1(4-1) = 3$.

$u_{2,0} = 2(4-2) = 4$

$u_{3,0} = 3(4-3) = 4$

To find 2nd row:- $u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$

$i = 1, \ j = 0$.

$u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right] = \frac{1}{2}\left[0+4\right] = 2$.

$u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = \frac{1}{2}\left[3+3\right] = 3$

$u_{3,1} = \frac{1}{2}\left[u_{2,0} + u_{4,0}\right] = \frac{1}{2}\left[4+0\right] = 2$.

3rd row $\quad i=1, j=1$.

$u_{1,2} = \frac{1}{2}\left[u_{0,1} + u_{2,1}\right] = \frac{1}{2}\left[0+3\right] = 3/2$

$u_{2,2} = \frac{1}{2}\left[u_{1,1} + u_{3,1}\right] = \frac{1}{2}\left[2+2\right] = 2$

$u_{3,2} = \frac{1}{2}\left[u_{2,1} + u_{4,1}\right] = \frac{1}{2}\left[3+0\right] = 1.5$

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4^{th} row $i=1$ $j=2$.

$$U_{1,3} = \frac{1}{2}\left[U_{0,2} + U_{2,2}\right] = \frac{1}{2}\left[0+2\right] = 1$$

$$U_{2,3} = \frac{1}{2}\left[U_{1,2} + U_{3,2}\right] = \frac{1}{2}\left[1.5+1.0\right] = 1.5$$

$$U_{3,3} = \frac{1}{2}\left[U_{2,2} + U_{4,2}\right] = \frac{1}{2}\left[2+0\right] = 1$$

5^{th} row $i=1$ $j=3$.

$$U_{1,4} = \frac{1}{2}\left[U_{0,3} + U_{2,3}\right] = \frac{1}{2}\left[0+1.5\right] = 0.75.$$

$$U_{2,4} = \frac{1}{2}\left[U_{1,3} + U_{3,3}\right] = \frac{1}{2}\left[1+1\right] = 1$$

$$U_{3,4} = \frac{1}{2}\left[U_{2,3} + U_{4,3}\right] = \frac{1}{2}\left[1.5+0\right] = 0.75$$

6^{th} row, $i=1$ $j=4$.

$$U_{1,5} = \frac{1}{2}\left[U_{0,4} + U_{2,4}\right] = \frac{1}{2}\left[0+1\right] = 0.5$$

$$U_{2,5} = \frac{1}{2}\left[U_{1,4} + U_{3,4}\right] = \frac{1}{2}\left[0.75+0.75\right] = 0.75$$

$$U_{3,5} = \frac{1}{2}\left[U_{2,4} + U_{4,4}\right] = \frac{1}{2}\left[1+0\right] = 0.5$$

$x \rightarrow$ / $t \downarrow$		x_0	x_1	x_2	x_3	x_4
		0	1	2	3	4
t_0	0	0	3	4	3	0
t_1	1	0	2	3	2	0
t_2	2	0	1.5	2	1.5	0
t_3	3	0	1	1.5	1	0
t_4	4	0	0.75	1	0.75	0
t_5	5	0	0.5	0.75	0.5	0

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(2) Solve that $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ Subject to the Conditions $u(x,0) = \sin \pi x$, $0 \le x \le 1$ $u(0,t) = u(1,t) = 0$. Carry out Computation two level taking $h = 1/3$, $k = 1/36$.

$\Rightarrow$ Given One dimensional heat Equation.

$$U_t = U_{xx}$$

$$\boxed{C^2 = 1}$$

and given that $h = 1/3$, $k = 1/36$

$\therefore$ $x = x_0 = 0$, $x_1 = 1/3$ $x_2 = 2/3$, $x_3 = 1$

$t = t_0 = 0$, $t_1 = 1/36$ $t_2 = 2/36$ ......

and also given $u(0,t) = u(1,t)$.

$t \downarrow$ \ $x \rightarrow$		x_0	x_1	x_2	x_3
		0	1/3	2/3	1
t_0	0	$U_{0,0}$	$U_{1,0}$	$U_{2,0}$	$U_{3,0}$
t_1	1/36	$U_{0,1}$	$U_{1,1}$	$U_{2,1}$	$U_{3,1}$
t_2	2/36	$U_{0,2}$	$U_{1,2}$	$U_{2,2}$	$U_{3,2}$
⋮	⋮	⋮	⋮	⋮	⋮
t_n		⋮	⋮	⋮	⋮

$U_{0,0} = 0$ $\qquad U_{3,0} = U(1,0) = 0$

$U_{0,1} = 0$ $\qquad U_{3,1} = U(1, 1/36) = 0$

$U_{0,2} = 0$ $\qquad U_{3,2} = U(1, 2/36) = 0$

$U_{0,3} = 0$

⋮

We have relation to find.

$U(x,0) = \sin(\pi x)$

$U_{1,0} = U(1/3, 0) = \sin(\pi/3) = \sqrt{3}/2 = 0.866$

$U_{2,0} = U(2/3, 0) = \sin(2\pi/3) = 0.866$

$u_{3,0} = u(1,0) = \sin\pi = 0.$

To find other rows using Schmidt Explicit formula.

$$u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha) u_{i,j} + \alpha u_{i+1,j} \longrightarrow (1)$$

where $\alpha = \frac{Kc^2}{h^2} = \frac{1/36(1)}{1/9} = 9/36 = 1/4$

$$(1) \Rightarrow u_{i,j+1} = \frac{1}{4} u_{i-1,j} + \frac{1}{2} u_{i,j} + \frac{1}{4} u_{i+1,j}$$

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j} + 2u_{i,j} + u_{i+1,j}\right]$$

let $i=1,\ j=0.$

$$u_{1,1} = \frac{1}{4}\left[u_{0,0} + 2u_{1,0} + u_{2,0}\right]$$

$$= \frac{1}{4}\left[0 + 2(0.866) + 0.866\right]$$

$$u_{1,1} = 0.6495$$

$$u_{2,1} = \frac{1}{4}\left[u_{1,0} + 2u_{2,0} + u_{3,0}\right]$$

$$= \frac{1}{4}\left[\sqrt{3}/2 + 2(0.866) + 0\right]$$

$$u_{2,1} = 0.6495$$

$i=1,\ j=1$

$$u_{1,2} = \frac{1}{4}\left[u_{0,1} + 2u_{1,1} + u_{2,1}\right]$$

$$= \frac{1}{4}\left[0 + 2(0.6495) + 0.6495\right]$$

$$u_{1,2} = 0.487125$$

$$u_{2,2} = \frac{1}{4}\left[u_{1,1} + 2u_{2,1} + u_{3,1}\right]$$

$$= \frac{1}{4}\left[0.6495 + 2(0.6495) + 0\right]$$

$$u_{2,2} = 0.487125$$

(3) Find the values of $u_t = 4u_{xx}$ and the boundary condition $u(0,t) = 0,\ = u(8,t)$ & $u(x,0) = 4x - \frac{x^2}{2}$ at the points $x = i$, $i = 0, 1, 2, 3, 4, 5, 6, 7, 8.$, $K = j/8$ $j = 0, 1, 2, 3, 4.$

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given one dimensional heat Equation.

$\rightarrow \quad U_t = 4U_{xx}$

$\boxed{c^2 = 4}$

$x = 0, 1, 2, 3, 4, 5, 6, 7, 8.$

$t = 0, 1/8, 2/8, 3/8, 4/8.$

1st row, we have $u(x,0) = 4x - \frac{x^2}{2}$

$U_{1,0} = 4 - \frac{1}{2}$

$= 3.5$

$U_{2,0} = 4(2) - \frac{2^2}{2}$

$= 8 - 2$

$U_{2,0} = 6.$

$U_{3,0} = 12 - \frac{9}{2} = 7.5$

$U_{4,0} = 16 - 16/2 = 8$

$U_{5,0} = 20 - 25/2 = 7.5$

$U_{6,0} = 24 - 36/2 = 6.$

$U_{7,0} = 28 - 49/2 = 3.5$

$U_{8,0} = 0.$

To find 2nd row:-

$U_{i,j+1} = \frac{1}{2}\left[U_{i-1,j} + U_{i+1,j}\right]$

$i=1, \; j=0.$

$U_{1,1} = \frac{1}{2}\left[U_{0,0} + U_{2,0}\right] = \frac{1}{2}\left[0+6\right] = 3$

$U_{2,1} = \frac{1}{2}\left[U_{1,0} + U_{3,0}\right] = \frac{1}{2}\left[3.5+7.5\right] = 5.5$

$U_{3,1} = \frac{1}{2}\left[U_{2,0} + U_{4,0}\right] = \frac{1}{2}\left[6+8\right] = 7$

$U_{4,1} = \frac{1}{2}\left[U_{3,0} + U_{5,0}\right] = \frac{1}{2}\left[7.5+7.5\right] = 7.5$

$U_{5,1} = \frac{1}{2}\left[U_{4,0} + U_{6,0}\right] = \frac{1}{2}\left[8+6\right] = 7.$

$U_{6,1} = \frac{1}{2}\left[U_{5,0} + U_{7,0}\right] = \frac{1}{2}\left[7.5+3.5\right] = 5.5$

$U_{7,1} = \frac{1}{2}\left[U_{6,0} + U_{8,0}\right] = \frac{1}{2}\left[6+0\right] = 3.$

3rd row:- $i=1, j=1$

$$U_{1,2} = \frac{1}{2}\left[U_{0,1} + U_{2,1}\right] = \frac{1}{2}\left[0+5.5\right] = 2.75$$

$$U_{2,2} = \frac{1}{2}\left[U_{1,1} + U_{3,1}\right] = \frac{1}{2}\left[3+7\right] = 5$$

$$U_{3,2} = \frac{1}{2}\left[U_{2,1} + U_{4,1}\right] = \frac{1}{2}\left[5.5+7.5\right] = 6.5$$

$$U_{4,2} = \frac{1}{2}\left[U_{3,1} + U_{5,1}\right] = \frac{1}{2}\left[7+7\right] = 7$$

$$U_{5,2} = \frac{1}{2}\left[U_{4,1} + U_{6,1}\right] = \frac{1}{2}\left[7.5+7.5\right] = 6.5$$

$$U_{6,2} = \frac{1}{2}\left[U_{5,1} + U_{7,1}\right] = \frac{1}{2}\left[7+3\right] = 5$$

$$U_{7,2} = \frac{1}{2}\left[U_{6,1} + U_{8,1}\right] = \frac{1}{2}\left[5.5+0\right] = 2.75.$$

4th row $i=1, j=2.$

$$U_{1,3} = \frac{1}{2}\left[U_{0,2} + U_{2,2}\right] = \frac{1}{2}\left[0+5\right] = 2.5$$

$$U_{2,3} = \frac{1}{2}\left[U_{1,2} + U_{3,2}\right] = \frac{1}{2}\left[2.75+6.5\right] = 4.675$$

$$U_{3,3} = \frac{1}{2}\left[U_{2,2} + U_{4,2}\right] = \frac{1}{2}\left[5+7\right] = 6$$

$$U_{4,3} = \frac{1}{2}\left[U_{3,2} + U_{5,2}\right] = \frac{1}{2}\left[6.5+6.5\right] = 6.5$$

$$U_{5,3} = \frac{1}{2}\left[U_{4,2} + U_{6,2}\right] = \frac{1}{2}\left[7+5\right] = 6$$

$$U_{6,3} = \frac{1}{2}\left[U_{5,2} + U_{7,2}\right] = \frac{1}{2}\left[6.5+2.75\right] = 4.625$$

$$U_{7,3} = \frac{1}{2}\left[U_{6,2} + U_{5,2}\right] = \frac{1}{2}\left[5+0\right] = 2.5$$

5th row $i=1, j=3.$

$$U_{1,4} = \frac{1}{2}\left[U_{0,3} + U_{2,3}\right] = \frac{1}{2}\left[0+4.625\right] = 0.23125$$

$$U_{2,4} = \frac{1}{2}\left[U_{1,3} + U_{3,3}\right] = \frac{1}{2}\left[2.5+6\right] = 4.25$$

$$U_{3,4} = \frac{1}{2}\left[U_{2,3} + U_{4,3}\right] = \frac{1}{2}\left[4.625+6.5\right] = 5.5625$$

$$U_{4,4} = \frac{1}{2}\left[U_{3,3} + U_{5,3}\right] = \frac{1}{2}\left[6+6\right] = 6$$

$$U_{5,4} = \frac{1}{2}\left[U_{4,3} + U_{6,3}\right] = \frac{1}{2}\left[6.5+4.625\right] = 5.5625$$

$$U_{6,4} = \frac{1}{2}\left[U_{5,3} + U_{7,3}\right] = \frac{1}{2}\left[6+2.5\right] = 4.25$$

$$U_{7,4} = \frac{1}{2}\left[U_{6,3} + U_{8,3}\right] = \frac{1}{2}\left[4.625+0\right] = 0.23125.$$

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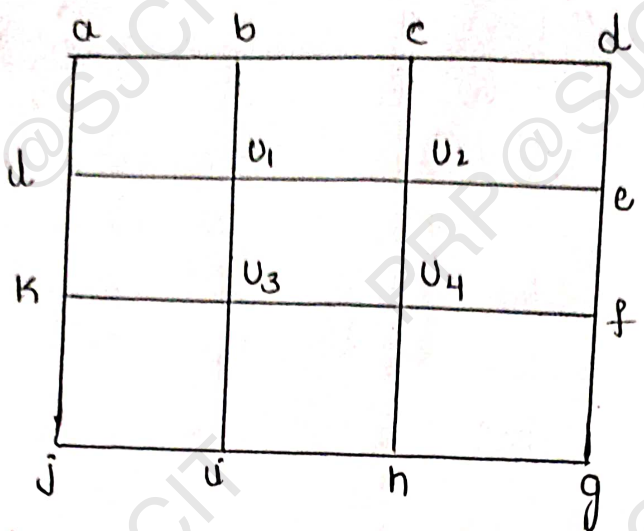
$t \downarrow$ \ $x \rightarrow$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0	1	2	3	4	5	6	7	8
t_0	$U_{0,0}$	$U_{1,0}$	$U_{2,0}$	$U_{3,0}$	$U_{4,0}$	$U_{5,0}$	$U_{6,0}$	$U_{7,0}$	$U_{8,0}$
t_1	$U_{0,1}$	$U_{1,1}$	$U_{2,1}$	$U_{3,1}$	$U_{4,1}$	$U_{5,1}$	$U_{6,1}$	$U_{7,1}$	$U_{8,1}$
t_2	$U_{0,2}$	$U_{1,2}$	$U_{2,2}$	$U_{3,2}$	$U_{4,2}$	$U_{5,2}$	$U_{6,2}$	$U_{7,2}$	$U_{8,2}$
t_3	$U_{0,3}$	$U_{1,3}$	$U_{2,3}$	$U_{3,3}$	$U_{4,3}$	$U_{5,3}$	$U_{6,3}$	$U_{7,3}$	$U_{8,3}$
t_4	$U_{0,4}$	$U_{1,4}$	$U_{2,4}$	$U_{3,4}$	$U_{4,4}$	$U_{5,4}$	$U_{6,4}$	$U_{7,4}$	$U_{8,4}$

Thus the required values of $U_{i,j}$ are tabulated.

$t \downarrow$ \ $x \rightarrow$		x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
		0	1	2	3	4	5	6	7	8
t_0	0	0	3.5	6	7.5	8	7.5	6	3.5	0
t_1	1/8	0	3	5.5	7	7.5	7	5.5	3	0
t_2	2/8	0	2.75	5	6.5	7	6.5	5	2.75	0
t_3	3/8	0	2.5	4.625	6	6.5	6	4.625	2.5	0
t_4	4/8	0	0.2123	4.25	5.5	6	5.5	4.25	0.23	0

Numerical Solution of the Laplace's Equation in two dimensions.

- Step 1 :- Write the given Laplace's Equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (or) $U_{xx} + U_{yy} = 0$.
- Step 2 :- It can be solved (or) Evaluate having nine square mesh as given below.



* **Step 3:-** To find U_1, U_2, U_3, U_4 the five point rule follows as.

$$U_1 = \frac{1}{4}\left[l + U_2 + U_3 + b\right]$$

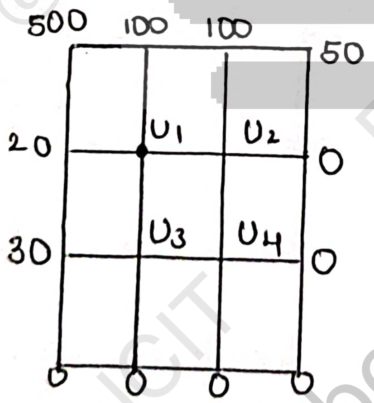
$$U_2 = \frac{1}{4}\left[U_1 + e + U_4 + c\right]$$

$$U_3 = \frac{1}{4}\left[K + U_4 + i + U_i\right]$$

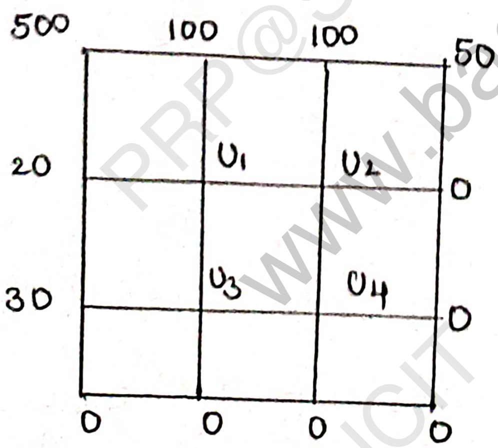
$$U_4 = \frac{1}{4}\left[U_3 + f + U_2 + h\right]$$

* **Step 4:-** Solve the above linear equation & find U_1, U_2, U_3, U_4...

① Solve $U_{xx} + U_{yy} = 0$ in the following square region with the boundary conditions as indicated in the figure.



$\Rightarrow$



Using standard five-point formula, we get.

$$U_1 = \frac{1}{4}\left[20 + U_2 + U_3 + 100\right]$$

$$U_1 = \frac{1}{4}\left[U_2 + U_3 + 120\right]$$

$$\Rightarrow 4U_1 = U_2 + U_3 + 120$$

$$\Rightarrow 4U_1 - U_2 - U_3 = 120 \longrightarrow ①$$

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$$U_2 = \frac{1}{4}\left[U_1 + 0 + 100 + U_4\right]$$

$$\Rightarrow 4U_2 = U_1 + U_4 + 100.$$

$$-U_1 + 4U_2 - U_4 = 100 \longrightarrow (2)$$

$$U_3 = \frac{1}{4}\left[U_1 + 0 + 30 + U_4\right]$$

$$\Rightarrow 4U_3 = U_1 + U_4 + 30$$

$$\Rightarrow -U_1 + 4U_3 - U_4 = 30 \longrightarrow (3)$$

$$U_4 = \frac{1}{4}\left[U_3 + 0 + U_2 + 0\right]$$

$$4U_4 - U_3 - U_2 = 0$$

$$-U_2 - U_3 + 4U_4 = 0 \longrightarrow (4)$$

By Evaluating U_1 from eq^n (1) and (2), (2) & (3)

$$(1) + (2)\,4 \Rightarrow 15U_2 - U_3 - 4U_4 = 520 \longrightarrow (5)$$

$$(2) - (3) \Rightarrow 4U_2 - 4U_3 = 70$$

$$\Rightarrow 2U_2 - 2U_3 + 0U_4 = 35 \longrightarrow (6)$$

$$\therefore U_2 = 40.4167$$

$$U_3 = 22.9167$$

$$U_4 = 15.8333$$

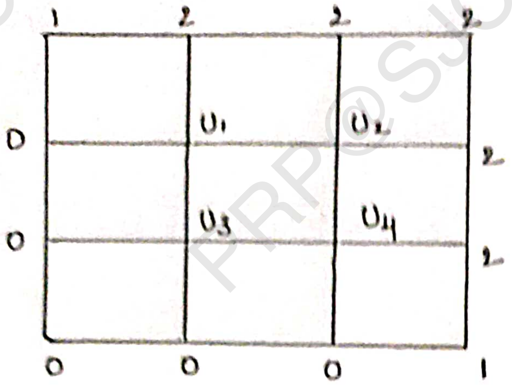
$$(1) \Rightarrow U_1 = \frac{1}{4}\left[U_2 + U_3 + 120\right]$$

$$= \frac{1}{4}\left[40.4167 + 22.9167 + 120\right]$$

$$\boxed{U_1 = 45.8333}$$

(2) Solve $U_{xx} + U_{yy} = 0$, for the square mesh with boundary values as given below. Iterate till the mesh values are correct to two decimal places.

→ Given the Laplaces Eq^n $U_{xx} + U_{yy} = 0$ with the boundary values of mesh as given with 9-squares.



WKT $U_1 = \frac{1}{4}[0 + U_2 + 2 + U_3]$

$\Rightarrow 4U_1 = U_2 + U_3 + 2$

$\Rightarrow 4U_1 - U_2 - U_3 = 2 \longrightarrow (1)$

$U_2 = \frac{1}{4}[U_1 + 2 + U_4 + 2]$

$\Rightarrow 4U_2 = U_1 + U_4 + 4$

$\Rightarrow -U_1 + 4U_2 - U_4 = 4 \longrightarrow (2)$

Let $U_3 = \frac{1}{4}[0 + U_4 + 0 + U_1]$

$\Rightarrow 4U_3 = U_1 + U_4 + 0$

$\Rightarrow -U_1 + 4U_3 - U_4 = 0 \longrightarrow (3)$

$U_4 = \frac{1}{4}[U_3 + 2 + 0 + U_2]$

$\Rightarrow 4U_4 = U_2 + U_3 + 2$

$\Rightarrow -U_2 - U_3 + 4U_4 = 2 \longrightarrow (4)$

Add eq^n (1) & (2) & Multiply $\times 4$ to eq^n (2)

$\Rightarrow 15U_2 - U_3 - 4U_4 = 18 \longrightarrow (5)$

Subtract eq^n (2) - (3)

$\Rightarrow 4U_2 - 4U_3 = 4 \quad \div$ by 4

$\Rightarrow U_2 - U_3 + 0.U_4 = 1 \longrightarrow (6)$

$\therefore U_2 = 1.50$

$U_3 = 0.50$

$U_4 = 1.00$

$\therefore (1) \Rightarrow U_1 = \frac{1}{4}[U_2 + U_3 + 2]$

$\Rightarrow U_1 = \frac{1}{4}[1.50 + 0.50 + 2]$

$\Rightarrow \boxed{U_1 = 1.00}$

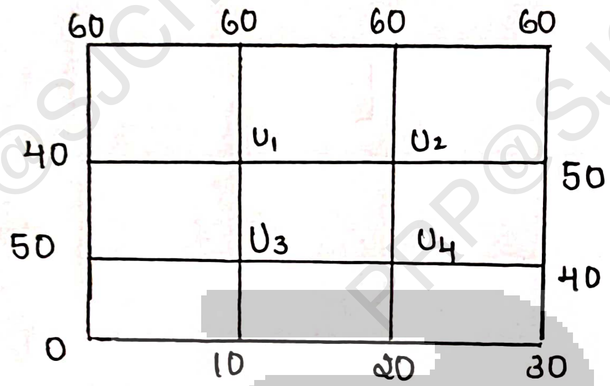
$\therefore U_1 = 1.00 \qquad U_4 = 1.00.$

$U_2 = 1.50$

$U_3 = 0.50$

(3) Given the Values of $u(x,y)$ with the boundary of Squares in the following fig. Evaluate the function $u(x,y)$. Satisfying the laplace eqⁿ $u_{xx}+u_{yy}=0$. At pivotal points of the figure.

→ Given :- The function $u(x,y)$ Satisfying the Laplace eqⁿ $u_{xx}+u_{yy}=0$ with 9- Square of mesh having the boundary Values as given below.



W.K.T

$$U_1 = \frac{1}{4}\left[40 + U_2 + 60 + U_3\right]$$

$$\Rightarrow 4U_1 = U_2 + U_3 + 100$$

$$\Rightarrow 4U_1 - U_2 - U_3 = 100 \longrightarrow (1)$$

$$U_2 = \frac{1}{4}\left[U_1 + 50 + 60 + U_4\right]$$

$$\Rightarrow 4U_2 = U_1 + U_4 + 110$$

$$\Rightarrow -U_1 + 4U_2 - U_4 = 110 \longrightarrow (2)$$

$$U_3 = \frac{1}{4}\left[10 + U_1 + 50 + U_4\right]$$

$$\Rightarrow 4U_3 = U_1 + U_4 + 60$$

$$\Rightarrow -U_1 + 4U_3 - U_4 = 60 \longrightarrow (3)$$

$$U_4 = \frac{1}{4}\left[U_3 + 40 + U_2 + 20\right]$$

$$\Rightarrow 4U_4 = U_3 + U_2 + 60$$

$$-U_2 - U_3 + 4U_4 = 60 \longrightarrow (4)$$

Add (1) + (2)4 $\Rightarrow 15U_2 - U_3 - 4U_4 = 540 \longrightarrow (5)$

Subtract (2) - (3)

$$\Rightarrow 4U_2 - 4U_3 = 50 \quad \div 2$$

$$\Rightarrow 2U_2 - 2U_3 + 0.U_4 = 25 \longrightarrow (6)$$

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By Solving, Eq^n we get (4), (5), (6) we get,

$$U_2 = 47.9166.$$
$$U_3 = 35.4166$$
$$U_4 = 35.833$$

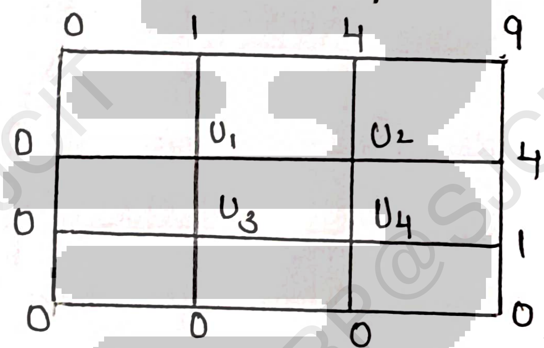
(1) $\Rightarrow$ $U_1 = \frac{1}{4}\left[U_2 + U_3 + 100\right]$

$\Rightarrow$ $U_1 = \frac{1}{4}\left[47.9166 + 35.4166 + 100\right]$

$\Rightarrow$ $\boxed{U_1 = 45.833}$

(4) Evaluate the function $u(x,y)$ Satisfying $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. at the pivodal points given the Values on the boundary as indicated in the figure.

$\Rightarrow$ Given the laplace's eq^n $u(x,y)$ Satisfying the laplace eq^n $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ with 9- Square of mesh.



W.K.T $U_1 = \frac{1}{4}\left[0 + U_2 + 1 + U_3\right]$

$\Rightarrow 4U_1 = U_2 + U_3 + 1$

$\Rightarrow 4U_1 - U_2 - U_3 = 1 \longrightarrow (1)$

$U_2 = \frac{1}{4}\left[U_1 + 4 + 4 + U_4\right]$

$\Rightarrow 4U_2 = U_1 + U_4 + 8$

$\Rightarrow -U_1 + 4U_2 - U_4 = 8 \longrightarrow (2)$

$U_3 = \frac{1}{4}\left[0 + U_1 + 0 + U_4\right]$

$\Rightarrow 4U_3 = U_1 + U_4 + 0$

$\Rightarrow -U_1 + 4U_3 - U_4 = 0 \longrightarrow (3)$

$U_4 = \frac{1}{4}\left[U_3 + 1 + U_2 + 0\right]$

$\Rightarrow 4U_4 = U_3 + U_2 + 1$

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$\Rightarrow -U_2 - U_3 + 4U_4 = 1 \longrightarrow (4)$

Add (1) + (2) Multiply 4 to eq^n (1), we get.

$\Rightarrow 15U_2 - U_3 + 4U_4 = 33 \longrightarrow (5)$

Subtract eq^n (2) & (3)

(2) - (3) $\Rightarrow 4U_2 - 4U_3 = 8 \quad \div 4$

$\Rightarrow U_2 - U_3 + 0.U_4 = 2 \longrightarrow (6)$

$\therefore$ By solving eq^n (4) (5) (6), we get

$U_2 = 2.5$

$U_3 = 0.5$

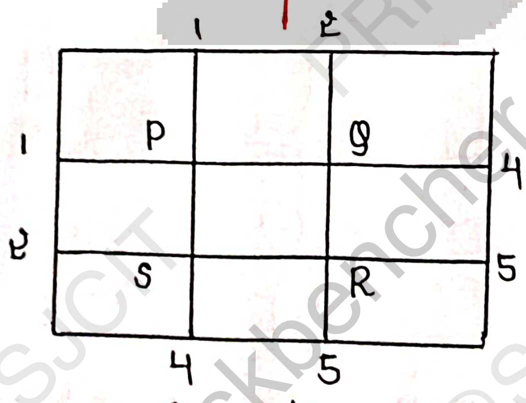
$U_1 = 1$

$\therefore$ (1) $\Rightarrow U_1 = \frac{1}{4}[U_2 + U_3 + 1]$

$\Rightarrow U_1 = \frac{1}{4}[2.5 + 0.5 + 1]$

$\Rightarrow \boxed{U_1 = 1}$

(5) Solve the Elliptic partial differential eq^n for the following square ~~matrix~~ mesh using the five point difference formula by setting up the linear eq^n at the unknown point P, Q, R, S.



Using five point formula, we have,

WKT $P = \frac{1}{4}[1 + Q + 1 + S]$

$\Rightarrow 4P = Q + S + 2$

$\Rightarrow 4P - Q - S = 2 \longrightarrow (1)$

$Q = \frac{1}{4}[P + 4 + R + 2]$

$\Rightarrow 4Q = P + R + 6$

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$\Rightarrow -P + 4Q - R = 6 \longrightarrow (2)$

$$S = \frac{1}{4}\left[2 + R + P + 4\right]$$

$\Rightarrow 4S = R + P + 6$

$\Rightarrow -P - R + 4S = 6 \longrightarrow (3)$

$$R = \frac{1}{4}\left[5 + S + Q + 5\right]$$

$\Rightarrow 4R = Q + S + 10$

$\rightarrow -Q + 4R + S = 10 \longrightarrow (4)$

Add Eqn (1) & (2) & Multiply (4) to eqn (2)

$\Rightarrow 15Q - 4R - S = 26 \longrightarrow (5)$

Subtract eqn (2) - (3)

$\Rightarrow 4Q - 4S = 0 \quad \div 4$

$\Rightarrow Q - 0 \cdot R - S = 0 \longrightarrow (6)$

$\therefore$ we have (4), (5), (6) By Solving we get.

$Q = 3$

$R = 4$

$S = 3$

$\therefore (1) \Rightarrow P = \frac{1}{4}\left[Q + S + 2\right]$

$$P = \frac{1}{4}\left[3 + 3 + 2\right]$$

$\boxed{P = 2}$

$\therefore \boxed{P = 2,\ Q = 3,\ R = 4,\ S = 3}$ //

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