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LECTURE NOTES

TRANSFORM CALCULUS , FOURIER SERIES & NUMERICAL TECHNIQUES

(21MAT31)

Module-5

(Numerical Solution of Second Ordinary Differential Equations & Calculus of Variations)

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→ Runge - Kutta Method of 4^{th} Order :-

- **Step 1 :-** Consider the 2nd Order differential Equations in the form

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0.$$

- **Step 2 :-** Write $\frac{dy}{dx} = y'(x) = z = f(x, y, z)$ and Write the Same in the given equation, we get $\frac{dz}{dx} = g(x, y, z)$, then the initial Condition become $y(x_0) = y_0$ $y'(x_0) = z(x_0) = z_0$.

Step 3:- Find $y(x_1)$ and $z(x_1)$ through given formulae.

$$y(x_0+h) = y(x_1) = y_0 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$
$$z(x_0+h) = z(x_1) = z_0 + \frac{1}{6}\left[l_1 + 2l_2 + 2l_3 + l_4\right]$$

$$K_1 = hf(x_0, y_0, z_0)$$
$$l_1 = hg(x_0\ y_0, z_0)$$

$$K_2 = hf(x_0 + h/2, y_0 + K_1/2, z_0 + l_1/2)$$
$$l_2 = hg(x_0 + h/2, y_0 + K_1/2, z_0 + l_1/2)$$

$$K_3 = hf(x_0 + h/2, y_0 + K_2/2, z_0 + l_2/2)$$
$$l_3 = hg(x_0 + h/2, y_0 + K_2/2, z_0 + l_2/2)$$

$$K_4 = hf(x_0 + h/2, y_0 + K_3, z_0 + l_3)$$
$$l_4 = hg(x_0 + h, y_0 + K_3, z_0 + l_3)$$

01. Using Runge - Kutta Method Solve $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$, for $x = 0.2$ Correct to 4 decimal places using the initial Condition $y(0)=1$ $y'(0)=0$.

$\Rightarrow$ Let $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2 \longrightarrow (1)$

$y(0)=1 \quad y'(0)=1$

Let $\frac{dy}{dx} = z, \ f(x, y, z) \longrightarrow (2)$

$(1) \Rightarrow \frac{dz}{dx} = xz^2 - y^2 = g(x, y, z) \longrightarrow (3)$

$\therefore y(0) = 1 \qquad y'(0) = 0 = z(0) = z_0$

$x_0 = 0 \qquad y_0 = 1 \qquad z_0 = 0.$

$\Rightarrow \quad x_0 = 0 \qquad y_0 = 1 \qquad z_0 = 0$

To find $y(0.2)$ let $h = 0.2$

$$K_1 = hf(x_0, y_0, z_0)$$
$$= (0.2) f(0, 1, 0)$$
$$= 0.2(0)$$
$$\boxed{K_1 = 0}$$

$$\Rightarrow \quad l_1 = hg(x_0, y_0, z_0)$$
$$= (0.2) g(0, 1, 0)$$
$$= 0.2(0(0)^2 - (1)^2)$$
$$= 0.2(-1)$$
$$\boxed{l_1 = -0.2}$$

$$K_2 = hf(x_0 + h/2, y_0 + K_1/2, z_0 + l_1/2)$$
$$= 0.2 f(0.1, 1, -0.1)$$
$$= (0.2)(-0.1)$$
$$\boxed{K_2 = -0.02}$$

$$l_2 = hg(x_0 + h/2, y_0 + K_1/2, z_0 + l_1/2)$$
$$= (0.2) g(0.1, 1, -0.1)$$
$$= (0.2)(0.1(-0.1)^2 - (1)^2)$$
$$= (0.2)(-1)$$
$$\boxed{l_2 = -0.2}$$

$$K_3 = hf(x_0 + h/2,\ y_0 + k_2/2,\ z_0 + l_2/2)$$
$$= (0.2)\, f(0.1,\ 1,\ -0.1)$$
$$= (0.2)(-0.1)$$
$$\boxed{K_3 = -0.02}$$

$$l_3 = hg(x_0 + h/2,\ y_0 + k_2/2,\ z_0 + l_2/2)$$
$$= (0.2)\, g(0.1,\ 1,\ -0.1)$$
$$= (0.2)(-1)$$
$$\boxed{l_3 = -0.2}$$

$$K_4 = hf(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$
$$= (0.2)\, f(0.2,\ 1,\ -0.2)$$
$$= (0.2)(-0.2)$$
$$\boxed{K_4 = -0.04}$$

$$l_4 = hg(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$
$$= 0.2\, g(0.2,\ 1,\ -0.2)$$
$$= 0.2(-1)$$
$$\boxed{l_4 = -0.2}$$

$\therefore$ WKT. $y(x_0 + h) = y(x_1) = y_0 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$$y(0.2) = 1 + \frac{1}{6}\left[0 - 0.04 - 0.04 - 0.04\right]$$
$$y(0.2) = 1 - 0.02$$
$$\boxed{y(0.2) = 0.98}$$

$$y'(x_0+h) = y'(x_1) = z(x_1) = z_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= 0 + \frac{1}{6}\left[-0.2 - 0.4 - 0.4 - 0.2\right]$$

$$\boxed{z(0.2) = -0.2 = y'(0.2)}$$

02. Given $y'' - xy' - y = 0$ with $y(0) = 1$, $y'(0) = 0$. Compute $y(0.2)$ using Runge - Kutta Method.

$\Rightarrow$ Given $y'' - xy' - y = 0 \longrightarrow (1)$

$$y' = z = f(x, y, z) \longrightarrow (2)$$

$$(1) \Longrightarrow \frac{dz}{dx} = -xz - y = 0$$

$$\frac{dz}{dx} = zx + y = g(x, y, z) \longrightarrow (3)$$

Given $y(0) = 1$, $y'(0) = 0$

$x_0 = 0 \quad y_0 = 1 \quad z_0 = 0$, Let $h = 0.2$

$$k_1 = hf(x_0, y_0, z_0)$$

$$= 0.2 f(0, 1, 0)$$

$$= 0.2(0)$$

$$\boxed{k_1 = 0}$$

$$l_1 = hg(x_0, y_0, z_0)$$

$$= 0.2\, g(0, 1, 0)$$

$$= 0.2(1)$$

$$\boxed{l_1 = 0.2}$$

$$k_2 = hf(x_0 + h/2,\ y_0 + k_1/2,\ z_0 + l_1/2)$$

$$= (0.2) f(0.1, 1, 0.1)$$

$$= (0.2)(0.1)$$

$$\boxed{K_2 = 0.02}$$

$$l_2 = hg(0.1, 1, 0.1)$$
$$= 0.2(1.01)$$
$$\boxed{l_2 = 0.202}$$

$$K_3 = hf(x_0 + h/2, y_0 + K_2/2, z_0 + l/2)$$
$$= 0.2f(0.1, 1.01, 0.101)$$
$$= 0.2(0.101)$$
$$\boxed{K_3 = 0.202}$$

$$l_3 = hg(0.1, 1.01, 0.101)$$
$$= (0.2)(1.0201)$$
$$\boxed{l_3 = 0.20402}$$

$$K_4 = hf(x_0+h, y_0+K_3, z_0 + l_3/2)$$
$$= 0.2f(0.2, 1.0202, 0.20402)$$
$$\boxed{K_4 = 0.040804}$$

W.K.T

$$y(x_0+h) = y(x_1) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$
$$y(0.2) = 1 + \frac{1}{6}[0 + 0.04 + 0.0404 + 0.040804]$$
$$= 1 + \frac{1}{6}[0.121204]$$
$$= 1 + 0.0202$$
$$\boxed{y(0.2) = 1.0202}$$

03. Solve $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$, for $x=0.1$. Corect to 4 decimal places using initial Condition. $y(0)=1$, $y'(0)=0$ using Runge - Kutta Method.

$\Rightarrow$ Given $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1 \longrightarrow (1)$

$$\frac{dy}{dx} = z = f(x,y,z) \longrightarrow (2)$$

$$(1) \Rightarrow \quad \frac{dz}{dx} = x^2z + 2xy + 1 \longrightarrow (3) = g(x,y,z)$$

Given $y(0)=1$, $y'(0)=1$ $\quad h=0.1$

$x_0 = 0 \quad y_0 = 0 \quad z_0 = 0 \quad y(0.1) = ?$

$$K_1 = hf(x_0, y_0, z_0)$$
$$= (0.1) f(0,1,0)$$
$$= (0.1)(0)$$
$$\boxed{K_1 = 0}$$

$$l_1 = hg(x_0, y_0, z_0)$$
$$= 0.1\, g(0,1,0)$$
$$= 0.1(1)$$
$$\boxed{l_1 = 0.1}$$

$$K_2 = hf(x_0 + h/2, y_0 + K_1/2, z_0 + l_1/2)$$
$$= (0.1) f(0.05, 1, 0.05)$$
$$= (0.1)(0.05)$$
$$\boxed{K_2 = 5 \times 10^{-3} = 0.005}$$

$$l_2 = hg(0.05, 1, 0.05)$$

$$= (0.1)(1.100125)$$

$$\boxed{l_2 = 0.11001}$$

$$K_3 = hf(x_0 + h/2, y_0 + K_2/2, z_0 + l_2/2)$$

$$= (0.1) f(0.05, 1.0025, 0.05505)$$

$$= (0.1)(0.05505)$$

$$\boxed{K_3 = 0.0055}$$

$$l_3 = hg(0.05, 1.0025, 0.05505)$$

$$= (0.1)(1.10038)$$

$$\boxed{l_3 = 0.1100}$$

$$K_4 = hf(x_0 + h, y_0 + K_3, z_0 + l_3)$$

$$= 0.1 f(0.1, 1.0055, 0.1100)$$

$$\boxed{K_4 = 0.011}$$

$$y(x_0 + h) = y(x_1) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$y(0.1) = 1 + \frac{1}{6}[0 + 2(0.005) + 2(0.0055) + 0.011]$$

$$\boxed{y(0.1) = 1.0053}$$

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04. Using R-K Method Solve $\frac{d^2y}{dx^2} = x^3\left(y + \frac{dy}{dx}\right)$. $y(0)=1$, $y'(0.1)=0.5$. find y at $x=0.1$.

$\Rightarrow$ given $\frac{d^2y}{dx^2} = x^3\left(y + \frac{dy}{dx}\right) \longrightarrow (1)$

Let $\frac{dy}{dx} = z = f(x,y,z) \longrightarrow (2)$

$(1) \Rightarrow \frac{dz}{dx} = x^3(y+z) = g(x,y,z) \longrightarrow (3)$

given $y(0)=1$, $y'(0.1) = 0.5 = z(0) = z_0$

$x_0 = 0$, $y_0 = 1$ $z_0 = 0.5$ & $h = 0.1$

$K_1 = hf(x_0, y_0, z_0)$

$= (0.1) f(0, 1, 0.5)$

$= (0.1)(0.5)$

$\boxed{K_1 = 0.05}$

$l_1 = hg(x_0, y_0, z_0)$

$= (0.1) g(0, 1, 0.5)$

$= (0.1)(0)$

$\boxed{l_1 = 0}$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2, z_0 + l_1/2)$

$= (0.1) f(0.05, 1.025, 0.5)$

$= (0.1)(0.5)$

$\boxed{K_2 = 0.05}$

$$l_2 = hg\left(x_0 + h/2,\ y_0 + k_1/2,\ z_0 + l_1/2\right)$$

$$= (0.1)\, g\left(0.05,\ 1.025,\ 0.5\right)$$

$$= (0.1)\left(1.90625 \times 10^{-4}\right)$$

$$\boxed{l_2 = 1.90625 \times 10^{-5} \approx 0}$$

$$k_3 = hf\left(x_0 + h/2,\ y_0 + k_2/2,\ z_0 + l_2/2\right)$$

$$= (0.1)\, f\left(0.05,\ 1.025,\ 0.5\right)$$

$$= (0.1)(0.5)$$

$$\boxed{k_3 = 0.05}$$

$$l_3 = hg\left(0.05,\ 1.025,\ 0.5\right)$$

$$= (0.1)(0.0002)$$

$$\boxed{l_3 = 0.00002 \approx 0}$$

$$k_4 = hf\left(x_0 + h,\ y_0 + k_3,\ z_0 + l_3\right)$$

$$= (0.1)\, f\left(0.1,\ 1.05,\ 0.5\right)$$

$$k_4 = (0.1)(0.5)$$

$$\boxed{k_4 = 0.05}$$

WKT,

$$y(x_0 + h) = y(x_1) = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$y(0.1) = 1 + \frac{1}{6}\left[0.05 + 0.1 + 0.1 + 0.05\right]$$

$$= 1 + \frac{1}{6}\left[0.3\right]$$

$$\boxed{y(0.1) = 1.05}$$

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05. Using R-K method of 4th Order to Solve $y'' = xy' - y$. $y(0) = 3$, $y'(0) = 0$, find y and z at $x = 0.1$.

$\Rightarrow$ Given $y'' = xy' - y \longrightarrow (1)$

$y' = z = f(x, y, z) \longrightarrow (2)$

$(1) \Rightarrow \dfrac{dz}{dx} = xz - y = g(x, y, z) \longrightarrow (3)$

and given $y(0) = 3$, $y'(0) = 0 = z_0$, $h = 0.1$

$x_0 = 0 \quad y_0 = 3 \quad z_0 = 0.$

$K_1 = hf(x_0, y_0, z_0)$

$= (0.1) f(0, 3, 0)$

$= (0.1)(0)$

$\boxed{K_1 = 0}$

$l_1 = hg(x_0, y_0, z_0)$

$= (0.1) g(0, 3, 0)$

$= (0.1)(-3)$

$\boxed{l_1 = -0.3}$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2, z_0 + l_1/2)$

$= (0.1) f(0.05, 3, -0.15)$

$= (0.1)(-0.15)$

$\boxed{K_2 = -0.015}$

$l_2 = hg(0.05, 3, -0.15)$

$= (0.1)(-3.0075)$

$\boxed{l_2 = -0.30075} \approx -0.3$

$$K_3 = hf\left(x_0 + h/2,\ y_0 + K_2/2,\ z_0 + l_2/2\right)$$

$$= 0.1\, f\,(0.05,\ 2.9925,\ -0.15)$$

$$K_3 = (0.1)\,(-0.15)$$

$$\boxed{K_3 = -0.015}$$

$$l_3 = hg\,(0.05,\ 3,\ -0.15)$$

$$= (0.1)\,(-3)$$

$$\boxed{l_3 = -0.3}$$

$$K_4 = hf\,(x_0 + h,\ y_0 + K_3,\ z_0 + l_3)$$

$$= (0.1)\, f\,(0.1,\ 2.985,\ -0.3)$$

$$= (0.1)\,(-0.3)$$

$$\boxed{K_4 = -0.03}$$

$$l_4 = hg\,(0.1,\ 2.985,\ -0.3)$$

$$= (0.1)\,(-3.015)$$

$$\boxed{l_4 = -0.3}$$

W.K.T,

$$y(x_0+h) = y(x_1) = y_0 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$y(0.1) = 3 + \frac{1}{6}\left[0 + (-0.03) - 0.03 - 0.03\right]$$

$$= 3 - 0.015$$

$$\boxed{y(0.1) = 2.985}$$

$$y'(x_0+h) = z(x_1) = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$

$$= 0 + \frac{1}{6}[-0.3 - 0.6 - 0.6 - 0.3]$$

$$= \frac{1}{6}[-1.8]$$

$$y'(0.1) = z(0.1) = -0.3$$

$$\boxed{z(0.1) = -0.3}$$ //

Milner's Predictor - Corrector Method:-

Q1. Apply milner's predictor- corrector method to compute $y(0.4)$, given the D.E $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$ and the following table of initial Values.

x	0	0.1	0.2	0.3
y	1	1.1103	1.2427	1.3990
y'	1	1.2103	1.4427	1.6990

$\Rightarrow$ given $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx} \longrightarrow (1)$

$$\frac{dy}{dx} = z = f(x, y, z) \longrightarrow (2)$$

$$\therefore (1) \Rightarrow \frac{dz}{dx} = 1 + z = g(x, y, z) \longrightarrow (3).$$

and given:-

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y	$y_0 = 1$	$y_1 = 1.1103$	$y_2 = 1.2427$	$y_3 = 1.3990$
$y = z$	$z_0 = 1$	$z_1 = 1.2103$	$z_2 = 1.4427$	$x_4 = 1.6990$

$$\therefore f_0 = f(x_0, y_0, z_0) = z_0 = 1$$

$$f_1 = f(x_1, y_1, z_1) = z_1 = 1.2103$$

$$f_2 = f(x_2, y_2, z_2) = z_2 = 1.4427$$

$$f_3 = f(x_3, y_3, z_3) = z_3 = 1.6990$$

$$\therefore g_0 = g_1(x_0, y_0, z_0) = 1 + z_0 = 2$$

$$g_1 = g(x_1, y_1, z_1) = 1 + z_1 = 2.2103.$$

$$g_2 = g(x_2, y_2, z_2) = 1 + z_2 = 2.4427$$

$$g_3 = g(x_3, y_3, z_3) = 1 + z_3 = 2.6990.$$

$$\therefore y_4^{(P)} = y_0 + 4h/3\,[2f_1 - f_2 + 2f_3]$$

$$= 1 + 4(0.1)/3\,[2(1.2103) - 1.4427 + 2(1.6990)]$$

$$\boxed{y_4^{(P)} = 1.5834}$$

$$\therefore z_4^{(P)} = z_0 + 4h/3\,[2g_1 - g_2 + 2g_3]$$

$$= 1 + 4(0.1)/3\,[2(2.2103) - 2.4427 + 2(2.6990)]$$

$$\boxed{z_4^{(P)} = 1.9834}$$

$$f_4^{(P)} = f[x_4, y_4^{(P)}, z_4^{(P)}] = z_4^{(P)} = 1.9834$$

$$y_4^{(c)} = y_2 + h/3\,[f_2 + 4f_3 + f_4^{(P)}]$$

$$= 1.2427 + 0.1/3\,[1.4427 + 4(1.6990) + 1.9834]$$

$$\boxed{y_4^{(c)} = 1.5834}$$

02. Given D.E $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ and the following initial Value is.

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	2.0657

Compute $y(1.4)$ by applying Milne's Predictor-Corrector formula.

$\rightarrow$ given $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx} \longrightarrow (1)$

$$\frac{dy}{dx} = z = f(x,y,z) \longrightarrow (2)$$

$(1) \Rightarrow$

$$2\frac{dz}{dx} = 4x + z$$

$$\frac{dz}{dx} = \frac{1}{2}\left[4x + z\right] = g(x,y,z) \longrightarrow (3)$$

and given.

x	$x_0 = 1$	$x_1 = 1.1$	$x_2 = 1.2$	$x_3 = 1.3$
y	$y_0 = 2$	$y_1 = 2.2156$	$y_2 = 2.4649$	$y_3 = 2.7514$
y'	$z_0 = 2$	$z_1 = 2.3178$	$z_2 = 2.6725$	$z_3 = 2.0657$

$\therefore f_0 = f(x_0\, y_0\, z_0) = z_0 = 2$ $\qquad g_0 = g(x_0\, y_0\, z_0) = 3$

$f_1 = f(x_1, y_1, z_1) = z_1 = 2.3178$ $\qquad g_1 = g(x_1\, y_1\, z_1) = 3.3589$

$f_2 = f(x_2, y_2, z_2) = z_2 = 2.6725$ $\qquad g_2 = g(x_2\, y_2\, z_2) = 3.7362$

$f_3 = f(x_3, y_3, z_3) = z_3 = 2.0657$ $\qquad g_3 = g(x_3\, y_3\, z_3) = 3.6328$

$$\therefore y_4^{(P)} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 2 + \frac{4(0.1)}{3}\left[2(2.3178) - 2.6725 + 2(2.0657)\right]$$

$$\boxed{y_4^{(P)} = 2.8126}$$

$$\therefore z_4^{(P)} = z_0 + \frac{4h}{3}\left[2g_1 - g_2 + 2g_3\right]$$

$$= 2 + \frac{4(0.1)}{3}\left[2(3.3589) - 3.7362 + 2(3.6328)\right]$$

$$\boxed{z_4^{(P)} = 3.3663}$$

$$f_4^{(P)} = f\left(x_4, y_4^{(P)}, z_4^{(P)}\right)$$

$$z_4^{(P)} = 3.3663$$

$$\therefore y_4^{(C)} = y_2 + h/3\left[f_2 + 4f_3 + f_4\right]$$

$$= 2.4649 + 0.1/3\left[2.6725 + 4(2.0657) + 3.3663\right]$$

$$\boxed{y_4^{(C)} = 2.94162} \quad //$$

03. Apply Milner's method to Compute $y(0.8)$ given that $y'' = 1 - 2yy'$ and the following table.

x	0	0.2	0.4	0.6
y	0	0.02	0.0795	0.1762
y'	0	0.1996	0.3972	0.5639

$\Rightarrow$ Given $y'' = 1 - 2yy' \longrightarrow (1)$

$$y' = z = f(x, y, z) \longrightarrow (2)$$

$$① \Rightarrow \frac{dz}{dx} = 1 - 2yz \longrightarrow ③ = g(x,y,z)$$

and given.

x	$x_0 = 0$	$x_1 = 0.2$	$x_2 = 0.4$	$x_3 = 0.6$
y	$y_0 = 0$	$y_1 = 0.02$	$y_2 = 0.0795$	$y_3 = 0.1762$
y'	$z_0 = 0$	$z_1 = 0.1996$	$z_2 = 0.3972$	$z_3 = 0.5634$

$f_1 = f(x_1 y_1 z_1) = z_1 = 0.1996$ $\quad g_1 = g(x_1 y_1 z_1) = 1 - 2y_1 z_1 = 0.9920$

$f_2 = f(x_2 y_2 z_2) = z_2 = 0.3972$ $\quad g_2 = g(x_2 y_2 z_2) = 1 - 2y_2 z_2 = 0.9368$

$f_3 = f(x_3 y_3 z_3) = z_3 = 0.5634$ $\quad g_3 = g(x_3 y_3 z_3) = 1 - 2y_3 z_3 = 0.8013.$

W.K.T

$$y_4^{(P)} = y^{(P)}_{(x_4)} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[2(0.1996) - 0.3972 + 2(0.8013)\right]$$

$$\boxed{y_4^{(P)} = 0.30128}$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left[2g_1 - g_2 + 2g_3\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[2 \times 0.9920 - 0.9368 + 2(0.8013)\right]$$

$$\boxed{z_4^{(P)} = 0.7066}$$

$$f_4^{(P)} = f(x_4, y_4^{(P)}, z_4^{(P)}) = z_4^{(P)} = 0.7066.$$

$$\therefore y^{(c)}_{x_4} = y_4^{(c)} = y_2 + h/3\left[f_2 + 4f_3 + f_4^{(P)}\right]$$

$$= 0.0795 + \frac{0.2}{3}\left[0.3972 + 4(0.5634) + 0.7066\right]$$

$$\boxed{y^{c}_{(0.8)} = 0.30346}$$

//

04. Use milner's method to obtain an approximate solution at the point $x=0.8$ of the problem.

$y'' = 2yy'$, given $y(0)=0$, $y'(0)=1$ $y(0.2)=0.2027$, $y'(0.2)=1.041$ $y(0.4)=0.4228$, $y'(0.4)=1.179$, $y(0.6)=0.6841$, $y'(0.6)=1.468$.

$\Rightarrow$ given $y'' = 2yy' \longrightarrow (1)$

$$y' = z = f(x,y,z) \longrightarrow (2)$$

$$(1) \Rightarrow \frac{dz}{dx} = 2yz = g(x,y,z) \longrightarrow (3)$$

and given table.

x	$x_0=0$	$x_1=0.2$	$x_2=0.4$	$x_3=0.6$
y	$y_0=0$	$y_1=0.2027$	$y_2=0.4228$	$y_3=0.6841$
$y'=z$	$z_0=1$	$z_1=1.041$	$z_2=1.179$	$z_3=1.468$

$f_1 = f(x_1\ y_1\ z_1) = z_1 = 1.041$ $\quad g_1 = g(x_1 y_1 z_1) = 2y_1z_1 = 0.42202$

$f_2 = f(x_2\ y_2\ z_2) = z_2 = 1.179$ $\quad g_2 = g(x_2 y_2 z_2) = 2y_2z_2 = 0.9969$

$f_3 = f(x_3\ y_3\ z_3) = z_3 = 1.468$ $\quad g_3 = g(x_3 y_3 z_3) = 2y_3z_3 = 2.0085$

W.K.T

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[2(1.041) - 1.179 + 2(1.468)\right]$$

$$\boxed{y_4^{(P)} = 1.0237}$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left[2g_1 - g_2 + 2g_3\right]$$

$$= 1 + \frac{4(0.2)}{3}\left[2(0.42202) - 0.9969 + 2(2.0085)\right]$$

$$\boxed{z_4^{(P)} = 2.0304}$$

$$f_4^{(P)} = f\left[x_4, y_4^{(P)}, z_4^{(P)}\right] = z_4^{(P)} = 2.0304.$$

$$\therefore y_4^{(c)} = y_{x_4}^{(c)} = y_2 + h/3\left[f_2 + 4f_3 + f_4^{(P)}\right]$$

$$= 0.4228 + \frac{0.2}{3}\left[1.179 + 4(1.468) + 2.0304\right]$$

$$\boxed{y^{(c)}_{(0.8)} = 1.0282}$$

05. Apply milne's method to Compute $y(0.4)$, given that $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$, $y'(0.1) = -0.0995$, $y(0.2) = 0.9802$, $y'(0.2) = -0.196$ $y(0.3) = 0.956$, $y'(0.3) = -0.2863$. $y(0.1) = 0.995$.

$\Rightarrow$ given $y'' + xy' + y = 0 \longrightarrow (1)$

$$y' = z \longrightarrow f(x, y, z) \longrightarrow (2)$$

$$(1) \Rightarrow \frac{dz}{dx} + xz + y = 0$$

$$\frac{dz}{dx} = -xz - y = g(x, y, z) \longrightarrow (3)$$

$$-(xz + y)$$

and table.

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y	$y_0 = 1$	$y_1 = 0.995$	$y_2 = 0.9302$	$y_3 = 0.956$
$y' = z$	$z_0 = 0$	$z_1 = -0.099$	$z_2 = -0.196$	$z_3 = -0.2863$

$f_1 = f(x_1 y_1 z_1) = z_1 = -0.0995$ $\qquad g_1 = g(x_1 y_1 z_1) = -(x_1 z_1 + y_1) = -0.98505$

$f_2 = f(x_2 y_2 z_2) = z_2 = -0.196$ $\qquad g_2 = g(x_2 y_2 z_2) = -(x_2 z_2 + y_2) = -0.941$

$f_3 = f(x_3 y_3 z_3) = z_3 = -0.2863$ $\qquad g_3 = g(x_3 y_3 z_3) = -(x_3 z_3 + y_3) = -0.87011$

$$y_4^{(P)} = y_0 + 4h/3\left[2f_1 - f_2 + 2f_3\right]$$

$$= 1 + 4(0.1)/3\left[2(-0.0995) + 0.196 - 2(0.2863)\right]$$

$$\therefore \boxed{y_4^{(P)} = 0.9232}$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left[2g_1 - g_2 + 2g_3\right]$$

$$= 0 + \frac{4(0.1)}{3}\left[2(-0.98505) + 0.941 - 2(0.8701)\right]$$

$$\boxed{z_4^{(P)} = -0.3693}$$

$$f_4^{(P)} = f(x_4, y_4^{(P)}, z_4^{(P)}) = z_4^{(P)} = -0.3693.$$

$$\therefore y_4^{(P)} = y^{(c)}_{(x_4)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4^{(P)}\right]$$

$$= 0.9802 + \frac{0.1}{3}\left[-0.196 - 4(0.2863) - 0.3693\right]$$

$$\boxed{y_4^{(c)} = y(0.4) = 0.9231}$$

⇒ Calculus of Variations :-

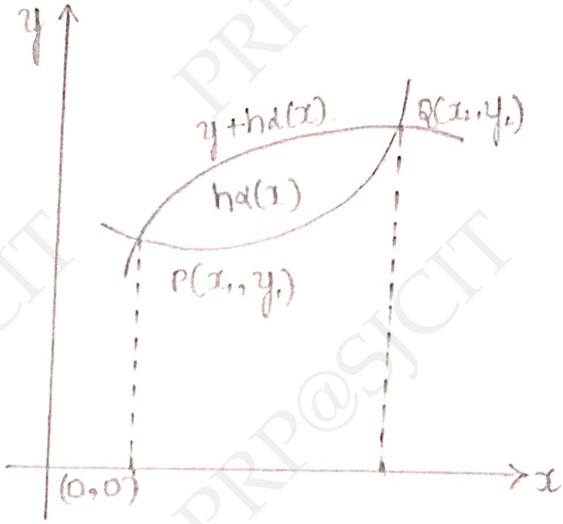
Euler's Theorem :-

Statement :- A necessary condition for the integral $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$, where $y(x_1) = y_1 ,\ y(x_2) = y_2$ to be as extimum of $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$.

Given :- $y = y(x)$ be a curve passing through the two points that be a $P(x_1, y_1)$ & $Q(x_2, y_2)$ passing and let the neighbouring curve of $y = y(x)$ passing through the same point P and Q as $y + h\alpha(x)$ and to extimize the neighbouring curve $\alpha(x_1) = 0$, $\alpha(x_2) = 0$ at P & Q.

Given :-

$$I = \int_{x_1}^{x_2} f(x,y,y')\,dx \longrightarrow (1)$$



$$= \int_{x_1}^{x_2} f(x, y+h\alpha(x), y'+h\alpha'(x))\,dx \rightarrow (2)$$

diff eqn (2) w.r.t. h, we get

$$(2) \Rightarrow \frac{dI}{dh} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial h}\right]dx$$

$$= \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial x}(0) + \frac{\partial f}{\partial y}\alpha(x) + \frac{\partial f}{\partial y'}\alpha'(x)\right]dx.$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\alpha(x) + \frac{\partial f}{\partial y'}\alpha'(x)\right]dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2}\frac{\partial f}{\partial y}\alpha(x)\,dx + \frac{\partial f}{\partial y'}\int_{x_1}^{x_2}\alpha'(x)\,dx - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\alpha'(x)\,dx.$$

$$= \int_{x_1}^{x_2}\frac{\partial f}{\partial y}\alpha(x)\,dx + \frac{\partial f}{\partial y'}\left[\alpha(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\alpha(x)\,dx.$$

$$= \int_{x_1}^{x_2}\frac{\partial f}{\partial y}\alpha(x)\,dx + \frac{\partial f}{\partial y'}\left[\alpha(x_1) - \alpha(x_2)\right] - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\alpha(x)\,dx.$$

$$= \int_{x_1}^{x_2}\frac{\partial f}{\partial y}\alpha(x)\,dx + \frac{\partial f}{\partial y'}(0-0) - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\alpha(x)\,dx.$$

$$= \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\alpha(x) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\alpha(x)\right]dx.$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\alpha(x)\,dx \longrightarrow (3)$$

For the extimum of I, we should make $\frac{dI}{dh}=0$

$$\Rightarrow \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\alpha(x)\,dx=0$$

whereas $\alpha(x)\neq 0$, therefore we have.

$$\boxed{\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0}$$

(1) Find the extimal for the function $\int_0^{\pi/2}(y^2-y'^2-2y\sin x)\,dx$. where $y(0)=0$, $y(\pi/2)=1$.

$$\Rightarrow \text{Let } \int_{x_1}^{x_2} f(x,y,y')\,dx=\int_0^{\pi/2}(y^2-y'^2-2y\sin x)\,dx \longrightarrow (1)$$

$$f=y^2-y'^2-2y\sin x$$

To Extimise (1), W.K.T

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$$

$$\Rightarrow (2y-2\sin x)-\frac{d}{dx}(-2y')=0$$

$$\Rightarrow y-\sin x+\frac{d}{dx}(y')=0$$

$$\Rightarrow y-\sin x+y''=0$$

$$\Rightarrow y''+y=\sin x$$

$$(D^2+1)y=\sin x.$$

A.E $\quad m^2+1=0 \Rightarrow m^2=-1 \Rightarrow m=\pm i$

The roots are $m=0\pm i$

$$y_c=c_1\cos x+c_2\sin x$$

$$\therefore y_p = \frac{\sin x}{D^2+1}$$

$$y_p = \frac{-x}{2(1)} \cos x$$

$$y_p = \frac{-x}{2} \cos x$$

$$y = y_c + y_p$$

$$= c_1 \cos x + c_2 \sin x - \frac{x}{2} \cos x \longrightarrow (3)$$

let $y(0) = 0$.

$$(3) \Rightarrow \quad 0 = c_1 \cos(0) + c_2 \sin(0) - 0/2 \cos(0)$$

$$0 = c_1$$

$$\boxed{c_1 = 0}$$

and $y(\pi/2) = 1$

$$(3) \Rightarrow \quad 1 = c_1 \cos(\pi/2) + c_2 \sin(\pi/2) - \frac{\pi/2}{2} \cos(\pi/2)$$

$$1 = 0 + c_2(1) - 0$$

$$\boxed{c_2 = 1}$$

$$\boxed{y = \sin x - \frac{x}{2} \cos x}$$

//

2] Find the extimal for the function of $\int_0^{\pi/2} [y'^2 - y^2 + 4y\cos x]\,dx$, for which $y(0) = 0$. $y(\pi/2) = 0$.

$\rightarrow$ Let $\int_{x_1}^{x_2} f(x, y, y')\,dx = \int_{x_1}^{x_2} (y'^2 + y^2 + 4y\cos x)\,dx \longrightarrow (1)$

$$f = y'^2 + y^2 + 4y\cos x$$

To Extimize (1), WKT.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

$$\Rightarrow \left(-2y + 4\cos x\right) - \frac{d}{dx}[2y'] = 0$$

$$\Rightarrow -y + 2\cos x - \frac{d}{dx} y' = 0$$

$$\Rightarrow y + 2\cos x - y'' = 0$$

$$\Rightarrow y'' + y = 2\cos x$$

$$\Rightarrow (D^2+1)y = 2\cos x$$

A.E $\quad m^2+1=0$

$m = 0 \pm i$

$$y_c = c_1\cos x + c_2\sin x$$

$$y_p = \frac{2\cos x}{D^2+1}$$

$$\Rightarrow y_p = \frac{2x\,\sin(1)x}{2(1)} \Rightarrow \cancel{2}\frac{x}{\cancel{2}}\sin x$$

$$y = y_c + y_p$$

$$\Rightarrow y = c_1\cos x + c_2\sin x + x\sin x \longrightarrow (3)$$

let $y(0)=0$

$$0 = c_1\cos(0) + c_2\sin(0) + \cancel{x\sin(0)}^{\,0}$$

$$0 = c_1 + 0 + 0$$

$$\boxed{c_1 = 0}$$

and $y(\pi/2)=0$

$$0 = c_1\cos(\pi/2) + c_2\sin(\pi/2) + \pi/2\sin(\pi/2)$$

$$0 = 0 + c_2 + \pi/2$$

$$\boxed{c_2 = -\pi/2}$$

$$\boxed{y = -\pi/2\,\sin x + x\sin x}$$

//

03. Find the Extimal of the function of $I = \int_0^1 [y'^2 - y^2 - ye^{2x}]$ that passes through the points $(0,0)$, $(1, 1/e)$.

$\Rightarrow$ Let $\int_{x_1}^{x_2} f(x, y, y')\,dx = \int_0^1 (y'^2 - y^2 - ye^{2x})\,dx$.

$$f = y'^2 - y^2 - ye^{2x}$$

W.K.T $\quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$= (-2y - e^{2x}) - \frac{d}{dx}(2y') = 0$$

$$\Rightarrow -2y - e^{2x} - 2y'' = 0$$

$$\Rightarrow 2y'' + 2y + e^{2x} = 0$$

$$\Rightarrow y'' + y = e^{2x}/2$$

$$(D^2 + 1)y = -1/2\, e^{2x} \longrightarrow (1)$$

The A.E $\quad m^2 + 1 = 0$

$$m = 0 \pm 1i$$

$$y_c = C_1 \cos x + C_2 \sin x.$$

$$y_p = -\frac{1}{2}\frac{e^{2x}}{D^2+1} = -\frac{1}{2}\frac{e^{2x}}{2^2+1}$$

$$= \frac{-1}{2}\frac{e^{2x}}{5} = \frac{-1}{10}e^{2x}$$

$$y = y_c + y_p$$

$$= C_1 \cos x + C_2 \sin x - \frac{1}{10}e^{2x} \longrightarrow (2)$$

Let $y = 0$ at $x = 0$

$(2) \Longrightarrow \quad 0 = C_1 \cos 0 + C_2 \sin 0 - \frac{1}{10}e^{0}$

$$C_1 - \frac{1}{10} = 0$$

$$\boxed{C_1 = 1/10}$$

and given $y = 1/e$ at $x=1$

$\Rightarrow 1/e = C_1\cos 1 + C_2\sin 1 - 1/10\, e^1$

$\Rightarrow 1/e = 1/10(0.5403) + C_2(0.8414) - e^2/10$

$\Rightarrow C_2(0.8414) = 1/e + e^2/10 - 0.05403.$

$\Rightarrow C_2(0.8414) = 0.3678 + 0.7389 - 0.05403$

$\Rightarrow C_2(0.8414) = 1.0526$

$\Rightarrow C_2 = 1.2510$

$y = 1/10\cos x + (1.2510)\sin x - \frac{1}{10}e^{2x}.$

4] Find the extremum for the function $I = \int_{x_1}^{x_2}[y^2+y'^2+2ye^x]dx$

$\rightarrow$ Let $\int_{x_1}^{x_2} f(x, y, y')dx = \int_{x_1}^{x_2}[y^2+y'^2+2ye^x]dx.$

$f = y^2+y'^2+2ye^x$

WKT $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$= (2y+2e^x) - \frac{d}{dx}(2y') = 0$

$\Rightarrow y + e^x - y'' = 0$

$\Rightarrow -y'' + y = -e^x$

$\Rightarrow y'' - y = e^x \longrightarrow (1)$

$\Rightarrow (D^2-1)y = e^x$

The A.E $m^2 - 1 = 0$

$\Rightarrow m^2 = 1$

$\Rightarrow m = 1, -1$

$y_c = C_1e^x + C_2e^{-x}$

$$y_p = \frac{e^x}{D^2-1}$$

$$y_p = \frac{e^x}{(D-1)(D+1)} = \frac{x^1}{x!}\frac{e^x}{D+1} = \frac{x^1 e^x}{(1+1)} = \frac{x}{2}e^x$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow y = c_1 e^x + c_2 e^{-x} + \frac{x}{2}e^x$$

5] On what curves can the functional $\int_0^1 \left[y'^2 + 12xy\right] dx$ with $y(0)=0$, $y(1)=1$ can be extimized.

→ Let $\int_{x_1}^{x_2} f(x,y,y')\,dx = \int_0^1 (y'^2 + 12xy)\,dx$

$$f = y'^2 + 12xy$$

W.K.T $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\Rightarrow 12x - \frac{d}{dx}(2y') = 0$$

$$\Rightarrow 6x - y'' = 0 \qquad \frac{d^2y}{dx^2} = 6x$$

$$y'' = 6x$$

Integral twice,

$$\Rightarrow \int \frac{d}{dx}\left(\frac{dy}{dx}\right) = \int 6x\,dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{6x^2}{2} + c_1$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 + c_1$$

$$\Rightarrow \int \frac{dy}{dx} = \int (3x^2 + c_1)\,dx$$

$$\Rightarrow y = \frac{3x^3}{3} + c_1 x + c_2$$

$$\Rightarrow y = x^3 + c_1 x + c_2 \longrightarrow ①$$

given $y(0)=0$

$(1) \Rightarrow \quad 0 = 0^3 + c_1(0) + c_2$

$$\boxed{c_2 = 0}$$

and $y(1)=1$.

$(1) \Rightarrow \quad 1 = 1^3 + c_1(1) + c_2$

$\Rightarrow 1 = 1 + c_1 \cancel{1} + \cancel{c_2} 0$

$\Rightarrow 1 = 1 + c_1(1)$

$\Rightarrow c_1(1) = 0$

$\Rightarrow c_1(1) = 0$

$\Rightarrow \boxed{c_1 = 0}$

$$\boxed{y = x^3}$$

6] On what curves can the functional $\int_0^1 [y'^2 + 2xy]\,dx$ with $y(0)=0$, $y(1)=1$ can be extemised.

$\rightarrow$ Let $\int_{x_1}^{x_2} f(x,y,y')\,dx = \int_0^1 [y'^2 + 2xy]\,dx$

$f = y'^2 + 2xy.$

WKT $\quad \dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$

$\Rightarrow 2x - \dfrac{d}{dx}(2y') = 0$

$\Rightarrow 2x - 2y'' = 0$

$\Rightarrow x - y'' = 0$

$\Rightarrow y'' = x$

$\Rightarrow \dfrac{d^2y}{dx^2} = x$

$\Rightarrow \int \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \int x\,dx.$

$$\frac{dy}{dx} = \frac{x^2}{2} + c_1$$

$$\int \cancel{\frac{dy}{dx}} = \int \left(\frac{x^2}{2} + c_1\right) dx$$

$$\int d \cdot y = \frac{x^3}{6} + c_1 x + c_2$$

given $y(0) = 0$

$$0 = 0 + c_1(0) + c_2$$

$$\Rightarrow \boxed{c_2 = 0}$$

and $y(1) = 1$

$$1 = 1/6 + c_1$$

$$\Rightarrow c_1 = -1/6 + 1$$

$$\Rightarrow \boxed{c_1 = 5/6}$$

$$\therefore \boxed{y = x^3/6 + 5/6\, x}$$

7] Find the Extimal of the function $\int_{x_1}^{x_2} [y' + x^2 y'^2] dx$

→ Let $\int_{x_1}^{x_2} f(x, y, y') dx = \int_{x_1}^{x_2} [y' + x^2 y'^2] dx$

$$f = y' + x^2 y'^2$$

W.K.T $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\Rightarrow 0 - \frac{d}{dx}(1 + 2x^2 y') = 0$$

$$\Rightarrow \frac{d}{dx}(1 + 2x^2 y') = 0$$

$$\Rightarrow 0 + 2(2xy' + x^2 y'') = 0$$

$$\Rightarrow 2xy' + x^2 y'' = 0$$

$$\Rightarrow x^2 y'' + 2xy' = 0$$

let $\log x = z$ $x = e^z$

and W.K.T $xy' = Dy$

$x^2y'' = D(D-1)y$

when $D = \frac{d}{dz}$

$(1) \Rightarrow D(D-1)y + 2Dy = 0$

$(D^2 - D + 2D)y = 0$

$(D^2 + D)y = 0$

when A.E is $m^2 + m = 0$

$m = 0, -1$

$y = c_1 e^{0(z)} + c_2 e^{-z}$

$\Rightarrow y = c_1 + c_2 e^{-z}$

$\Rightarrow y = c_1 + \frac{c_2}{e^z}$

$$\boxed{\therefore y = c_1 + \frac{c_2}{x}}$$

8] Prove that geodesic of a plane surface are a straight line.

→ WKT, the length of the arc geodesic $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$\frac{ds}{dx} = \sqrt{1 + (y')^2}$ and to maximize the arc length.

We have $\int_{x_1}^{x_2} f(x, y, y')dx = \int_{x_1}^{x_2} \frac{ds}{dx} . dx$

$= \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx.$

$\therefore f = \sqrt{1 + y'^2}$

WKT $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$

$$\Rightarrow \quad 0 - \frac{d}{dx}\left[\frac{1}{2\sqrt{1+y'^2}}\,2y'y''\right] = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0$$

$$\Rightarrow \quad \frac{\left(\sqrt{1+y'^2}\right)(y'') - y'\frac{1}{\cancel{2}\sqrt{1+y'^2}} \times \cancel{2}y'y''}{1+y'^2}$$

$$\Rightarrow \quad y''\sqrt{1+y'^2} - \frac{y'^2 y''}{\sqrt{1+y'^2}} = 0$$

$$\Rightarrow \quad \frac{y''(1+y'^2)}{} - \frac{y'^2 y''}{\sqrt{1+y'^2}} = 0$$

$$\Rightarrow \quad \frac{y''(1+y'^2) - y'^2 y''}{\sqrt{1+y'^2}} = 0$$

$$\Rightarrow \quad y'' + \cancel{y'^2 y''} - \cancel{y'^2 y''} = 0$$

$$\Rightarrow \quad y'' = 0$$

$$\Rightarrow \quad D^2 y = 0$$

A.E $m^2 = 0$

$m = 0, 0$

$$y = \cancel{Ce}\,(C_1 + C_2 x)\, e^{0.x}$$

$\Rightarrow y = C_1 + C_2 x$ is a straight line //